This is the Pre-Published Version.

Minimizing Nondecreasing Separable Objective Functions for the Unit-time Open Shop Scheduling Problem

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Abstract

In this paper we study both the nonpreemptive and preemptive versions of the popular unit-time open shop scheduling problem. For the set of feasible schedules which satisfy a predetermined order of job completion times, we construct the linear description of the convex hull of the vectors of the job completion times. Based on the properties of the resulting scheduling polyhedron, we show that the problem of constructing an optimal schedule minimizing an arbitrary nondecreasing separable cost function of job completion times is polynomially solvable.

Keywords: Scheduling; Open shop scheduling; Parallel machine scheduling, Separable nondecreasing function

1 Introduction

We consider an open shop scheduling problem with n jobs and m machines. Each job has to be processed by each machine exactly once in an arbitrary order and has a given processing time p_{ik} for its operation O_{ik}, $i = 1, \dots, n$, $k = 1, \dots, m$. Each machine can handle at most one operation at a time and each job can be processed by at most one machine at a time. If preemption is forbidden, then a schedule can be specified either by the starting or completion times of all operations. If preemption is allowed, then the processing of any operation O_{ik} may be interrupted at any time and resumed later, provided that the total length of all parts of the operation is equal to p_{ik}. A preemptive schedule can be specified by the starting or completion

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times of all parts of the operations. In either the preemptive or nonpreemptive case, a schedule is characterized by the vector $\mathbf{C} = (C_1, \ldots, C_n)$ of the completion times of the jobs, where $C_i = \max\{C_{ik} | k = 1, \ldots, m\}$ is the completion time of job i and C_{ik} is completion time of operation O_{ik}.

In this paper we focus on the special case of the problem with unit-time operations, i.e. $p_{ik} = 1$ for $i = 1, \ldots, n,\ k = 1, \ldots, m$. This case is closely related to the scheduling problem with m identical parallel machines and n identical jobs with preemption allowed. In the latter problem, each job i, $1 \leq i \leq n$, can be processed by any machine k, $1 \leq k \leq m$, which requires $p_i = m$.

The objective for both open shop and parallel machine problems is to construct a schedule which minimizes a given function $F = \sum_{i=1}^{n} w_i f(C_i)$ depending on the completion times of the jobs, where w_i is the weight of job i and $f(C_i)$ is an arbitrary nondecreasing function. The known examples of separable nondecreasing functions which have been studied in the scheduling literature are

- $F_1 = \sum_{i=1}^{n} w_i C_i$,
- $F_2 = \sum_{i=1}^{n} w_i C_i^2$,
- $F_3 = \sum_{i=1}^{n} w_i (C_i - p_i)^2$,
- $F_4 = \sum_{i=1}^{n} w_i {T_i}^2$, where $T_i = \max\{C_i - D_i, 0\}$ is a tardiness of job i with respect to due date D_i,
- $F_5 = \sum_{i=1}^{n} w_i (1 - e^{-rC_i})$, where r is a discount rate per unit time, $0 < r < 1$,
- $F_6 = \sum_{i=1}^{n} w_i e^{\alpha(C_i - D_i)}$, where α is a positive constant and D_i is the due date of job i.

The first function F_1 is the total weighted completion time and it is very well studied for the unit-time open shop problem (see [1, 6, 8, 24]). The second function F_2 was introduced by Townsend (1978) for the single machine problem with unequal job processing times. His paper [30] was the first one to address a scheduling problem with the quadratic performance measure. Since then various modified versions of the Townsend's branch and bound algorithm and the other approaches have been developed [2, 9, 12, 26]. The next function F_3 generalizes function F_2. It was introduced by Szwarc and Mukhopadhyay [27, 28] in order to minimize the weighted sum of squared waiting times, that is to reduce the inventory costs of raw materials and work-in-process inventories. Function F_4 is the total weighted squared tardiness introduced in [25] as a compromise of the maximum tardiness and the total tardiness. The exponential function F_5 was introduced by Rothkopf in [20] and was also studied in [15, 21]. It can be treated as the discounted total weighted completion time (see [22]). The last function F_6 discussed in [8] is related to F_5 and it can be considered as the total weighted exponential lateness. Observe that functions F_2, F_3, F_4 and F_6 are convex, F_5 is concave and the linear function F_1 is both convex and concave.

While the problem of minimizing the objective function $F = \sum_{i=1}^{n} w_i f(C_i)$ has attracted much attention of scheduling researchers, it has been investigated mainly for the single machine case: polynomial-time algorithms are known for F_1, F_5, and F_6. For the multi-machine case with arbitrary processing times, it is unlikely that a polynomial-time algorithm can be developed for any of the functions F_1, $F_2, \ldots, F_6$ since the problem with two parallel machines and function F_1 is NP-hard [7]. On the other hand, the multi-machine problem with equal processing times has been extensively studied for the objective function F_1, but it has not been investigated for any of the functions $F_2, F_3, \ldots, F_6$.

Thus our first objective is to study the multi-machine problem of minimizing any of the functions $F_2, F_3, \ldots, F_6$ or, in general, an arbitrary nondecreasing separable function $F = \sum_{i=1}^{n} w_i f(C_i)$. Second, we aim to advance the research on the unit-time open shop scheduling problem from a polyhedral point of view by presenting a new mathematical formulation of its scheduling polyhedron.

