https://doi.org/10.1016/j.trb.2016.07.010

Exploring alternative service schemes for busy transit corridors

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Abstract
Transit systems in which buses or trains always visit each and every stop along corridors are compared against those that feature two alternative vehicle-dispatching schemes. The alternatives entail so-called skip-stop and express/local services. Continuous models found in the literature are expanded so that the alternatives could be compared under a wider array of options. Comparisons are separately drawn for systems that feature buses, BRT and metro-rail trains, both for cities that are wealthy and for those that are not. Idealizations in regard to travel demand and route symmetry are assumed in pursuit of insights useful for high-level planning.

Two rounds of parametric comparisons are conducted. In the first round, optimally-designed all-stop systems are presumably converted to furnish instead the alternative strategies without altering the original stop locations. In the second round, alternative schemes are designed in fully-optimized fashion from scratch. In both rounds, alternative dispatching schemes often bring lower generalized costs than do their optimally-designed all-stop counterparts. Estimated savings can reach 10% even in the first round where the alternative schemes are hampered by sub-optimal stop locations. If designed from scratch, the savings can reach 30%. Skip-stop service is found most often to be the lowest-cost option of the three.

Keywords: transit corridor; transit operations; skip-stop service; express service

1 Introduction
The high travel demands that characterize a busy transit corridor are often served by buses or trains that visit each and every stop along the way. Transit patrons may view this all-stop service as a convenience. Yet, systems that operate in this way are often plagued by poor service quality, and may face budgetary deficits to boot (Lave, 1980; Gomez-Ibanez et al., 1994). Alternatives have therefore been tried, including: (i) skip-stop service in which all of a corridor's multiple routes visit transfer stops, while service to the remaining stops are shared in such way that each of those stops is visited by only one of the routes; and (ii) express/local service with some routes that visit all stops, and others that serve only designated express stops.

There have been a number of theoretical efforts to explore these alternatives; see Ibarra-Rojas et al. (2015) for a review of those efforts. Most of them have relied on empirical methods or data-hungry analytical models that are computationally inefficient (e.g. Black, 1962; Afanasiev and Liberman, 1982; Ercolano, 1984; Conlon et al., 2001; Ulusoy et al., 2010; Tetreault and El-Geneidy, 2010; Leiva et al., 2010). Those efforts focused on case studies of specific sites where stop locations are fixed. The feasible scope of parametric analysis was therefore limited. Our work relies instead upon parsimonious,

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computationally-efficient models of the kinds found, for example, in: Holroyd (1967), Newell (1971), Wirasinghe and Ghoneim (1981), Vaughan (1986), Estrada et al. (2011) and Chen et al. (2015). These all: took travel demand to be exogenous to a transit system's details of design; assumed that demand was spatially and temporally homogeneous; and overlooked physical and societal constraints to route and stop placements. Very importantly, the models treated inputs, such as travel demand, and outputs, such as route and stop spacing, as smooth, continuous functions. Models of this continuous type can furnish transit-system designs and service schemes that minimize the generalized costs to transit agencies and their patrons. And they can often take simple analytical forms that readily unveil trade-offs between competing cost components.

Models of this type were used in Freyss, et al. (2013) to study a limited case of skip-stop service in which metro trains on each of two routes served every other non-transfer stop along a single-track corridor, as well as every transfer stop. The work assumed that stop spacing was given and trains had infinite passenger-carrying capacity. Continuous models were similarly used in Daganzo (2010a) to study express/local service for the highly idealized case in which each passenger travels a local-express-local route, such that all trips require two transfers. Each effort found that, under certain conditions, its studied alternative generated lower generalized costs than would all-stop service. Yet, head-to-head comparisons across the two alternatives were not pursued.

The present study expands the continuous models in Freyss, et. al (2013) and Daganzo (2010a) so that they can furnish optimal designs for much more general forms of skip-stop and express/local service. Specifically, we develop new models that optimize a broader set of design variables including the stop densities, vehicle headway, and for skip-stop service, the number of routes. The models account for characteristics that include the vehicle's passenger-carrying capacity for three transit modes: ordinary bus, bus rapid transit (BRT), and rail. The models are sufficiently detailed to enable examination of features that include schedule coordination between routes on skip-stop systems; and patrons' route selections to reduce transfers on express/local systems. We are thus able to draw meaningful comparisons between the alternative schemes and their more traditional all-stop counterpart for a wide array of circumstances.

2 Models

Continuous models for the above three schemes will be presented in sequence. The models for each scheme generate designs that minimize the generalized costs of an average transit trip, as imparted both to a patron and the operating agency. The same models are used for all the transit modes, since at the system-planning level, the 3 modes are strikingly similar (e.g., see Daganzo 2010a, b). The patron costs entail those of: accessing the nearest origin stop and egressing from the nearest destination stop; waiting for a transit vehicle at the origin stop; transferring between vehicles that serve distinct routes; and the time spent traveling inside the vehicle(s). Agency costs include: staff wages; fuel costs; vehicle maintenance cost; vehicle purchase costs amortized over a vehicle's lifetime; and similarly amortized infrastructure costs.

All costs are expressed in units of time, just as in Daganzo (2010b), Sivakumaran et al. (2014) and Chen et al. (2015). Time is arguably a more intuitive means for making assessments, given that monetary systems vary across the globe. A value of time, μ (\$/h), is used to convert monetary costs to temporal ones.

Much as in Daganzo (2010b) and Chen et al. (2015), the present objective is to unveil general insights that can be useful when planning transit systems at a high level. To the extent possible, we therefore dispense with details that tend to be site-specific and focus instead on idealized systems by assuming that: (i) routes run along bi-directional, closed-loop corridors; (ii) travel demand in each direction is invariant to time and homogeneous along the corridor at density, λ (passenger/km/h), just as in Daganzo (2010b) and Estrada et al. (2011); (iii) passenger trip lengths are uniformly distributed over $[0, 2\bar{l}]$, where $\bar{l}$ is the average trip length (km); (iv) vehicles dwell for a constant time at each stop, τ (h), including the time lost to the vehicle's deceleration and acceleration when it arrives at and departs from a stop, as in Sivakumaran et al. (2014); and (v) patrons arrive randomly to an origin stop independent of service schedule, as in Daganzo (2010b) and Sivakumaran et al. (2014). Note that as a consequence of (ii), the optimally located stops are evenly spaced. We denote the optimal stop spacing as s (km).

Additional variables will be defined as needed. A table summarizing all variables used in this work is provided in Appendix A.

2.1 All-Stop (AS) Service

The present models for this traditional vehicle-dispatching scheme are similar to ones that already appear in the literature (Daganzo, 2010a; 2010b).[1] Our all-stop models are nonetheless described below in full, largely because they form the bases for the more complicated models of skip-stop and local/express services. Presenting the all-stop models will thus simplify the presentations of the alternative models to come thereafter.

Start with the costs to the average patron. Consideration shows that for accessing an origin stop, and for egressing from a destination stop, each have a cost of $\frac{s}{4v_w}$, where v_w is the patron's walking speed and s, as previously noted, is stop spacing. Since the patron arrives randomly to her origin stop, her average wait time there is half the vehicle headway, i.e. $\frac{H}{2}$. Her in-vehicle travel time is the sum of her time spent moving along her trip of length $\bar{l}$ at the vehicle's cruise speed, v, which is simply $\frac{\bar{l}}{v}$, and her time spent dwelling at intervening stops along her route, $\bar{l}\frac{\tau}{s}$. Hence, the cost of all-stop service for the average patron is:[2]

$$C_{P\_AS} = \frac{s}{2v_w} + \frac{H}{2} + \frac{\bar{l}}{v} + \bar{l}\frac{\tau}{s}. \quad (1)$$

On the agency side, we formulate the cost models again in the same fashion as in Daganzo (2010) and Sivakumaran et al. (2014). We define π_v as the cost per vehicle-km of service, which includes both vehicle operation and maintenance costs. The distance-based operating cost per patron is therefore $\frac{\pi_v}{\lambda\mu H}$, since: the total vehicle-kms traveled each direction and each hour along a 1-km segment is $\frac{1}{H} \times 1$; the

[1] The components of patron costs are presently modeled much as in Daganzo (2010a), except that models in the former case describe costs incurred by an average patron, and the latter pertain to worst-case conditions. Agency costs are presently modeled in similar manner to Daganzo (2010b).

[2] For all-stop transit services (and for the skip-stop services to be presented next) with fixed demand, each patron has only one route option available to her. Thus, the transit fare will not affect patrons' decisions. Fare is instead only a transfer of money from the patrons to the transit agency. This transfer is therefore not included in our formulation of the generalized cost.

number of patron-trips generated in that segment is λ; and recall that μ is the patron's value of time. The total vehicle-hours traveled per hour of service along a 1-km segment is $\frac{1}{H}\left(\frac{1}{v}+\frac{\tau}{s}\right)$, such that the time-based operating cost per trip served is $\frac{\pi_m}{\lambda\mu H}\left(\frac{1}{v}+\frac{\tau}{s}\right)$, where π_m is the unit cost per vehicle-h of service.

We define π_i to be the amortized construction and maintenance cost per km of link infrastructure (in one direction) per hour of service, such that the infrastructure cost per trip served is $\frac{\pi_i}{\lambda}$. Similarly, the amortized construction and maintenance cost per stop is π_s, such that the cost per trip served is $\frac{\pi_s}{2\lambda s}$. Here we assume that a stop serves both travel directions along the corridor.

Thus for all-stop systems, the agency cost per trip served, in units of time, is:

$$C_{A\_AS} = \frac{\pi_v}{\lambda\mu H} + \frac{\pi_m}{\lambda\mu H}\left(\frac{1}{v}+\frac{\tau}{s}\right) + \frac{\pi_i}{\lambda\mu} + \frac{\pi_s}{2\lambda\mu s}; \quad (2)$$

and the generalized cost-minimization problem can thus be formulated as:

$$\min GC_{AS} = C_{P\_AS} + C_{A\_AS} = \frac{s}{2v_w} + \frac{H}{2} + \frac{\bar{l}}{v} + \bar{l}\frac{\tau}{s} + \frac{\pi_v}{\lambda\mu H} + \frac{\pi_m}{\lambda\mu H}\left(\frac{1}{v}+\frac{\tau}{s}\right) + \frac{\pi_i}{\lambda\mu} + \frac{\pi_s}{2\lambda\mu s} \quad (3a)$$

subject to: $\lambda\bar{l}H \leq K$ (3b)

$H \geq H_{min}$, (3c)

where constraint (3b) ensures that the number of patrons on-board a transit vehicle does not exceed the vehicle's passenger-carrying capacity, K; and constraint (3c) ensures that the selected H will not fall below some minimum. This H_{min} would likely be set to satisfy a safety standard in the case of rail transit, or would be the inverse of the system's bus-carrying capacity otherwise. The decision variables in (3a) are H and s.