The polyhedral aspects of scheduling problems have been extensively studied since the pioneering work by Balas [3]. Most of the research in this area, however, concentrates on the single machine problem (see [17, 19, 32]) with the exception of [18], which investigates the non-preemptive unit-time parallel machine problem with general assumptions on the release dates and machine speeds. That paper shows that the scheduling polyhedron is supermodular, which implies the possibility of minimizing function F_1 by a greedy algorithm.

In this research we construct a description of the scheduling polyhedron for the preemptive parallel machine and open shop problems with zero release dates and equal machine speeds, which leads to polynomial time algorithms for variants of the problems involving an arbitrary nondecreasing objective function F including the functions F_1-F_6 as the special cases. The description consists of simultaneous linear inequalities that specify the convex hull of the characteristic vectors associated with the feasible open shop schedules. We investigate some properties of the polyhedron and show that the problem can be solved in polynomial time in both continuous and discrete variables or, equivalently, the scheduling problem with or without preemption is polynomially solvable.

To denote scheduling problems, we follow the standard classification scheme $\alpha|\beta|\gamma$ (see, e.g., [8]), where α describes the machine environment, β stands for the job characteristics, and γ

is the objective function. For the original open shop problem, $\alpha = O$, and for the auxiliary parallel machine problem, $\alpha = P$. The job characteristic β may include one of the conditions $p_{ik} = 1$ or $p_i = m$. The parameter *pmtn* in the second field denotes that preemption is allowed and [*pmtn*] denotes that preemption is allowed at integer times. The third field γ specifies the objective function F. The nonpreemptive and preemptive variants of our open shop problem are denoted by $O|p_{ik} = 1|F$ and $O|p_{ik} = 1, pmtn|F$, respectively; while the two variants of the parallel machine problem are denoted by $P|p_i = m|F$ and $P|p_i = m, pmtn|F$, respectively.

The paper is organized as follows. In Section 2, we first determine the order of job completion times in the optimal schedule which is induced by the objective function F. Using the result from [10, 13], we specify linear inequalities that describe the convex hull of the feasible completion time vectors which satisfy the specified job order. In fact, the convex hull is determined for an auxiliary problem $P|p_i = m, pmtn|F$ with each point of that hull corresponding to a feasible open shop schedule (perhaps with preemption) and the integer points corresponding to the nonpreemptive open shop schedules. In Section 3 we examine the properties of the scheduling polyhedron. Based on these properties, we demonstrate in Section 4 how an arbitrary nondecreasing function F can be minimized in polynomial time. Directions worthy of further research are suggested in Section 5.

2 The solution region of the problem $O|p_{ik} = 1|F$

In this section, we construct the linear description of the scheduling polyhedron for the problems $O|p_{ik} = 1|F$ and $O|p_{ik} = 1, pmtn|F$, which is defined as the convex hull of the job completion time vectors $\mathbf{C}(s)$ associated with the feasible schedules s. The dominating integer points of the scheduling polyhedron correspond to the characteristic vectors of the nonpreemptive problem $O|p_{ik} = 1|F$ and all noninteger points correspond to the characteristic vectors of the preemptive problem $O|p_{ik} = 1, pmtn|F$. We say that vector $\mathbf{X} = (x_1, x_2, \cdots, x_n)$ dominates vector $\mathbf{Y} = (y_1, y_2, \cdots, y_n)$ if $x_i \le y_i$ for $i = 1, \ldots, n$, and at least one of these inequalities is strict.

In order to derive the analytic formulas for the scheduling polyhedron, we use the following two results:

A. The equivalence of the problems $O|p_{ik} = 1, pmtn|F$ and $P|p_i = m, pmtn|F$ (see [5]);

B. The necessary and sufficient conditions for the existence of a feasible schedule that respects the deadlines $D_1, D_2, \ldots, D_n$ for the parallel machine problem $P|p_i = m$, $pmtn|C_i \le D_i$ (see [10, 13]).

The results can be formulated as follows.

Result A *Any feasible schedule for the problem $O|p_{ik} = 1, pmtn|F$ can be transformed into a feasible schedule for the problem $P|p_i = m, pmtn|F$ without changing the job completion times and vice versa.*

Observe that the equivalence of problems $O|p_{ik} = 1|F$ and $P|p_i = m, [pmtn]\ |F$ is proved in [6].

Result B *Let the jobs be numbered in accordance with $D_1 \leq D_2 \leq \ \ldots \ \leq D_n$. A feasible schedule for the problem $P|p_i = m,\ pmtn|C_i \leq D_i$ exists if and only if the following inequalities hold:*

$$\begin{aligned} D_1 &\geq m, \\ \sum_{i=\mu-m+1}^{\mu} D_i &\geq \mu m, \qquad \mu = m+1, \ldots, n. \end{aligned}$$

Due to Result A, the scheduling polyhedra for the problems $O|p_{ik} = 1, pmtn|F$ and $P|p_i = m, pmtn|F$ coincide. Due to Result B, the completion times $(C_1, \ldots, C_n)$ of an arbitrary feasible schedule for the problem $P|p_i = m, pmtn|F$, if renumbered in nondecreasing order, satisfy the inequalities

$$\mathcal{P}^1 : \begin{cases} C_1 \leq C_2 \leq \ldots \leq C_n, \\ C_1 \geq m, \\ \sum\limits_{i=\mu-m+1}^{\mu} C_i \geq \mu m, \qquad \mu = m+1, \ldots, n, \end{cases} \tag{1}$$

which specify the scheduling polyhedron $\mathcal{P}^1$.

The following lemma establishes a useful property of any optimal schedule for a nondecreasing separable function $F = \sum_{i=1}^{n} w_i f(C_i)$.

Lemma 1 *If $f(x)$ is a nondecreasing function and all jobs have equal processing times, then the minimum of the separable function $F = \sum_{i=1}^{n} w_i f(C_i)$ over all permutations $\pi = (\pi(1), \ldots, \pi(n))$ is attained for*

$$C_{\pi(1)} \leq C_{\pi(2)} \leq \ldots \leq C_{\pi(n)}, \qquad w_{\pi(1)} \geq w_{\pi(2)} \geq \ldots \geq w_{\pi(n)}. \tag{2}$$

The lemma can be proved by the standard pairwise job interchange argument.

Based on Results A,B and Lemma 1, we can reduce the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$ to the problem:

$$\text{Minimize} \sum_{i=1}^{n} w_i f(C_i), \quad \text{subject to (1).} \tag{3}$$

We study the relationship between the points of the polyhedron $\mathcal{P}^1$ and feasible schedules of the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$. In the next theorem we demonstrate that any point of the scheduling polyhedron $\mathcal{P}^1$ can be approximated with a point corresponding to a feasible schedule for the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$. More precisiely, we show that for any point $\mathbf{C} = (C_1, \ldots, C_n)$ of the scheduling polyhedron $\mathcal{P}^1$ there exists a schedule $\widetilde{\mathbf{C}}$ with job completion times $\tilde{C}_1, \ldots, \tilde{C}_n$ such that $|C_i - \tilde{C}_i| \leq \varepsilon$ for $i = 1, 2, \ldots, n$ and for any positive ε.

Theorem 1 *The points $\mathbf{C} = (C_1, \ldots, C_n)$ of the scheduling polyhedron $\mathcal{P}^1$ satisfy:*

1) For any point $\mathbf{C} \in \mathcal{P}^1$, there exists a "non-worse" point $\widetilde{\mathbf{C}} = (\tilde{C}_1, \ldots, \tilde{C}_n) \in \mathcal{P}^1$ with $\tilde{C}_i \leq C_i,\ i = 1, \ldots, n$, that determines a feasible schedule for the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$.

2) If a point $\mathbf{C} = (C_1, \ldots, C_n)$ of the scheduling polyhedron $\mathcal{P}^1$ is not dominated by any other point from $\mathcal{P}^1$, then it specifies a feasible schedule for the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$.

3) An arbitrary point $\mathbf{C}$ of the scheduling polyhedron $\mathcal{P}^1$ can be approximated with a point corresponding to a feasible schedule for the problems $P|p_i = m, pmtn|F$ and $O|p_{ik} = 1, pmtn|F$.

Observe that the known descriptions of scheduling polyhedra (see [17, 18]) do not have property 3).

Proof. 1) The existence of the "non-worse" point $\widetilde{\mathbf{C}}$ for the problem $P|p_i = m, pmtn|F$ immediately follows from Result B is the values of $C_1, C_2, \ldots, C_n$ are used instead of deadlines $.D_1, D_2, \ldots, D_n$. The corresponding optimal schedule can be constructed by means of the $O(n \log mn)$ algorithm from [23]. The number of preemptions in the resulting schedule is not larger than $n - 2$ (see [23]). That schedule can be transformed into a feasible schedule for the problem $O|p_{ik} = 1, pmtn|F$ by the $O(qm^2)$ algorithm from [5], where q is the number of states in the preemptive parallel machine schedule and it is not greater than the number of preemptions $n - 2$.

2) As it follows from 1), there exists a feasible schedule $\widetilde{\mathbf{C}}$ for $P|p_i = m, pmtn|F$ whose completion times $(\tilde{C}_1, \ldots, \tilde{C}_n)$ satisfy: $\tilde{C}_i \leq C_i$. Due to the assumption, there is no component j for which $\tilde{C}_j < C_j$. It means that $\tilde{C}_i = C_i,\ i = 1, \ldots, n$. The schedule itself can be constructed as described in 1).

3) As described above, we can always construct a feasible schedule $\widetilde{\mathbf{C}}$ with job completion times $\tilde{C}_i \leq C_i, i = 1, \ldots, n$. To construct a feasible schedule approximating $\mathbf{C}$, we transform a feasible schedule $\widetilde{\mathbf{C}}$ by inserting idle times and processing infinitesimal parts of each job i, $i = 1, \ldots, n$, as close to C_i as possible. ■

We illustrate Theorem 1 by the following example with $n = 3$ jobs, $m = 2$ parallel machines and processing times $p_i = m$, $i = 1, \ldots, n$. The scheduling polyhedron $\mathcal{P}^1$ is given by the simultaneous inequalities

$$\begin{cases} C_1 \leq C_2 \leq C_3, \\ C_1 \geq 2, \\ C_2 + C_3 \geq 6. \end{cases} \tag{4}$$

The two integer vectors $\mathbf{C}^1 = (2, 2, 4)$ and $\mathbf{C}^2 = (2, 3, 3)$ satisfy (4). They are not dominated by any other vector satisfying (4) and they determine the feasible schedules represented in Figure 1. Vector $\mathbf{C}^3 = (3, 3, 3)$ also satisfies simultaneous inequalities (4) but it is dominated by vector $\mathbf{C}^2$ and it does not determine a feasible schedule: two machines cannot complete processing three jobs at time $C_1 = C_2 = C_3 = 3$. On the other hand, there exists a noninteger vector $\mathbf{C}^4 = (3, 3, 3 + \varepsilon)$ which approximates vector $\mathbf{C}^3$ and determines a feasible preemptive schedule with inserted idle time. Schedules $\mathbf{C}^1$, $\mathbf{C}^2$ and $\mathbf{C}^4$ are illustrated in Fig. 1.