2.2 Skip-Stop (SS) Service

The present models for this alternative service scheme are distinct from those in Freyss et al. (2013) in several respects. First, the spacing between non-transfer stops and vehicle headways are both decision variables in the present models, but were inputs in Freyss et al. The present models are further distinct in that they: consider for each transit mode a vehicle's passenger-carrying capacity; and allow for more than two routes to coexist along a corridor.

An example of the latter feature is shown in Figure 1 for the case of $m = 3$ routes, labeled A, B and C. Each dot in the figure accompanied by a letter represents a stop that is visited by vehicles from the associated route (only). Hence while traveling between two consecutive transfer stops, a vehicle visits only $k = 2$ designated stops, and skips $(m-1)k$ intermediate ones.

No more than one transfer is required of any patron. Examples of the kind of transfer that might occur is shown in Figure 1. The dashed arrows describe a trip that originates at a stop along Route A and terminates at another along B. The dotted arrows illustrate a case in which patrons must back-track.

The expected costs of transfers are, of course, highest when schedules are not coordinated across routes. This was the case for the metro-rail systems studied in Freyss et al. (2013). Since trains from each route visit a distinct set of stops, the dispatch headway between those trains at a transfer station, H, has a lower bound, $\tau + H_{min}$, as can be discerned in Figure 2a.

The present models are further distinct from those in Freyss et al. (2013) in that the former allow for schedule coordination of distinct routes at transfer stops. For rail systems, the new models can be used on corridors served by even numbers of routes (i.e. $m = 2, 4, \ldots$) and when those routes are coordinated to arrive at transfer stops in pairwise fashion; i.e., routes 1 and 2 will arrive at each transfer stop simultaneously, and the same for routes 3 and 4, 5 and 6, etc. A second track for each direction of travel is needed for this brand of coordination, and the present models can account for the attendant infrastructure costs, as will be shown momentarily.

When a corridor is served instead by ordinary buses or BRT, vehicles from each of the m routes (m being either even or odd) can be scheduled to arrive together at each transfer stop without need for added infrastructure.[3] This full-form coordination requires that the buses travel in convoy and in select sequence to avert the need for overtaking. Consideration shows that for the example shown in Figure 1, the vehicle sequence in the convoy would have to be: a vehicle serving route C, followed by one serving B, followed by another serving A; see Figure 2b. Key impacts of this coordination can be accounted for in the present models, as will be demonstrated.

We turn first to the model of patron cost. It bears similarities to all-stop model (1), but with additional terms to account for the expected costs of: waiting at an origin stop for the arrival of a vehicle that serves a suitable route; and waiting again at a transfer stop. The expected durations of these episodes are accounted for by increasing the patron's expected wait time at her origin stop by the factor α_1, where:

$$\alpha_1 = \begin{cases} \frac{2m^2k - mk + 1}{mk+1}, & \text{for systems without schedule coordination} \\ \frac{2(m^2k - mk + 1)}{mk+1}, & \text{for coordinated double-track rail systems } (m \text{ is even}) \\ m, & \text{for coordinated bus/BRT systems} \end{cases} \qquad (4a)$$

Derivations for α_1 are furnished in Appendix B; see especially parts B.1 – B.4.

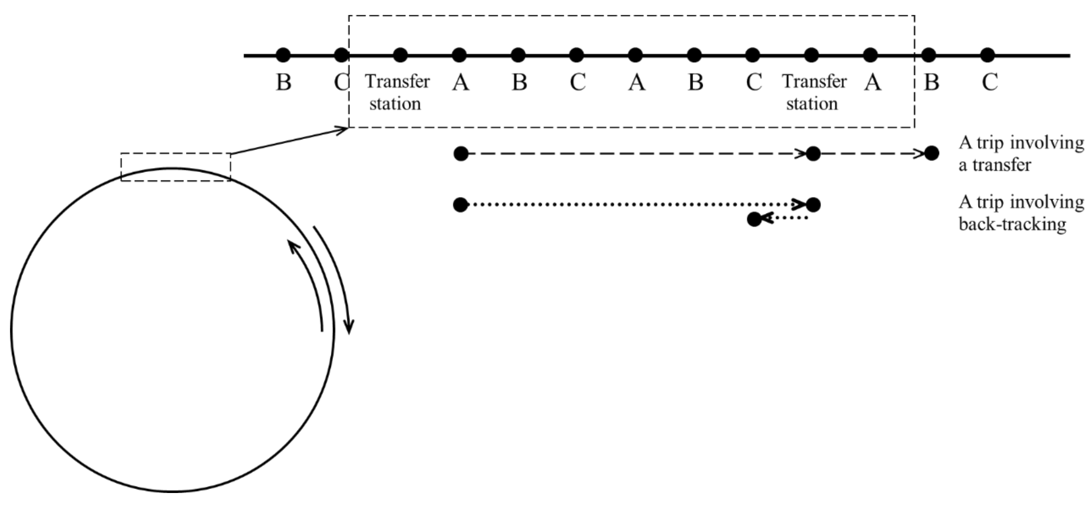

Figure 1. Skip-stop service with $m = 3$ routes

[3] A transfer stop is assumed to have a sufficient number of berths to serve m buses concurrently.

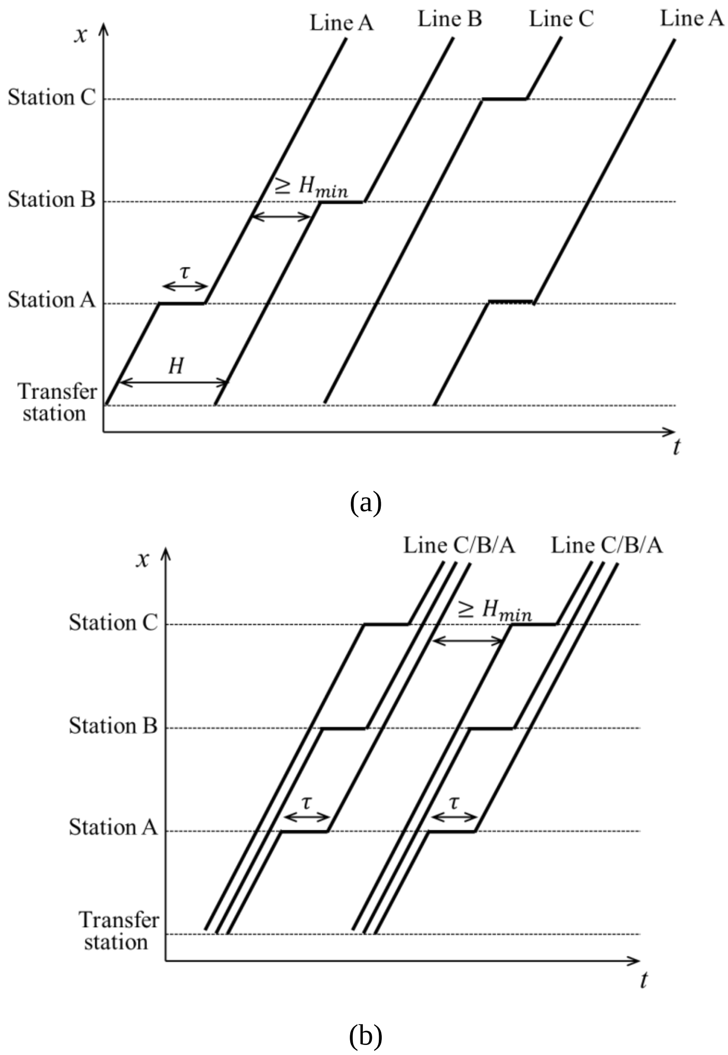

Figure 2. (a) Minimum headway for skip-stop rail service on a single track; (b) schedule coordination for ordinary bus and BRT systems.

Transfers can also entail the patron's inconvenience of walking from one route's loading area to another. This cost is accounted for by means of a specified time penalty. If the two connecting routes share the same platform, the transfer penalty is denoted C_t. If the two routes instead use different platforms (which occurs in a double-track rail system when a patron needs to transfer to a train traveling in the opposite direction), the penalty is assumed to be $2C_t$. More generally, a trip's expected transfer penalty is expressed as $\alpha_2 C_t$, where α_2 is given by:

$$\alpha_2 = \begin{cases} \frac{(m-1)mk^2}{(mk+1)^2}, & \text{for uncoordinated systems and coordinated bus or BRT systems} \\ \left(\frac{(m-1)mk^2}{(mk+1)^2} + \frac{mk}{mk+1} \times \frac{(m-1)k}{4\bar{l}/s}\right), & \text{for coordinated, double-track rail systems } (m \text{ is even}) \end{cases} \quad (4b)$$

Derivations are furnished in Section B.5 of Appendix B.

The model of patron cost under skip-stop service is therefore:

$$C_{P\_SS} = \frac{s}{2v_w} + \alpha_1 \frac{H}{2} + \left(\bar{l} + \frac{m(m-1)k^2}{12\bar{l}/s} s\right)\left(\frac{1}{v} + \frac{\tau}{s} \cdot \frac{k+1}{mk+1}\right) + \alpha_2 C_t \; . \tag{4c}$$

The term $\frac{m(m-1)k^2}{12\bar{l}/s} s$ in (4c) accounts for the expected distance of back-tracking, as explained in Appendix B.6. The $\frac{k+1}{mk+1}$ is the proportion of stops visited by an average trip, where $k+1$ is the number of stops visited by any vehicle traveling from one transfer stop to the next; and $mk+1$ is the total number of stops in that same segment that spans neighboring transfer stops.