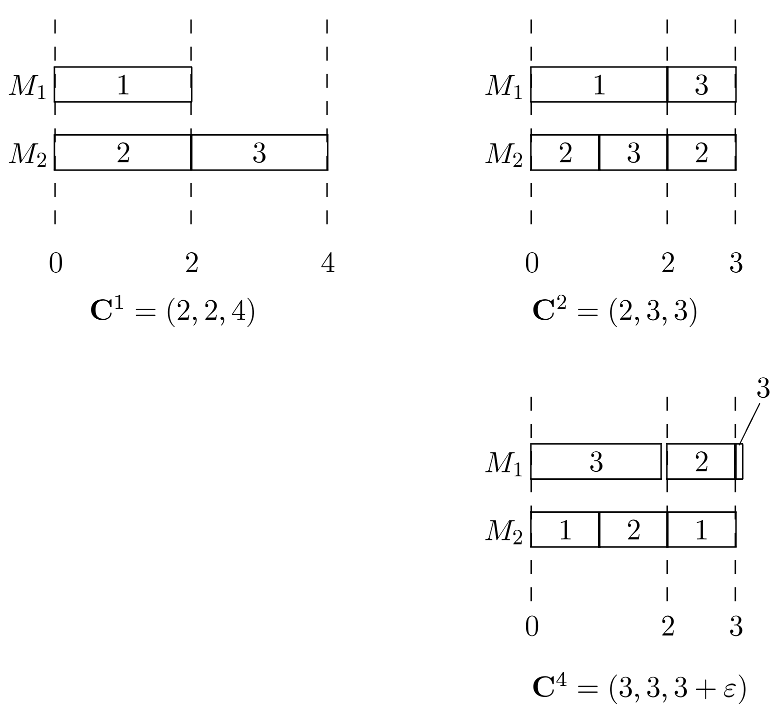

Figure 1: Schedules $\mathbf{C}^1$, $\mathbf{C}^2$ and $\mathbf{C}^4$

Consider now the nonpreemptive problem $O|p_{ik} = 1|F$ and the parallel machine problem with preemptions at integer times $P|p_i = m, [pmtn]\ |F$. The next theorem establishes the properties similar to 1)-2).

Theorem 2 *The integer points* $\mathbf{C} = (C_1, \ldots, C_n)$ *of the scheduling polyhedron* $\mathcal{P}^1$ *satisfy:*

1) *For an arbitrary integer point* $\mathbf{C}$ *of the scheduling polyhedron* $\mathcal{P}^1$*, there exists a "non-worse" integer point* $\widetilde{\mathbf{C}} = (\tilde{C}_1, \ldots, \tilde{C}_n)$ *with* $\tilde{C}_i \leq C_i,\ i = 1, \ldots, n$*, that determines a feasible schedule for the problems* $P|p_i = m, [pmtn]|F$ *and* $O|p_{ik} = 1|F$*.*

2) *If an integer point* $\mathbf{C}$ *of the scheduling polyhedron* $\mathcal{P}^1$ *is not dominated by any other integer point from* $\mathcal{P}^1$*, then it specifies a feasible schedule for the problems* $P|p_i = m, [pmtn]|F$ *and* $O|p_{ik} = 1, pmtn|F$*.*

Proof. Property 1) can be proved in a similar way as in the previous theorem. As above, the schedule with completion times $(\tilde{C}_1, \ldots, \tilde{C}_n)$ can be constructed by the algorithm from [10, 23] and in the resulting schedule preemptions occur at integer times. A feasible nonpreemptive open shop schedule can now be obtained in $O(nm \log^2(nm))$ time using the approach of Brucker et al. [6] based on the edge coloring algorithm. ∎

Thus we can conclude that the problems $O|p_{ik} = 1|F$ and $P|p_i = m, [pmtn]\ |F$ can be reduced to problem (3) with additional constraints on the integer values of C_i.

Observe that verifying whether a vector $\mathbf{C}$ of the scheduling polyhedron $\mathcal{P}^1$ is not dominated by any other vector from $\mathcal{P}^1$ can be done in $O(n)$ steps for both Theorems 1 and 2.

3 Some properties

In this section, we study the properties of the scheduling polyhedron and an optimal schedule.

3.1 Lower and upper bounds on job completion times

Simultaneous inequalities (1) determine an unbounded polyhedron $\mathcal{P}^1$ with an infinitely large number of points corresponding to the schedules with inserted idle times. The importance of limiting the search space by replacing the unbounded polyhedron with a closed one with regard to combinatorial problems was stressed in [19]. To address this problem and to reduce the search space, we may consider only a class of the so called "dense" schedules, for which a machine is idle if and only if there is no job waiting for this machine [4]. Clearly, for any nondecreasing objective function there exists an optimal schedule in the class of dense schedules.

The following lemma specifies the lower and upper bounds for the completion time C_i of each job i. It is assumed that, for a given schedule, the jobs are numbered in nondecreasing order of their completion times.

Theorem 3 *For any dense schedule, the job completion times satisfy the inequalities:*

$$\max\{i, m\} \leq C_i \leq i + m - 1, \quad i = 1, \ldots, m. \tag{5}$$

The proof is given in the Appendix. Observe that, as follows from Theorem 3, $C_1 = m$ for any dense schedule and hence for the optimal one as well.

3.2 Integral property

Integral polyhedra play an important role in combinatorial optimization. As we will show, the polyhedron of problem (3) is not integral. On the other hand, we can enlarge the minimization space of problem (3) retaining the same optimum solution while incorporating the integral property in the enlarged polyhedron.

First we give an example justifying that not all extreme points of the polyhedron $\mathcal{P}^1$ are integers. Let $n = 4$ and $m = 3$, then (1) becomes

$$C_1 \geq 3, \tag{6}$$

$$C_2 \geq C_1, \tag{7}$$

$$C_3 \geq C_2, \tag{8}$$

$$C_4 \geq C_3, \tag{9}$$

$$C_2 + C_3 + C_4 \geq 12. \tag{10}$$

It is easy to check that the extreme point defined by (6), (7), (9) and (10) treated as equalities has a form $(3, 3, 4.5, 4.5)$, i.e., it is not integral.

Let us now consider the alternative formulation of the minimization problem (3) obtained by replacing the inequalities

$$C_1 \leq C_2 \leq \ldots \leq C_n \tag{11}$$

by the lower bound on job completion time determined in Theorem 3:

$$C_i \geq \max\{i, m\}, i = 1, \ldots, n,$$

i.e. the problem of minimizing F over the polyhedron

$$\mathcal{P}^2 : \begin{cases} \quad C_i \geq \max\{i, m\}, & i = 1, \ldots, n, \\ \sum\limits_{i=\mu-m+1}^{\mu} C_i \geq \mu m, & \mu = m+1, \ldots, n. \end{cases} \tag{12}$$

Due to Lemma 1, the minimum of the nondecreasing separable function $F = \sum_{i=1}^{n} w_i f(C_i)$ satisfies the ordering (11) and, in this sense, the problem with the enlarged minimization space (12) is equivalent to problem (3).

In the matrix form, we represent this polyhedron as $Ax \geq b$ with the matrix A given by

$$A = \left(\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & & 1 \\
\hline
1 & 1 & 1 & 1 & & & & & & & \\
 & 1 & 1 & 1 & 1 & & & & & & \\
 & & 1 & 1 & 1 & 1 & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & 1 & 1 & 1 & 1 & & \\
 & & & & & & 1 & 1 & 1 & 1 & \\
 & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & \multicolumn{4}{c}{\underbrace{\hphantom{1\quad 1\quad 1\quad 1}}_{m\ \text{‘1’s}}}
\end{array}\right)
\begin{array}{l}
C_i \geq m, \quad i = 1, \ldots, n, \\[8em]
\sum\limits_{i=\mu-m+1}^{\mu} C_i \geq \mu m, \quad \mu = m+1, \ldots, n.
\end{array}$$

where in the first submatrix (rows $i = 1, \ldots, n$), ‘1’ is in the i-th position, and in the second submatrix (rows $i = n+1, \ldots, 2n-m$), ‘1’s are in positions $i-n+1, \ldots, i-n+m$. Since any row of the matrix A has the consecutive-ones-property, i.e., it has the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, A is totally unimodular or, equivalently, the polyhedron $\mathcal{P}^2$ is integral (see [11]).

The integral property of the polyhedron $\mathcal{P}^2$ and the equivalence of the problems with the simultaneous inequalities (12) and (1) will be used essentially in Section 4 dealing with the algorithmic issues.

We illustrate the polyhedra $\mathcal{P}^1$ and $\mathcal{P}^2$ by an example with $n = 3$ jobs and $m = 2$ machines (see Fig. 2). In accordance with Theorem 3, we restrict the search space to the parts of the polyhedra $\mathcal{P}^1$ and $\mathcal{P}^2$ with $C_1 = 2$. The polyhedron $\mathcal{P}^1$ is given by the hatched cone, while the polyhedron $\mathcal{P}^2$ is the union of the two cones. The black dots represent those integer points which are not dominated by others.

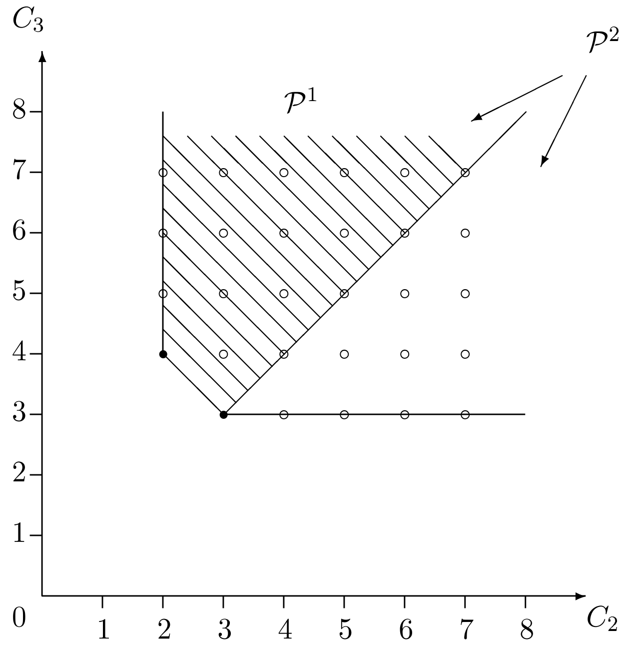

Figure 2: Polyhedra $\mathcal{P}^1$, $\mathcal{P}^2$

Finally we observe that the convex hull of all feasible schedules regardless of the ordering (11) can be defined as a union of $n!$ polyhedra $\mathcal{P}^1$ or $\mathcal{P}^2$, each of which satisfies its particular order of job completion times $C_{\pi(1)} \leq C_{\pi(2)} \leq \ldots \leq C_{\pi(n)}$. The relationships among the scheduling polyhedra are illustrated schematically in Fig. 3.