The agency's operating costs are akin to those incurred under all-stop service as expressed in (2):

$$C_{O\_SS} = \frac{\pi_v}{\lambda\mu H} + \frac{\pi_m}{\lambda\mu H}\left(\frac{1}{v} + \frac{\tau}{s} \cdot \frac{k+1}{mk+1}\right) . \tag{5}$$

The amortized cost of the additional infrastructure needed for a skip-stop scheme (relative to all-stop service) is also akin to (2). In the present case, we have:

$$C_{I\_SS} = \alpha_3 \frac{\pi_i}{\lambda\mu} + \alpha_4 \frac{\pi_s}{2\lambda\mu s} , \tag{6}$$

where the factors α_3 and α_4 account for infrastructure costs, as described in Table 3.

The generalized cost-minimization problem for skip-stop systems thus takes s, m, k, and H as the decision variables and has the form:

$$\min GC_{SS} = C_{P\_SS} + C_{O\_SS} + C_{I\_SS} = \frac{s}{2v_w} + \alpha_1 \frac{H}{2} + \left(\bar{l} + \frac{m(m-1)k^2}{12\bar{l}/s} s\right)\left(\frac{1}{v} + \frac{\tau}{s} \cdot \frac{k+1}{mk+1}\right) + \alpha_2 C_t + \frac{\pi_v}{\lambda\mu H} + \frac{\pi_m}{\lambda\mu H}\left(\frac{1}{v} + \frac{\tau}{s} \cdot \frac{k+1}{mk+1}\right) + \alpha_3 \frac{\pi_i}{\lambda\mu} + \alpha_4 \frac{\pi_s}{2\lambda\mu s} \tag{7a}$$

$$\text{subject to: } \lambda H\left(\bar{l} + \frac{m(m-1)k^2}{4\bar{l}/s} s\right) \leq K \tag{7b}$$

$$\begin{cases} mH \geq H_{min} & \text{for coordinated bus and BRT} \\ H \geq H_{min} + \tau & \text{for uncoordinated rail with single track} \\ 2H \geq H_{min} & \text{for coordinated rail with double tracks and } m = 2 \\ 2H \geq H_{min} + \tau & \text{for coordinated rail with double tracks and } m \geq 4 \end{cases} . \tag{7c}$$

Constraint (7b) pertains to each vehicle's passenger-carrying capacity, where the left-hand-side is the maximum number of on-board passengers per vehicle, as derived in Appendix B.7. Constraint (7c) furnishes the lower bound on vehicle headway. Note that H always denotes the inverse of bus/train flow. For coordinated bus and BRT systems, H_{min} represents the minimum headway required between distinct bus *convoys* to avoid bus queueing at transfer stops, and mH the *convoy*'s dispatch headway; for uncoordinated single-track rail systems, the minimum headway constraint is the same as in Freyss et al. (2013), which is described by Figure 2a; for double-track rail systems, the minimum headway constraint is the same as a single-track system that serves $\frac{m}{2}$ routes.

2.3 Express/Local (EL) Service

The example in Figure 3 features: (i) a route that serves all local stops along a corridor (labeled L in the figure) and express stops (labeled E); and (ii) an express route that serves every $N = 4$ stops, which are of the express type. A patron's fare is assumed invariant to whether she travels via local or express vehicles.

She can therefore choose her route so as to minimize her expected trip time between origin and destination stops.[4]

Some patrons may choose to travel by a combination of local and express vehicles that entails two transfers, as was assumed always to be the case in Daganzo (2010a). The present models are more general. They too consider trips with two transfers, as well as: trips that combine local and express vehicles in ways that necessitate only a single transfer; and shorter-length trips made solely via local vehicles to dodge the need for transfers entirely. Trips that start and end at express stops are considered as well.

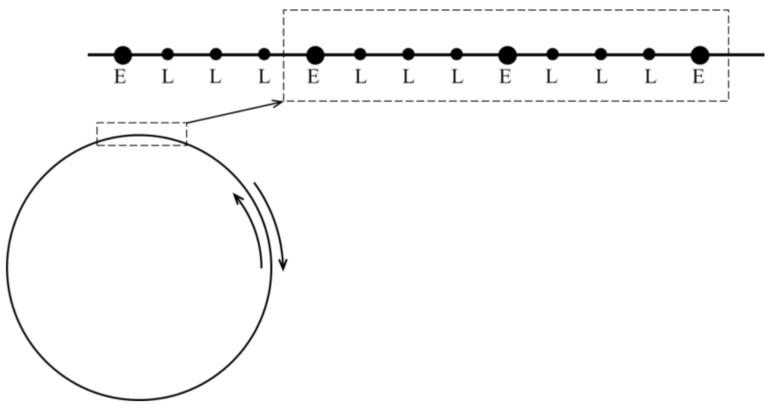

Figure 3. Express-local system with ratio $N = 4$

With the above in mind, we formulate an expression for a patron's cost when traveling between: (i) two express stops, $C_{P\_EL\_EE}$; (ii) one express and one local stop, $C_{P\_EL\_EL}$; and (iii) two local stops, $C_{P\_EL\_LL}$. These are given by Equations (C1-C3) in Appendix C. The patron's cost model is:

$$C_{P\_EL} = \frac{1}{N^2} C_{P\_EL\_EE} + \frac{2(N-1)}{N^2} C_{P\_EL\_EL} + \left(\frac{N-1}{N}\right)^2 C_{P\_EL\_LL}, \tag{8}$$

as is also formulated in Appendix C.

The agency's cost is much the same as given in (2) for all-stop service, except that there are now distinct headways for express and local vehicles, H_E and H_L, respectively. Thus,

$$C_{A\_EL} = \frac{\pi_v}{\lambda\mu}\left(\frac{1}{H_E} + \frac{1}{H_L}\right) + \frac{\pi_m}{\lambda\mu}\left(\frac{1}{H_L}\left(\frac{1}{v} + \frac{\tau}{s}\right) + \frac{1}{H_E}\left(\frac{1}{v} + \frac{\tau}{Ns}\right)\right) + \beta_1 \frac{\pi_i}{\lambda\mu} + \beta_2 \frac{\pi_s}{2\lambda\mu s}, \tag{9}$$

where the values of β_1 and β_2 are listed in Table 4.

The generalized cost-minimization model takes s, N, H_E, and H_L as the decision variables and has the form:

$$\min GC_{EL} = C_{P\_EL} + C_{A\_EL} \tag{10a}$$

$$\text{subject to: } Q_L H_L \le K \tag{10b}$$

[4] This is true when the transit agency adopts a constant fare for both the express and local services, and transfers between services are free of charge.

$$Q_E H_E \leq K \tag{10c}$$

$$H_E, H_L \geq H_{min} \tag{10d}$$

where Q_L and Q_E are the maximum cross-sectional passenger flows served by local and express vehicles, respectively. Their formulas are given by (C7) and (C8) in Appendix C.4.

3. Solution Methods

The objective function for all-stop service (3a) is convex,[5] and constraints (3b) and (3c) are both boundary ones. The mathematical program for all-stop service can therefore be solved via the gradient descent method.

In contrast, programs for skip-stop and express/local systems, (7a-c) and (10a-d) respectively, have non-convex forms. They are difficult to solve using ordinary search methods, primarily because of the integer nature of the decision variables m, k, and N. Given that in real settings the distance between consecutive transfer stops or between consecutive express stops must be held to some reasonable limit, we reckon that the above three decision variables will take values that fall within commensurately limited ranges. We therefore stipulate that $2 \leq m \leq 4$, since a larger m will entail more transfers which can confuse patrons and degrade service quality. We further stipulate that $1 \leq k \leq 4$, and that $2 \leq N \leq 8$. The following steps are then performed to obtain the optimal solutions to (7a-c) and (10a-d).

Step 1. For skip-stop systems, the values of m and k are fixed and program (7a-c) is solved in its reduced form. For express/local systems, N is fixed and (10a-d) is solved in similarly reduced form. In both instances, a gradient-based search method is used to find 10 local optima from 10 randomly-selected initial values for the reduced program's continuous (i.e. non-fixed) decision variables. The lowest cost of these 10 cases is identified. The above process was repeated 9 additional times, which invariably confirmed that the least-cost solution was reproducible. The solution was therefore taken as the global optimum of the reduced program.

Step 2. Step 1 is repeated for all possible integer values of m and k (7a-c), and of N (10a-d). The minimum-cost solution is found among the optima of all the repetitions.

4. Numerical Analysis

We first select suitable inputs for the models. We then consider what occurs when: skip-stop and express-local systems are converted from their more traditional all-stop counterparts; and the conversions are burdened by remnants (i.e. the stop locations) of the originals. Finally, we consider what can occur when a city's transit system is designed from scratch, free from the influence of predecessor systems.

4.1 Parameter values

The costs of a transit system will vary across cities, owing to factors that include local wage rates, land procurement prices and construction practices (Flyvbjerg et al., 2008; Halcrow, Fox & Associates, 2000). Inputs used in the present work to accommodate these variations were taken largely from Sivakumaran et al. (2014). Those parameter values are presented in Tables 1 and 2.

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[5] The reader can verify that each monomial of (3a) is a convex function.

Note from the first of these tables how the values for unit operating costs, π_v and π_m, and infrastructure costs, π_i and π_s, are: lowest for ordinary buses; higher for BRT; and higher still for rail. Further note from Table 2 how the values for vehicle cruise speed, v, and passenger-carrying capacity, K, ascend across modes in that same sequence.

Further notice from Table 1 how operating cost (in \$/vehicle-h) is linked to a city's wealth by treating the π_m for each mode as a linear function of μ. This makes sense given that π_m is affected primarily by wage rate, which is proportional to μ, and by amortized vehicle purchase cost, which is largely invariant across rich and poor cities. Note too how capital and labor costs are accounted for by also treating π_i and π_s as linear functions of μ. In contrast, the unit cost of vehicle-kms of service, π_v, primarily entails energy costs, which tend to be invariant to μ (GIZ, 2011; IEA, 2007).