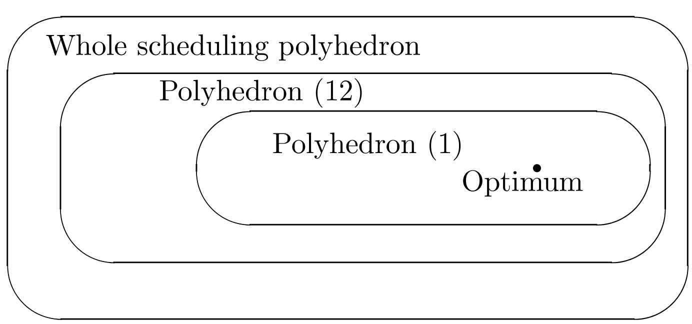

Figure 3: The relationships between the scheduling polyhedra

4 Algorithmic and complexity aspects

Based on the results of the previous section, we now describe polynomial-time algorithms to solve our scheduling problems $P|p_i = m, pmtn|F$, $O|p_{ik} = 1, pmtn|F$ $P|p_i = m, [pmtn]\ |F$, and $O|p_{ik} = 1|F$. We will distinguish the continuous and discrete cases. As we have shown, the optimal integer solution of the problem

$$\text{Minimize } F, \quad \text{subject to (12)}, \tag{13}$$

determines the optimal nonpreemptive schedule for problems $O|p_{ik} = 1|F$ and $P|p_i = m, [pmtn]|F$; the optimal continuous solution determines the optimal preemptive schedules for the problems $O|p_{ik} = 1, pmtn|F$ and $P|p_i = m, pmtn\ |F$.

In order to simplify problem (13), we use the lower and upper bounds (5) and substitute C_i by the new variables $x_i = C_i - \alpha_i$, where $\alpha_i = \max\{i, m\}$, thus replacing problem (13) by the problem

$$\begin{array}{ll} \text{Minimize} & w_1 f(m) + \sum\limits_{i=2}^{n} w_i f(x_i + \alpha_i) \\ \text{subject to} & \left\{ \begin{array}{ll} \sum\limits_{i=\mu-m+1}^{\mu} x_i \geq h_\mu, & \text{for } \mu = m+1, \ldots, n, \\ 0 \leq x_i \leq \beta_i, & \text{for } i = 1, \ldots, n, \end{array} \right. \end{array} \tag{14}$$

where

$$\beta_i = \min\{i, m\} - 1,$$
$$h_\mu = \left\{ \begin{array}{ll} (\mu - m)(3m - \mu - 1)/2, & \text{for } \mu = m+1, \ldots, 2m-1, \\ m(m-1)/2, & \text{for } \mu = 2m, \ldots, n. \end{array} \right.$$

Observe that $h_\mu \leq m(m-1)/2$ for any $\mu = m+1, \ldots, n$.

4.1 Nondecreasing convex separable objective function: Integer solution

We describe now an approach based on the reduction of problem (14) with additional integral restrictions on the variables x_i, $i = 1, \ldots, n$, to linear programming problem (LP).

In order to formulate problem (14) with additional integral restrictions as a linear program, we replace $f(x_i + \alpha_i)$ in (14) by the linear approximation on the integer grid

$$\overline{f}(x_i + \alpha_i) = \xi_{ik} x_i + \eta_{ik}$$

with coefficients ξ_{ik}, η_{ik} determined for successive unit-length slots $[k-1, k]$, $k = 1, \ldots, \beta_i$:

$$\xi_{ik} = f(k + \alpha_i) - f(k + \alpha_i - 1),$$

$$\eta_{ik} = kf(k + \alpha_i - 1) - (k-1)f(k + \alpha_i).$$

It is easy to check that both functions f and $\overline{f}$ have equal values in the integer points $x_i = k$, $k = 1, \ldots, \beta_i$, and thus finding an integer optimal solution of F is equivalent to finding an integer optimum solution of the piecewise-linear function $\overline{F} = \sum_{i=2}^{n} w_i \overline{f}(x_i + \alpha_i)$.

Define new variables y_{ik} for $k = 1, \ldots, \beta_i$, such that

$$0 \leq y_{ik} \leq 1,$$

and replace each variable x_i by

$$x_i = \sum_{k=1}^{\beta_i} y_{ik}, \tag{15}$$

thus obtaining a new problem:

$$\begin{array}{ll} \text{Minimize} & \overline{F} = \sum\limits_{i=2}^{n} \sum\limits_{k=1}^{\beta_i} w_i \xi_{ik} y_{ik} \\ \text{subject to} & \left\{ \begin{array}{ll} \sum\limits_{i=\mu-m+1}^{\mu} \sum\limits_{k=1}^{\beta_i} y_{ik} \geq h_\mu, & \mu = m+1, \ldots, n, \\ 0 \leq y_{ik} \leq 1, & i = 1, \ldots, n, \ k = 1, \ldots, \beta_i. \end{array} \right. \end{array} \tag{16}$$

If y^*_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, \beta_i$, determine the integer optimal solution, then substituting y^*_{ik} into (15) yields the optimum values for x_i because f is convex. Moreover, the matrix of (16) is totally unimodular since it has the consecutive-ones-property (see Section 3.2). Hence the continuous optimum solution of the linear programming problem coincides with the integer one and determines the solution of the initial nonpreemptive scheduling problem.