Table 1. Cost parameters for each mode

	π_v (\$/veh·km)	π_m (\$/veh·h)	π_i (\$/km/h)	π_s (\$/station/h)
Bus	0.59	$2.66 + 3\mu$	$6 + 0.2\mu$	$0.42 + 0.014\mu$
BRT	0.66	$3.81 + 4\mu$	$162 + 5.4\mu$	$4.2 + 0.14\mu$
Rail	2.20	$101 + 5\mu$	$594 + 19.8\mu$	$294 + 9.8\mu$

Table 2. Operating parameters for each mode

	τ (sec)	v (km/h)	K (passenger/veh)	H_{min} (min)
Bus	30	25	80	1
BRT	30	40	160	1
Rail	45	60	3000	1.5

A patron's walking speed, v_w, is specified to be only 2 km/hr. This is to account for her access and egress delays when crossing streets, and for the inconvenience of having to walk (Daganzo, 2010b). The time-valued penalty for transferring between routes served by the same platform, C_t, is assumed to be 1 min.

All that remains in the present context is to select appropriate numeric values for the infrastructure cost coefficients α_3, α_4, β_1 and β_2 used by the models for skip-stop and express/local service. These are given in Tables 3 and 4. Note that for double-track skip-stop rail systems, and for express/local systems that involve a second track/bus lane, the cost coefficients are estimated with some degree of subjectivity. Footnotes are included with these tables to explain the reasoning behind our numeric selections. Moreover, sensitivity analysis presented in later sections shows that estimation errors on the coefficients produce only modest effects.

Table 3. Values of α_3 and α_4 for skip-stop systems

		α_3	α_4
Converted systems*	Coordinated bus and BRT, and uncoordinated rail with single track (no additional infrastructure is needed)	0	0
	Coordinated rail with double tracks (with a second track added and stops expanded)	1#	1#
Systems from scratch	Coordinated bus and BRT, and uncoordinated rail with single track	1	1
	Coordinated rail with double tracks	2§	1.5§

[*] It is assumed that a converted BRT or rail system is equipped with two dedicated bus lanes or tracks, with each serving one direction of travel.

[#] We set $\alpha_3 = \alpha_4 = 1$ to account for the high costs of reconstruction.

[§] We set $\alpha_4 = 1.5$ because building a new stop from scratch to serve two tracks in each travel direction is less expensive than building a stop that initially serves a single-track system and then expanding the stop to serve double tracks.

[#§] To account for the uncertainty in the values of α_3 and α_4, we will allow them to vary within a range of $[80\%, 120\%]$ of the tabulated values.

Table 4. Values of β_1 and β_2 for express/local systems

		β_1	β_2
Converted systems	Ordinary bus	0[ǂ]	0
	BRT	1[*]	0
	Rail	1[#]	1[#]
Systems from scratch	Ordinary bus	1	1
	BRT	2[§]	1
	Rail	2[§]	1.5[§]

[ǂ] We assume that ordinary express buses can use adjacent travel lanes to overtake local buses, such that additional infrastructure is not needed for converted bus systems.

[*] This number accounts for the construction cost of a dedicated passing lane in each travel direction.

[#] These numbers account for the cost of constructing a second track in each direction and expanding stops to serve all of the tracks.

[*#§] To account for the error in parameter estimation, we will allow all of these numbers to vary within a range of $[80\%, 120\%]$ of the tabulated values.

4.2 System conversions

Suppose that converted skip-stop and express/local systems retain the same stops used by their predecessor (all-stop) systems. This would seem realistic in cases when stop relocation is costly or inconvenient, e.g. due to a lack of available space.

Analysis is performed in parametric fashion for: $\bar{l} \in [3,15]$ km; both low-wage cities like Shanghai with $\mu = 5$ \$/h, and high-wage ones like Barcelona with $\mu = 20$ \$/h; all three transit modes; and ranges of λ that vary with the passenger-carrying capacity of each mode.

For each scenario, the optimal H and s for the all-stop systems are obtained from (3a-c). The optimal m, k and H for each converted skip-stop system are obtained via (7a-c), and stop spacing, s, is inherited from the optimal all-stop counterpart. In similar fashion, optimal N, H_E and H_L for each converted express/local system are obtained from (10a-d), with s again handed-down from the original all-stop system.

Attendant generalized costs of converted alternatives and of the original all-stop system are compared in three-way fashion. Outcomes are summarized below by mode.

4.2.1 Ordinary bus systems

Since ordinary buses share road space with the rest of traffic, it is assumed that bus systems are converted to accommodate either skip-stop or express/local service without adding new infrastructure. Outcomes for a low-wage (i.e. low-μ) city are summarized in Figures 4a-c.

Select contour lines in Figure 4a delineate the combinations of λ and $\bar{l}$ for which an encircled vehicle-dispatching scheme produced the lowest generalized cost. The figure shows that the original all-stop service remains the lowest-cost option for only the smallest of λ, as delineated by the thick, dark contour line. This is because the relatively high agency costs of alternative schemes make them less competitive when demand is low. Note how this already small range of λ becomes even smaller as $\bar{l}$ grows large. This trend occurs because a large $\bar{l}$ favors skip-stop schemes in which more stops can be skipped per trip.

Coordinated skip-stop service emerges as the best option for all other values of the λ and $\bar{l}$ jointly examined. As λ or $\bar{l}$ increases, converted skip-stop service brings steadily greater percent reductions in generalized cost, as compared against the original service. Maximum differences exceed 6%, as evident from the solid contour lines in Figure 4a. Results further indicate that the optimal number of routes used to provide skip-stop service, m, increases from 2 to 4 as λ and $\bar{l}$ increase; see Figure 4b.

The above trends are unsurprising: a transit agency's cost per person-km served diminishes as λ and $\bar{l}$ grow large. This trend can render the higher-frequency and more differentiated skip-stop service the preferred alternative in many cases. Interestingly, the threshold of λ that renders skip-stop service optimal is surprisingly small. For example, Figure 4a shows that for $\bar{l} = 6$ km, a coordinated skip-stop scheme trumps all-stop service when $\lambda > 70$ trips/km/h, which translates to a cross-sectional flow of 420 passengers/h. Many of the world's busy bus corridors have much higher passenger flows. Thus, coordinated skip-stop service can seemingly be successful in wide-ranging cases.[6]

Of further interest, converted express/local bus service also outperforms all-stop service for all but the smallest of λ. Yet, the cost savings of the former are not as great as those obtained under skip-stop operations. All this is reasonable: an express/local scheme may outperform skip-stop service when $\bar{l}$ is large, as will be demonstrated momentarily for rail transit. A large ratio of the total number of stops to the number of express stops, N, means that more stops can be skipped under express service. However, the remnant of all-stop service (i.e. the invariant and sub-optimal stop spacing) precluded any express/local bus scheme from realizing its full potential.

We further find that neither the original all-stop systems, nor their converted express/local ones furnish sufficient capacity to serve patrons under the highest values of λ and $\bar{l}$ jointly examined. When joint values fall to the right of the thin, dashed contour line in Figure 4a, only conversions to coordinated skip-stop service can prevent residual queues of patrons from steadily expanding at the stops.

Moreover, converted skip-stop service can sometimes even save costs to the bus agency, as well as to its patrons. Instances in which savings accrue to the agency are shown for the case of a low-wage city in Figure 4c. Note how agency cost diminishes (as shown by the positive percentage savings in the figure) under moderate to high values of λ and $\bar{l}$ that lie to the right of the bold dashed contour curve. When patrons travel long distances in large numbers, the higher vehicle speeds enjoyed under skip-stop service enable the agency to serve demand using fewer buses.

[6] The range of (low) demands for which all-stop bus service is the preferred option might have expanded had the analysis allowed buses to skip visiting any stop where no patron seeks to board or alight. Still, the present analysis unveils how alternatives to the traditional vehicle-dispatching scheme become preferable as demand grows large.

Outcomes similar to those above are unveiled for the case of a high-wage city, as exemplified in Figure 4d. Note how the contour lines in that figure are shifted closer to its lower-left corner, as compared against the contours in Figure 4a. This indicates that converted skip-stop service enjoys even broader appeal in the high-wage city than in the lower-wage one. The distinction underscores the value of differentiated transit service in affluent cities where patrons have high values of time.

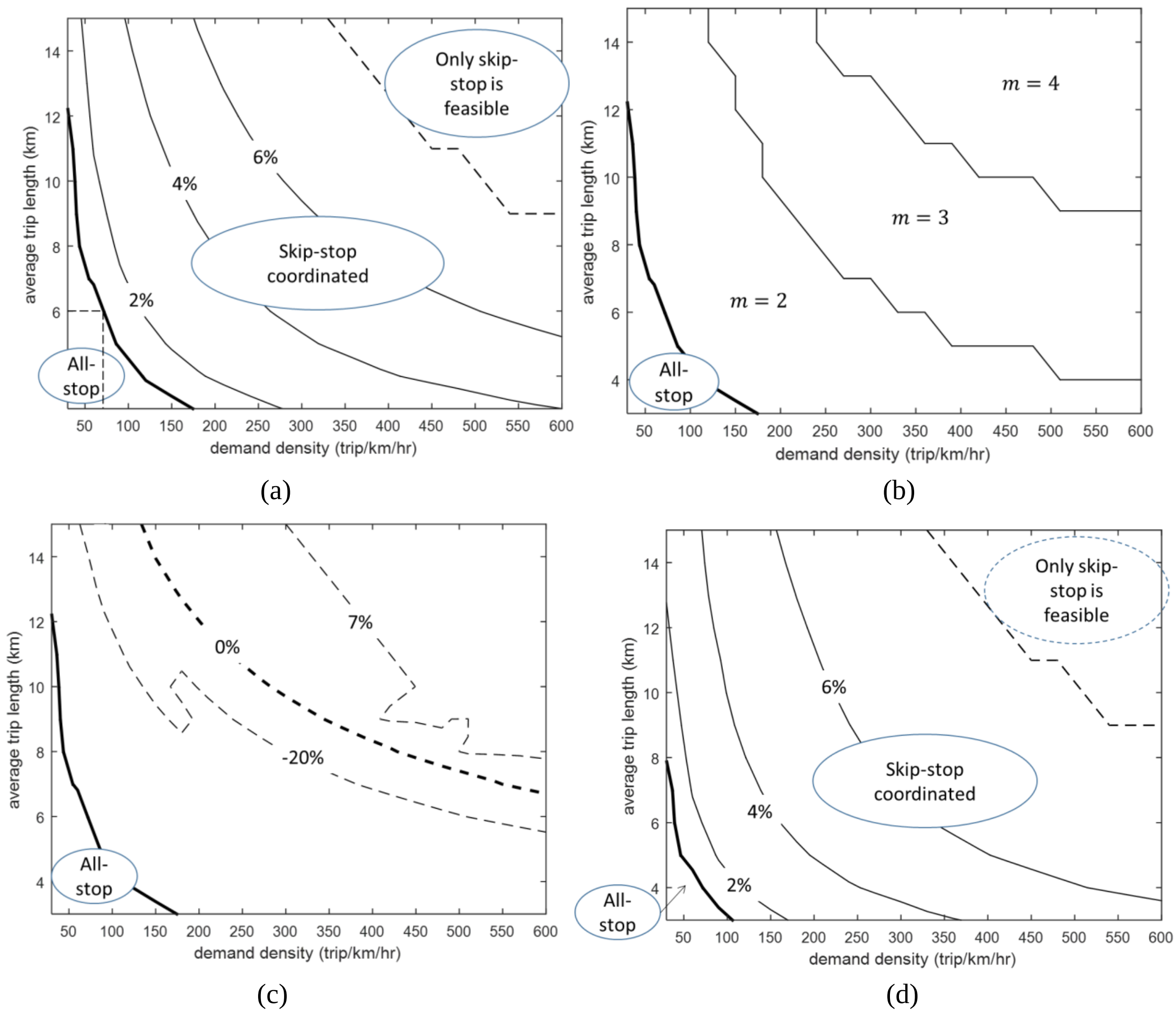

Figure 4. Conversions for ordinary bus systems: (a) lowest-cost schemes for a low-wage city ($\mu = 5$ \$/h); (b) optimal m for a low-wage city; (c) agency cost savings for a low-wage city; (d) lowest-cost schemes for a high-wage city ($\mu = 20$ \$/h)

4.2.2 BRT systems

The analyses to follow assume that BRT systems are converted to coordinated skip-stop ones without the infusion of new infrastructure. In contrast, the analyses of converted express/local systems assume that new passing lanes are to be installed at each stop.

Cost comparisons for the low- and high-wage cities are summarized Figures 5a and b, respectively. Note the larger range of λ now examined, as compared against what was used in Figures 4a-

d. The decision to now explore this greater range makes sense, given that cities typically resort to BRT service when travel demands exceed what can be accommodated by ordinary buses. Further note that for both the low- and high-wage cities, coordinated skip-stop service is the superior option for nearly the full ranges of λ and $\bar{l}$ examined[7]; and that cost savings relative to the original all-stop service can exceed 10%. Of further interest are the wide ranges of λ and $\bar{l}$ for which all-stop service would produce steadily expanding patron queues at the stops. And note the especially broad appeal of skip-stop systems for the high-wage city. As an interesting aside, converted express/local service once again tended to outperform all-stop systems, but not the converted skip-stop ones.

Varying the value of the coefficient β_1 (see again Table 4) to account for the infrastructure cost of converted express/local service had no effect on the outcomes shown in Figures 5a and b. Even when β_1 was diminished from 1.0 to 0.8, skip-stop conversions continued to be the superior service option of the three, and there was no change in the feasibility regions delineated by dashed lines in the figures.

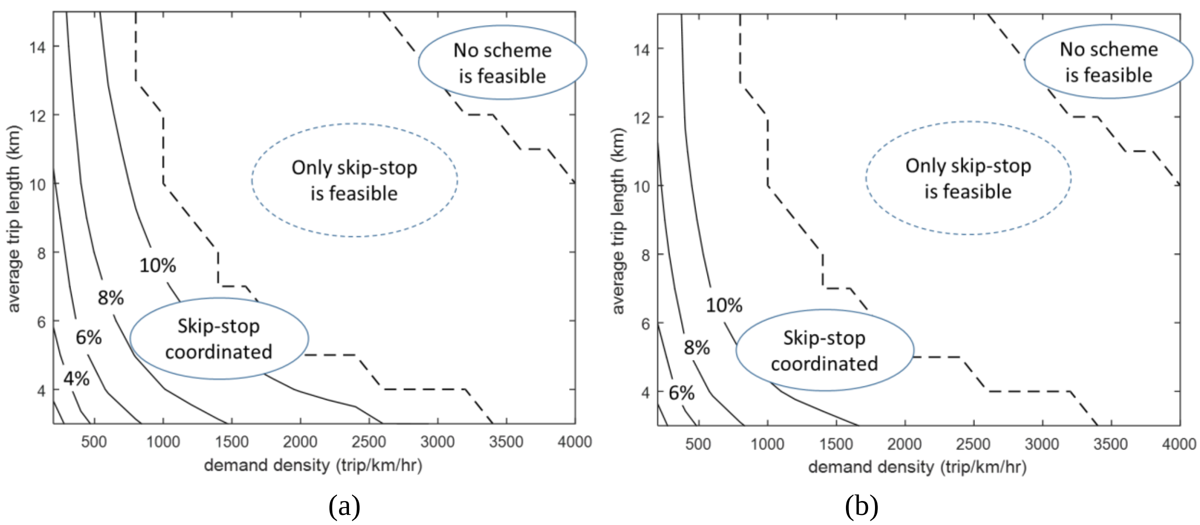

Figure 5. Conversions for BRT systems: (a) lowest-cost schemes for a low-wage city ($\mu = 5$ \$/h); (b) lowest-cost schemes for a high-wage city ($\mu = 20$ \$/h)

4.2.3 Rail systems

Assume now that new infrastructure: is not added when converting all-stop rail systems to *uncoordinated* skip-stop ones; but is required of the conversions to coordinated (i.e. double-track) skip-stop systems and to express/local ones. The added infrastructure entails a second track in each travel direction, and enlarged stops to accommodate those new tracks. Outcomes are summarized in Figures 6a and b.

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[7] All-stop service still produces the lowest cost in the very bottom-left corner, but for a range that is too small to show in Figure 5.

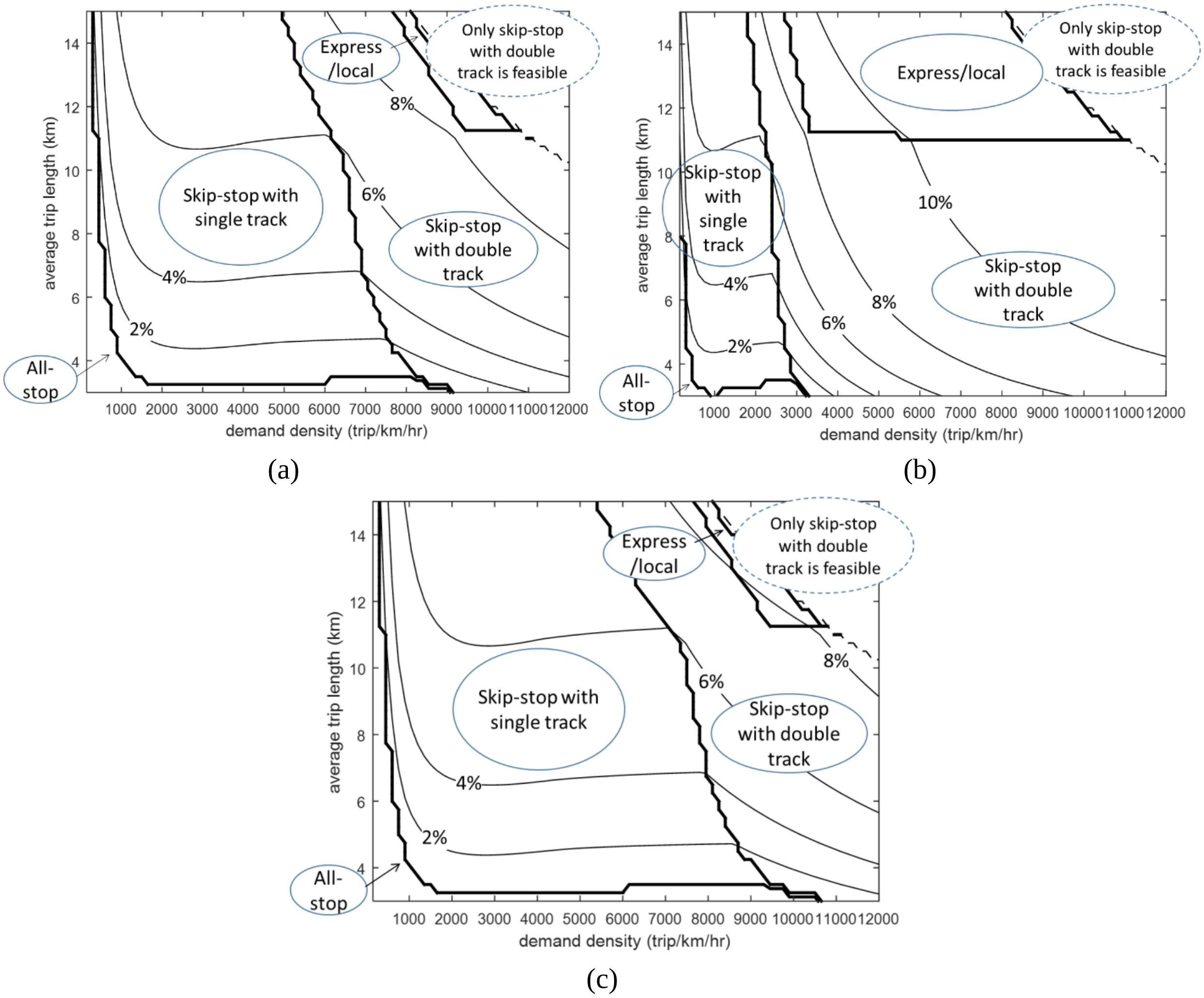


Figure 6. Conversions for rail systems: (a) lowest-cost schemes for a low-wage city ($\mu = 5$ \$/hr); (b) lowest-cost schemes for a high-wage city ($\mu = 20$ \$/hr) ; (c) when $\alpha_3 = \alpha_4 = \beta_1 = \beta_2 = 1.2$ for a low-wage city

Visual comparison of the figures reveals that, once again, all-stop service is preferable for only the lower range of λ, and that this range diminishes with increasing $\bar{l}$. Skip-stop conversions continue to dominate, but less overwhelmingly so than for bus and BRT systems. This is because skip-stop trains, in contrast to buses and BRT, cannot be fully coordinated at transfer stops, even with the second tracks added to the system. In consequence: i) the all-stop rail systems enjoy slightly more appeal, especially when λ is low; and ii) express/local systems outperform skip-stop ones when λ and $\bar{l}$ are both large, especially in the high-wage city.