Different algorithms and approaches can be used to solve the linear programming problem (16). In particular, the algorithm from [31] has the complexity $O(\overline{n}^{4.5}\overline{m}^2 c(A))$, where A is $\overline{m} \times \overline{n}$ matrix and $c(A)$ is a constant depending only on A. For the constraints matrix corresponding to (16), $\overline{m} < n + nm$, $\overline{n} < nm$, and $c(A)$ can be estimated as $O(\log \overline{n})$. Thus the complexity of solving LP (16) by the algorithm from [31] is $O((nm)^{6.5} \log(nm))$. Indeed, other specialized algorithms can be developed with other complexity bounds. We have chosen the algorithm from [31] just to illustrate that LP problem (16) is solvable in strongly polynomial time.

Recall that after the optimal solution $\mathbf{C}^*$ is obtained, the corresponding schedule can be constructed in $O(nm \log^2(nm))$ time as described in the proof of Theorem 2.

4.2 Nondecreasing convex separable objective function: Continuous solution

The continuous problem of minimizing the convex function F over the polyhedron $\mathcal{P}^2$ can be solved by the standard techniques of convex minimization. For instance, applying the $O(n^3 L)$

algorithm by Monteiro and Adler [16] for solving the convex quadratic programming problem with n variables and input size $L = n\log W + n^2$ (here $W = \max\{w_i | i = 1, \ldots, n\}$) results in an $O(n^4(\log W + n))$ algorithm for the scheduling problem (14).

Recall that after the optimal solution $\mathbf{C}^*$ is obtained, the corresponding schedule can be constructed in $O(n\log n + nm^2)$ as described in the proof of Theorem 1. Thus the complexity of solving the preemptive scheduling problem is dominated by the complexity of solving the minimization problem (14).

4.3 Nondecreasing concave objective function

Our scheduling problems with a concave objective function are proved to be much easier to deal with than with a convex function. The crucial property for concave minimization is the existence of an optimal nonpreemptive schedule for the preemptive parallel machine problem $P|pmtn|F$. This property was first established for the linear objective function [14], then for the exponential function [20] and finally for an arbitrary nondecreasing concave function [29].

Due to this property and Lemma 1, the optimal solution is given by the vector

$$\mathbf{C}^* = \Bigg(\underbrace{m, \ldots, m,}_{m \text{ components}} \quad \underbrace{2m, \ldots, 2m,}_{m \text{ components}} \quad \ldots, \quad \underbrace{km, \ldots, km,}_{m \text{ components}} \quad \ldots, \quad \underbrace{\left\lceil \frac{n}{m} \right\rceil m, \ldots, \left\lceil \frac{n}{m} \right\rceil m}_{n - \left\lfloor \frac{n}{m} \right\rfloor m \text{ components}} \Bigg)$$

consisting of $\left\lfloor \frac{n}{m} \right\rfloor$ m-tuples of equal components and the remaining $n - \left\lfloor \frac{n}{m} \right\rfloor m$ components being equal to $\left\lceil \frac{n}{m} \right\rceil m$. This implies an $O(n\log n)$ algorithm for constructing the optimum vector $\mathbf{C}^*$ for the preemptive and nonpreemptive problems $P|p_i = m, pmtn|F$, $P|p_i = m, [pmtn]|F$, $P|p_i = m|F$, $O|p_{ik} = 1, pmtn|F$, and $O|p_{ik} = 1|F$. The total complexity (including the construction of the optimal schedule) can be estimated as $O(n\log n)$ for the three parallel machine problems and as $O(n\log n + nm)$ for the two open shop problems.

5 Conclusions

In this paper we have constructed the linear description of the scheduling polyhedron for the unit-time parallel machine and open shop scheduling problems. Based on numerous scattered results obtained for the unit-time problems, most of which were published more than 20 years ago, we have provided new mathematical programming formulations for these well-studied scheduling problems. We have demonstrated that the nonpreemptive unit-time scheduling problem is solvable in strongly polynomial time in case of a general objective function. An interesting topic for the research is further investigation of the constructed polyhedral model in order to develop the algorithms with better complexity bounds.

Acknowledgements

This research was partially supported by The Hong Kong Polytechnic University under grant numbers 351-493-A3-014, B-Q274 and G-T397. The second author was supported in part by the Belarusian Foundation of Fundamental Research (Projects M96-052 and Ф97-147) and by INTAS (Project INTAS-96-0820).

Appendix

Proof of Theorem 3 We use simultaneous inequalities (1) as the description of the vectors of job completion times for feasible schedules.

The lower bound for C_i is trivial. To prove the upper bound we show that if j is the first index for which

$$C_j \geq j + m, \tag{17}$$

then C_j can be decreased by 1 and the resulting schedule is feasible.

Case 1: $j = 1$.

If $C_1 \geq m + 1$, then it is easy to see from (1) that C_1 can be decreased to m.