In the low-wage city, uncoordinated (single track) skip-stop service ($m = 2$) is preferable for a wide range of λ, as is evident in Figure 6a. When the value of μ is low, the expense of adding tracks to foster coordinated service (where m varies from 2 to 4) only makes sense when λ is quite high, particularly for $\bar{l} < 10$ km. In the high-wage city, however, added tracks can be justified even when λ is only moderately high. This too is evident by visually comparing Figures 6a and b.

To parametrically explore the effects of added infrastructure costs, we varied the ranges of α_3, α_4, β_1, and β_2 from 80% to 120% of their tabulated values; see again Tables 3 and 4. Outcomes did not change qualitatively. As an illustration, Figure 6c presents outcomes for a low-wage city with $\alpha_3 = \alpha_4 = \beta_1 = \beta_2 = 1.2$, meaning that double-track options become more expensive than previously considered. Visual comparison of the figure with its counterpart Figure 6a reveals that skip-stop conversions with single tracks now trump other options for wider ranges of λ and $\bar{l}$. This was to be expected. Little else changes, however.

4.3 Systems from scratch

Now suppose that transit systems are to be designed from blank slates. Fully-optimized systems and attendant generalized costs are obtained for: all-stop service via program (3a-c); skip-stop service via (7a-c); and express/local service via (10a-d). The parametric analysis to follow feature systems with: $\bar{l} \in [3,15]$ km; $\lambda \le 12000$ trips/km/hr; each of the three modes; and $\mu = 5$, 20 \$/hr.

As regards the low-wage city, Figure 7a reveals that an all-stop system of ordinary buses attains the lowest generalized cost for only the smallest of λ, and only when $\bar{l} < 9$ kms. As λ and $\bar{l}$ grow larger, the lowest-cost option shifts from bus, to BRT, and eventually to rail. Similarly, the preferred vehicle-dispatching scheme shifts from all-stop, to skip-stop, and then to express/local. Exceptions occur for only the largest λ and $\bar{l}$. This is because only coordinated (double-track) skip-stop rail systems can serve these highest demands and trip lengths without creating ever-expanding patron queues. Further note that express/local systems now outperform skip-stop ones for a wider range of cases, as compared against the earlier conversion scenarios; see again Figure 6a.

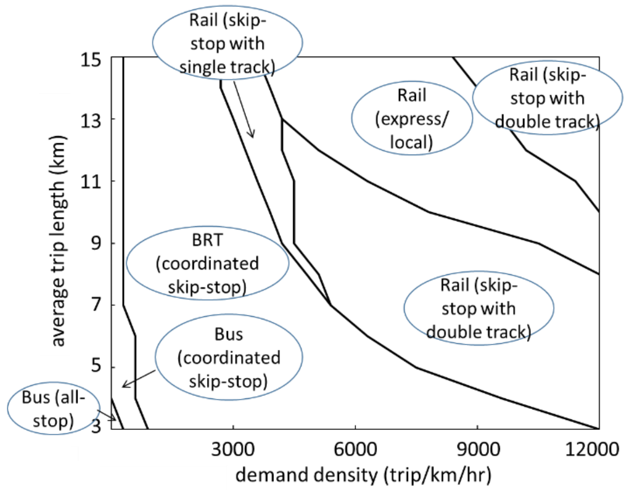
Figure 7b also pertains to a low-wage city. The figure is offered because all-stop service is often the default option when cities plan their future transit systems. It presents the lowest-cost modes for providing that traditional service. Visually comparing Figures 7a and b reveals that by opting for skip-stop schemes, BRT becomes a preferred mode to (all-stop) rail for wide ranges of λ and $\bar{l}$. This would seem useful information, given that BRT can be cheaper to build and operate than rail. Calculations indicate that resorting to skip-stop BRT service can shed generalized costs relative to all-stop rail systems by as much as 30%.

Parametric tests of varying the coefficients of infrastructure costs again had little effect on outcomes. As an illustration, Figure 7c shows the lowest-cost modes and service schemes for a low-wage city with: $\beta_1 = 1.6$ for BRT (i.e. when the infrastructure cost of an express/local BRT system becomes cheaper than in Figure 7a); and $\alpha_3 = \beta_1 = 2.4$ and $\alpha_4 = \beta_2 = 1.8$ for rail (i.e. when the infrastructure cost of a rail system with double-track becomes more expensive). By comparison with Figure 7a, the only noticeable change is the expanded ranges of λ and $\bar{l}$ for which single-track skip-stop rail systems now produce the lowest costs. This was to be expected given the high infrastructure costs. Other than this, Figures 7a and c look rather similar.

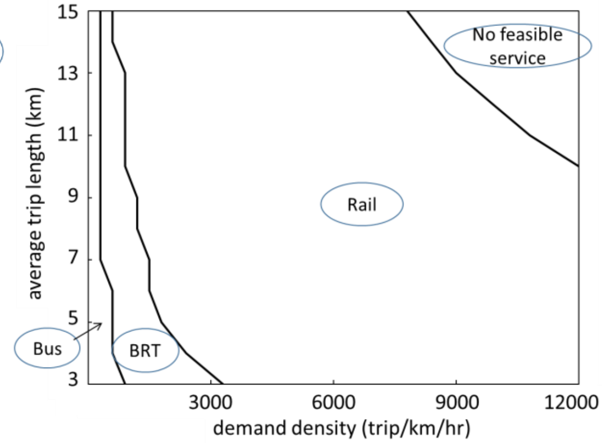
Finally, outcomes for the high-wage city are presented in Figures 8a and b. Note from the first of these figures how express/local rail service is the lowest-cost option for an especially wide range of cases.[8] Travelers with higher values of time, in effect, call for the forms of higher-quality transit service

[8] We find that the optimal ratio, N, ranged from 5 to 7 for the ranges of λ and $\bar{l}$ explored.

that require costlier outlays. Still, visual comparison of Figures 8a and b reveals once again that BRT systems with coordinated skip-stop service are often preferable to all-stop rail systems. Cost savings in the high-wage case are estimated to be as high as 25%. Predictably, the savings are now a bit lower than for the case of the low-wage city, where the patrons' lower value of time gives BRT the greater advantage over rail.



(a)



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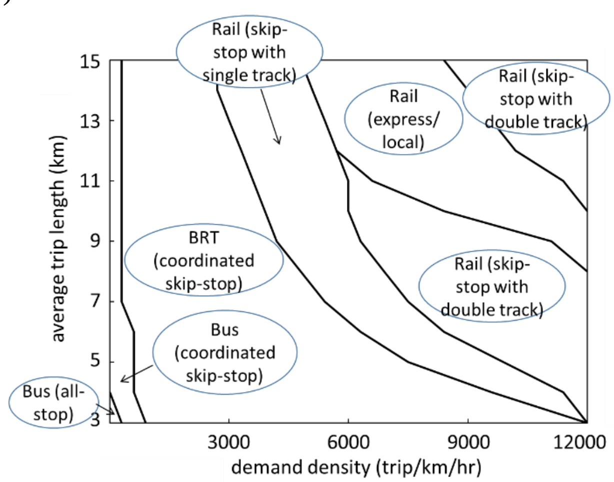
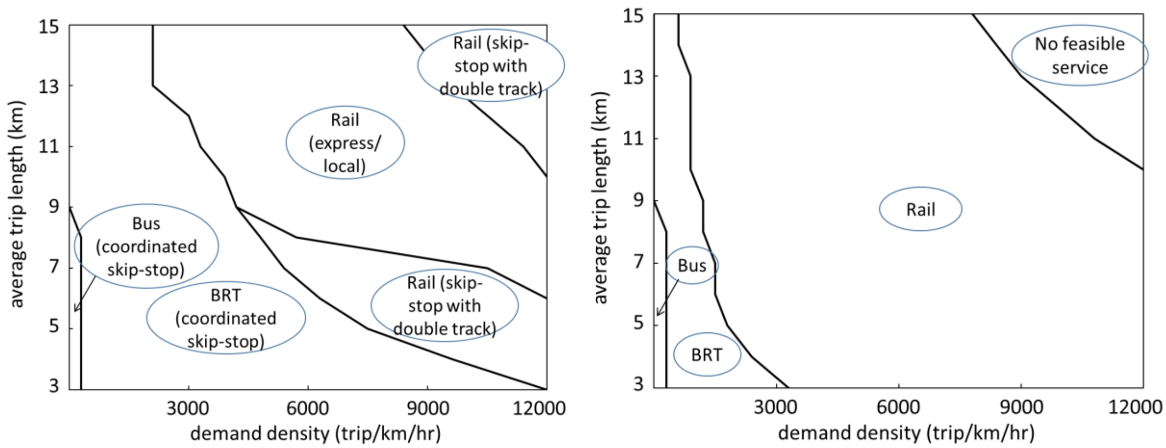


(c)

Figure 7. Lowest-cost, fully-optimal designs for a low-wage city ($\mu = 5$ \$/h): (a) lowest-cost modes and service schemes; (b) lowest-cost all-stop modes; (c) lowest-cost modes and service schemes when $\beta_1 = 1.6$ for BRT, $\alpha_3 = \beta_1 = 2.4$ and $\alpha_4 = \beta_2 = 1.8$ for rail.



(a) (b)

Figure 8. Lowest-cost, fully-optimal designs for a high-wage city ($\mu = 20$ \$/h): (a) lowest-cost modes and service schemes; (b) lowest-cost all-stop modes.

5. Conclusions

Design models that minimize the generalized cost of skip-stop and express/local transit service were developed for idealized corridors. The models extend the literature on the subject by allowing for more general service designs that feature: arbitrary stop spacings and vehicle headways; arbitrary numbers of skip-stop routes; the imposition of a vehicle's passenger-carrying capacity; schedule coordination to lessen the transfer costs that skip-stop service imposes on patrons; and more realistic route-choice behavior among patrons to capture their aversion to transfer between express and local lines.

Parametric analysis indicates that both alternate schemes - and skip-stop service in particular - can reduce costs as compared against more traditional all-stop service. This was found to be the case for a remarkably large range of conditions that spanned bus routes with moderately high travel demands to the busiest of rail systems. When all-stop systems were converted to feature alternative dispatching schemes, estimated cost savings reached 10%. This was the case even though the converted systems were hampered by remnants of the original all-stop designs. When fully-optimized alternative schemes were designed from scratch, estimated savings reached 30%. Savings came thanks to the higher average vehicle speeds, which even diminished the agency costs when patrons' travel demands and average trip lengths were sufficiently high.

Skip-stop service emerged as the lowest-cost option for the broadest range of cases. Costs were diminished in large part due to schedule coordination to reduce the patrons' costs of transferring between vehicles. This often made skip-stop service feasible even when more than two routes coexisted along the corridor. Of particular interest, the findings show that coordinated skip-stop service on fully-optimized BRT systems can often generate lower costs than can optimally-designed all-stop service via rail.

Express/local service also enjoyed an important niche, namely: rail systems that are optimally designed from scratch to serve high demands. This was the case even though express/local service required the installation of additional tracks.

Estimated savings from the alternative schemes might have been larger had we considered elasticities in transit demand. The higher vehicle speeds obtained with alternative strategies could induce more travelers to take transit, thereby diminishing the cost to serve each one. Greater savings might also have come had we considered travel demand that is spatially-inhomogeneous over the corridors. In those cases, skip-stop and express services can be fine-tuned to better serve origin-destination pairs of high demand.

Still greater savings might have occurred had we designed systems that are asymmetric and arbitrary in forms; e.g. see Ulusoy et al. (2010), Leiva et al. (2010), Tetreault and El-Geneidy (2010). Yet, our present symmetric designs reduce the number of transfers required of patrons, which makes trip planning easier.

The present idealizations, including the assumption of fixed vehicle dwell times at each stop, allowed for the formulation of simpler optimization models. Parametric analysis could thus be performed over wider ranges of operating conditions. A greater variety of high-level insights resulted.

That alternative service schemes were often shown to be favorable options despite our conservative assumptions underscores the potential advantages of these alternatives. We think that these outcomes can be of interest to the many communities that have a stake in promoting higher-quality transit service.

Acknowledgements

The authors are indebted to Carlos Daganzo, UC Berkeley, for his valuable insights, particularly in the early stages of this study. The research was partially supported by grants provided by the Hong Kong Polytechnic University (1-ZE38 and 1-ZVDB), and by the University of California Center for Economic Competitiveness in Transportation (UCCONNECT).

Appendix A. Table of Notation

Notation	Description
$\boldsymbol{K}$	Vehicle's passenger-carrying capacity (passenger/veh)
$\boldsymbol{C_t}$	Patron transfer penalty (h)
$\boldsymbol{\pi_v}$	Cost of vehicle-km of service ($/veh·km)
$\boldsymbol{\pi_m}$	Cost of vehicle-hour of service ($/veh·h)
$\boldsymbol{\pi_s}$	Amortized cost of a stop ($/stop/h)
$\boldsymbol{\pi_i}$	Amortized line-infrastructure cost ($/km/h)
$\boldsymbol{v_w}$	Patron's average walking speed (km/h)
$\boldsymbol{v}$	Vehicle's average cruising speed (km/h)
$\boldsymbol{H}$	Average vehicle headway for all-stop and skip-stop schemes (h)
$\boldsymbol{H_E}$	Average headway for express vehicles (h)
$\boldsymbol{H_L}$	Average headway for local vehicles (h)
$\boldsymbol{\bar{l}}$	Patron's average trip length (km)
$\boldsymbol{\tau}$	Patron loading/unloading time (h)
$\boldsymbol{s}$	Spacing between two neighboring stops (km)
$\boldsymbol{m}$	Number of routes in skip-stop service
$\boldsymbol{k}$	Number of non-transfer stops per route between two neighboring transfer stops
$\boldsymbol{N}$	Integer ratio between the total number of stops and the number of express ones
$\boldsymbol{\lambda}$	Total trip demand density for a single travel direction (trip/km/h)
$\boldsymbol{\mu}$	Patron value of time ($/h)
$\boldsymbol{C_{P\_AS}}$	Patron cost for all-stop service (h)
$\boldsymbol{C_{A\_AS}}$	Agency cost for all-stop service (h)
$\boldsymbol{GC_{AS}}$	Generalized cost for all-stop service (h)
$\boldsymbol{C_{P\_SS}}$	Patron cost for skip-stop service (h)
$\boldsymbol{C_{O\_SS}}$	Operating cost for skip-stop service (h)
$\boldsymbol{C_{I\_SS}}$	Infrastructure cost for skip-stop service (h)
$\boldsymbol{GC_{SS}}$	Generalized cost for skip-stop service (h)
$\boldsymbol{C_{P\_EL}}$	Patron cost for express/local service (h)

$C_{P\_EL\_EE}$	Patron cost of a trip between two express stops (h)
$C_{P\_EL\_EL}$	Patron cost of a trip between an express and a local stop (h)
$C_{P\_EL\_LL}$	Patron cost of a trip between two local stops (h)
C_{A_EL}	Agency cost for express/local service (h)
GC_{EL}	Generalized cost for express/local service (h)
Q_L	Passenger flow transported on the local route (passenger/h)
Q_E	Passenger flow transported on the express route (passenger/h)

Appendix B. Added Costs of Skip-Stop Service

This appendix furnishes derivations to account for the added costs that patrons incur under skip-stop service as compared against all-stop service.

B.1 Trip types and their probabilities of occurrence

Trips on skip-stop systems are of four types. We describe these and define their probabilities of occurrence, recognizing that in the present study, origins and destinations are uniformly distributed along a transit corridor.

Type 1: A trip's origin and destination stops are both transfer stops. In this instance, a patron waiting at her origin stop boards the first arriving vehicle, regardless of the route that the vehicle serves. Since $\frac{1}{mk+1}$ of the stops are transfer stops, $\frac{1}{(mk+1)^2}$ is a reasonable approximation of this trip type's probability of occurrence when $\bar{l} \gg s$.

Type 2: A trip's origin and destination stops both reside along the same route, but one or both of these stops are not transfer stops. In this instance, a patron boards the first arriving vehicle that serves her route and completes her trip without transferring. The probability that trips of this type both start and end at a non-transfer stop is $m \times \frac{k}{mk+1} \times \frac{k}{mk+1}$, where $\frac{k}{mk+1}$ is the probability that a stop is a non-transfer one served by route i $(1 \leq i \leq m)$. The probability that a trip of Type 2 either starts or ends (but does not both start and end) at a transfer stop is $2 \times \frac{1}{mk+1} \times \frac{k}{mk+1}$. Thus, trips of type 2 occur with a probability of approximately $\frac{2mk+mk^2}{(mk+1)^2}$.

Type 3: A trip's origin and destination stops both reside within the same segment bounded by neighboring transfer stops, but reside along distinct routes. A trip of this type requires a patron to back-track; see again the dotted arrows in Figure 1. Since trip lengths are uniformly distributed in $[0, 2\bar{l}]$, the number of stops visited in a trip is (approximately) uniformly distributed in $\left[0, \frac{2\bar{l}}{s}\right]$. For a randomly-selected (non-transfer) origin stop, the number of possible destination stops in both travel directions is therefore $\frac{4\bar{l}}{s}$, and the number of these that would entail a trip of type 3 is $(m-1)k$. Thus, the probability of this trip type's occurrence is the product of the probability that (i) the origin is not a transfer stop; and

(ii) a transfer stop does not reside between the trip's origin and destination, conditioned on (i). That probability is $\frac{mk}{mk+1} \times \frac{(m-1)k}{4\bar{l}/s}$.

Type 4: A trip's origin and destination stops reside on distinct routes and are separated by one or more transfer stops. The patron must therefore perform a transfer like the one illustrated with dashed arrows in Figure 1. The probability of this trip type's occurrence is simply the difference between 1 and the summed probabilities for the other three trip types: $\frac{(m-1)mk^2}{(mk+1)^2} - \frac{mk}{mk+1} \times \frac{(m-1)k}{4\bar{l}/s}$.

B.2 Patron's added wait under uncoordinated service

If service is not coordinated across routes, then a patron's average wait time at her origin stop is: $\frac{H}{2}$ for a type-1 trip; and $\frac{mH}{2}$ for a type-2 trip. Patrons with trips of types 3 and 4 each experience wait times at both, origin and transfer stops. These times sum to mH on average.

Thus, the patron's expected wait time under uncoordinated skip-stop service is:

$$\frac{1}{(mk+1)^2} \cdot \frac{H}{2} + \frac{2mk+mk^2}{(mk+1)^2} \cdot \frac{mH}{2} + \frac{(m-1)mk^2}{(mk+1)^2} \cdot mH = \frac{H\left(2m^2k-mk+1\right)}{2(mk+1)}.$$

This explains the first value of α_1 given in (4a).

B.3 Added wait under coordinated (double-track) rail service

In this instance, the patron's average wait time at her origin stop is: H for a type-1 trip (note in this instance that the headway between two consecutive train pairs is $2H$); and $\frac{mH}{2}$ for a type-2 trip. Average wait times at origin and transfer stops for trips of types 3 and 4 are $\frac{mH}{2} + \frac{mH}{2} \times \left(1 - \frac{1}{m-1}\right)$. There are in total $m(m-1)$ combinations of transfers from route i to j ($1 \leq i, j \leq m$ and $i \neq j$), among which m combinations are coordinated. Thus, $\frac{1}{m-1}$ of trips of types 3 and 4 enjoy coordinated transfers.

The patron's expected wait time under coordinated (double-track) rail service is therefore:

$$\frac{1}{(mk+1)^2} \times H + \frac{2mk+mk^2}{(mk+1)^2} \times \frac{mH}{2} + \frac{(m-1)mk^2}{(mk+1)^2} \times \frac{mH}{2} \times \frac{2m-3}{m-1} = \frac{H(m^2k-mk+1)}{mk+1},$$

which explains the second value of α_1 given in (4a).

B.4 Added wait under coordinated bus and BRT services

The patron's average wait time in these instances is $\frac{mH}{2}$, which is the third value of α_1 given in (4a).