Case 2: $2 \leq j \leq m + 1$.

As follows from (1), the component C_j of vector $\mathbf{C}$ can be decreased by 1 if inequalities

$$\sum_{i=\mu-m+1}^{\mu} C_i \geq \mu m$$

that have C_j hold as strict inequalities, i.e. if

$$\sum_{i=j+k-m+1}^{j+k} C_i > (j+k)m \quad \text{for} \quad k = m - j + 1, \ldots, m - 1. \tag{18}$$

Indeed, using the lower bound $C_i \geq m$ for the completion times of jobs $i = j + k - m + 1, \ldots, j - 1$, the assumption $C_j \geq j + m$, and (11) we conclude:

$$\begin{aligned}\sum_{i=j+k-m+1}^{j+k} C_i &= \sum_{i=j+k-m+1}^{j-1} C_i + \sum_{i=j}^{j+k} C_i \geq (m-k-1)m + (k+1)(j+m) \\ &= m^2 + jk + j.\end{aligned}$$

We show that $m^2 + jk + j > (j+k)m$:

$$(m^2+jk+j)-m(j+k) = m^2-j(m-k-1)-mk \geq m^2-(m+1)(m-k-1)-mk = k+1 > 0.$$

Observe that $j \leq m+1$ and $m-k-1 \geq 0$.

<u>**Case 3:** $m+2 \leq j \leq n$**.**</u>

We prove that inequalities (18) hold for $k = 0, \ldots, m-1$. First we show that (18) holds for $k = 0$. Suppose it is not the case, i.e.

$$\sum_{i=j-m+1}^{j} C_i \leq mj. \tag{19}$$

Then $C_{j-m} = \sum_{i=j-m}^{j-1} C_i - \sum_{i=j-m+1}^{j} C_i + C_j$. Since vector $\mathbf{C}$ corresponds to a feasible schedule, it satisfies (1) and $\sum_{i=j-m}^{j-1} C_i \geq m(j-1)$. Using this inequality, together with (17) and (19), we obtain:

$$C_{j-m} \geq m(j-1) - mj + (j+m) = j,$$

and this contradicts the assumption that j is the first index for which (17) holds.

Now we prove that inequalities (18) hold for any k, $k = 1, \ldots, m-1$. Consider the m-tuple $(C_{j+k-m+1}, \ldots, C_{j-1}, C_j, \ldots, C_{j+k})$ with $C_{j+k} \geq \ldots \geq C_j \geq j+m$. If the first components $C_{j+k-m+1}, \ldots, C_{j-1}$ of this m-tuple are not less than j, then

$$\begin{aligned}\sum_{i=j+k-m+1}^{j+k} C_i &= \sum_{i=j+k-m+1}^{j-1} C_i + \sum_{i=j}^{j+k} C_i \geq (m-k-1)j + (k+1)(j+m) \\ &= m(j+k+1) > m(j+k).\end{aligned}$$

Otherwise, i.e. if there are some components which are less than j, we represent this m-tuple in a form consisting of three components, the first components being less than j, the middle components being not less than j, and the last components being not less than $j+m$:

$$(\underbrace{j-\delta_{j+k-m+1}, \ldots, j-\delta_{j+l-m},}\ \underbrace{j+\delta_{j+l-m+1}, \ldots, j+\delta_{j-1},}\ \underbrace{C_j, \ldots, C_{j+k}}).$$

Here $k+1 \leq l \leq m-2$ and all values δ_i are nonnegative. Using this notation, we obtain:

$$\begin{aligned}\sum_{i=j+k-m+1}^{j+k} C_i &= \sum_{i=j+k-m+1}^{j+l-m} C_i + \sum_{i=j+l-m+1}^{j-1} C_i + \sum_{i=j}^{j+k} C_i \\ &\geq mj - \sum_{i=j+k-m+1}^{j+l-m} \delta_i + \sum_{i=j+l-m+1}^{j-1} \delta_i + (k+1)m.\end{aligned}$$

To estimate $\sum_{i=j+l-m+1}^{j-1} \delta_i - \sum_{i=j+k-m+1}^{j+l-m} \delta_i$, we use the inequality $\sum_{i=j-m}^{j-1} C_i \geq m(j-1)$, which follows from the feasibility of the schedule with the completion time vector $\mathbf{C}$. Besides, $C_{j-m} \leq j-1$ since according to our assumption, j is the first index for which (17) holds. Hence we obtain:

$$\begin{aligned} m(j-1) &\leq \sum_{i=j-m}^{j-1} C_i = C_{j-m} + \sum_{i=j-m+1}^{j+l-m} C_i + \sum_{i=j+l-m+1}^{j-1} C_i \\ &\leq (j-1) + (m-1)j - \sum_{i=j-m+1}^{j+l-m} \delta_i + \sum_{i=j+l-m+1}^{j-1} \delta_i. \end{aligned}$$

It follows that

$$\sum_{i=j+l-m+1}^{j-1} \delta_i - \sum_{i=j+k-m+1}^{j+l-m} \delta_i \geq \sum_{i=j+l-m+1}^{j-1} \delta_i - \sum_{i=j-m+1}^{j+l-m} \delta_i \geq -m+1.$$

Substituting the latter condition in the formula for $\sum_{i=j+k-m+1}^{j+k} C_i$, we obtain:

$$\sum_{i=j+k-m+1}^{j+k} C_i \geq mj - m + 1 + (k+1)m = m(j+k) + 1.$$

■

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