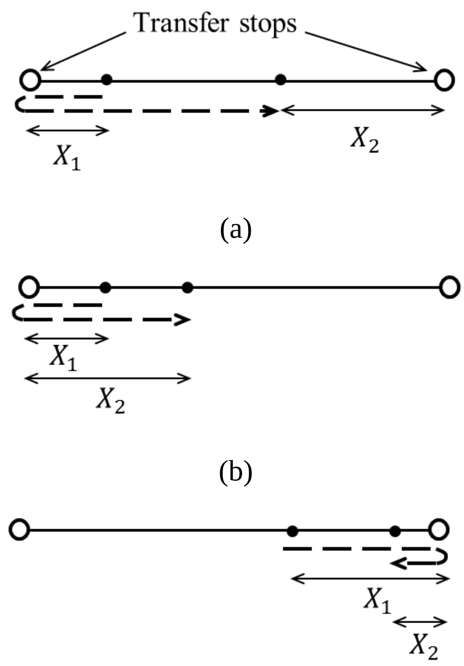
B.5 Added transfer penalties

Transfer penalty C_t is incurred only once for each trip of type 3 and 4. For bus and BRT services, and when rail service comes with only a single track per travel direction, a central-island platform can be used at each transfer stop to serve both travel directions. The average transfer cost is thus $\frac{(m-1)mk^2}{(mk+1)^2} \times C_t$, where $\frac{(m-1)mk^2}{(mk+1)^2}$ is the combined probability of trips of type 3 and 4. This explains the first α_2 in (4b).

For rail service with double tracks in each travel direction, each type-3 trip (with its expected back-tracking) includes a transfer from one platform to another, which costs $2C_t$. Thus, the average transfer cost is $\left(\frac{(m-1)mk^2}{(mk+1)^2} + \frac{mk}{mk+1} \times \frac{(m-1)k}{4\bar{l}/s}\right) \times C_t$. This explains the second α_2 in (4b).

B.6 Expected back-tracking distance

The origin and destination of an arbitrary type-3 trip (see the dotted arrows in Figure 1 for example) can be approximated as two random selections without replacement. These selections would come from the mk non-transfer stops that reside between two neighboring transfer stops. We denote X_1 and X_2 as the distances that the trip's origin and destination reside relative to the nearest transfer stop, as shown in Figures 9a-c, where the black dots denote trip origin and destination. Inspection of those figures reveals that the backtrack distance is $Z \equiv 2\min(X_1, X_2)$.



(a)

(b)

(c)

Figure 9. Three cases of backtrack trips.

We approximate the discrete variables X_1 and X_2 as continuous random variables (denoted with the same notation for simplicity) that follow the uniform distribution $U\left(0, \frac{(mk+1)s}{2}\right)$. Thus, we have:

$$E[Z] \approx 2\int_{x=0}^{\frac{(mk+1)s}{2}} \Pr(\min(X_1, X_2) \geq x)\, dx = 2\int_{x=0}^{\frac{(mk+1)s}{2}} \Pr(X_1 \geq x) \Pr(X_2 \geq x)\, dx$$
$$= 2\int_0^{\frac{(mk+1)s}{2}} \left(1 - \frac{x}{\frac{(mk+1)s}{2}}\right)^2 dx = \frac{(mk+1)s}{3}.$$

Hence, the average back-tracking distance is the probability of trips of type 3 times $E[Z]$, i.e., $\frac{mk}{mk+1} \times \frac{(m-1)k}{4\bar{l}/s} \times \frac{(mk+1)s}{3} = \frac{m(m-1)k^2}{12\bar{l}/s} s$, which explains the back-tracking distance in (4c).

B.7 Maximum number of on-board patrons per vehicle

Cross-sectional passenger flow in this rotationally symmetric corridor will be uniformly distributed if the back-tracking distances traveled are not included. This flow is $\lambda\bar{l}$.

On the other hand, the backtrack trips tend to have the highest cross-sectional flow near a transfer stop, because every type-3 trip will first arrive at a transfer stop and then turn around. This added passenger flow is equal to the number of type-3 trips divided by the number of transfer stops, i.e.,

$$\frac{\lambda \frac{mk}{mk+1} \cdot \frac{(m-1)k}{4\bar{l}/s}}{1/(mk+1)s},$$

where the numerator is the number of type-3 trips per km per hour in each travel direction, and the denominator is the density of transfer stops.

Hence, the maximum number of on-board patrons per vehicle is equal to the maximum cross-sectional passenger flow multiplying the headway:

$$H \cdot \left(\lambda\bar{l} + \frac{\lambda \frac{mk}{mk+1} \cdot \frac{(m-1)k}{\frac{4\bar{l}}{s}}}{\frac{1}{(mk+1)s}} \right) = \lambda H \left(\bar{l} + \frac{m(m-1)k^2 s^2}{4\bar{l}} \right).$$

Appendix C. Patron Costs and Flows (Q_L and Q_E) for Express/Local Service

This appendix furnishes derivations to account for (i) the patron costs of the three trip types (EE, EL, and LL) under express/local service as described in Section 2.3; and (ii) the passenger flows on express and local routes, Q_L and Q_E, respectively.

C.1 Patron cost for trips between two express stops

These passengers travel via express service only. Thus, the patron cost per trip is similar to that of trips under all-stop service:

$$C_{P\_EL\_EE} = \frac{s}{2v_w} + \frac{H_E}{2} + \bar{l}\left(\frac{1}{v} + \frac{\tau}{Ns}\right). \tag{C1}$$

This explains how $C_{P\_EL\_EE}$ was calculated in (8).

C.2 Patron cost for trips between an express and a local stops

Consider a patron traveling from an express stop to a local one[9]. The patron can choose between two route options: taking the local route only, or taking the express and transferring to the local at the last express stop on her trip. With the first option, her trip time is simply: $\text{access/egress cost} + \frac{1}{2}H_L + \frac{l}{v} + m\tau$, where l is the trip length and m the number of local stops encountered on that trip. With the express-

[9] Trips from a local stop to an express one would have the same average trip cost, thanks to symmetry.

local route option, the patron's trip time would be $\text{access/egress cost} + \frac{1}{2}(H_E + H_L) + \frac{l}{v} + (M + m')\tau + C_t$, where M and m' are the numbers of express and local stops encountered, respectively. We have $m = MN + m'$ and $1 \le m' \le N - 1$. The patron would choose the local-only option if and only if $M(N-1)\tau \le \frac{1}{2}H_E + C_t$. It follows that $M \le M_0 \equiv \left\lfloor \frac{\frac{1}{2}H_E + C_t}{(N-1)\tau} \right\rfloor$, or equivalently, $m \le N(M_0 + 1) - 1$, where $\lfloor x \rfloor$ is the largest integer not greater than x.

Trips of the above type fall into two classes: those on the local route (only) account for a fraction p_0 of the trips; and those that include the express route account for $1 - p_0$ of them. Given that the number of stops visited in a trip approximately follows a uniform distribution in $\left[0, \frac{2\bar{l}}{s}\right]$, we have:

$$p_0 \approx \min\left\{1, \frac{N(M_0+1)-1}{2\bar{l}/s}\right\}. \quad \text{(C2)}$$

The average trip time of those local-only trips would be $\frac{s}{2v_w} + \frac{H_L}{2} + \frac{E[l|\text{local-only}]}{v} + \tau \times E[m|m \le N(M_0 + 1) - 1]$, where the last term is equal to $\frac{1}{2}N(M_0 + 1)\tau$. Trips that include the express route have an average trip time of $\frac{s}{2v_w} + \frac{1}{2}(H_E + H_L) + \frac{E[l|\text{express-first}]}{v} + C_t + \tau \times E\left[M + m'|M > M_0\right]$, where the last term is equal to $\frac{M_0 + 1 + \frac{2\bar{l}}{Ns} + N}{2}\tau$. This is because both M and m' approximately follow uniform distributions but with different supports: $\left[M_0 + 1, \frac{2\bar{l}}{Ns}\right]$ for M, and $[1, N-1]$ for m'.

Given the above, the average patron cost for trips between express and local stops is:

$$C_{P\_EL\_EL} = p_0\left(\frac{s}{2v_w} + \frac{H_L}{2} + \frac{E[l|\text{local-only}]}{v} + \frac{1}{2}N(M_0+1)\tau\right) + (1 - p_0)\left(\frac{s}{2v_w} + \frac{1}{2}(H_E + H_L) + \frac{E[l|\text{express-first}]}{v} + C_t + \frac{M_0 + 1 + \frac{2\bar{l}}{Ns} + N}{2}\tau\right)$$

$$= \frac{s}{2v_w} + \frac{H_L}{2} + \frac{\bar{l}}{v} + p_0 \cdot \frac{1}{2}N(M_0+1)\tau + (1 - p_0) \cdot \left(\frac{H_E}{2} + C_t + \frac{M_0 + 1 + \frac{2\bar{l}}{Ns} + N}{2}\tau\right) \quad \text{(C3)}$$

This explains how $C_{P\_EL\_EL}$ was calculated in (8).

C.3 Patron cost for trips between two local stops

Trips of this type are also divided into two classes: those on the local route only; and those that entail a local-express-local route. The probability and average patron cost of trips of each class are formulated below.

Consider an arbitrary patron with trip length l, whose trip covers m local stop spacings (each of length s) and M express stop spacings (of length Ns). We have $m = MN + m'$, and $2 \le m' \le 2N - 2$. The trip time for the local-only route is $\text{access/egress cost} + \frac{1}{2}H_L + \frac{l}{v} + m\tau$; and the local-express-local has a trip time of access/egress cost $+ H_L + \frac{H_E}{2} + \frac{l}{v} + 2C_t + (M + m')\tau$. Thus, the patron would choose the local-only option if and only if $M(N-1)\tau \leq \frac{1}{2}(H_E + H_L) + 2C_t$; i.e., $M \leq M_1 \equiv \left\lfloor \frac{\frac{1}{2}(H_E+H_L)+2C_t}{(N-1)\tau} \right\rfloor$.

To compute the probability and patron cost of the local-only trips, we plot part of the O-D pairs of an express/local system with $N = 4$ in Figure 10. The vertical and horizontal axes display the origin and destination stops, respectively, where those labeled as E_i are express stops, and those labeled as L_i are local ones ($i = 1,2,3, \ldots$). Each dark dot where two grid lines intersect represents a potential O-D pair between two local stops. Without loss of generality, we focus on the O-D pairs that originate from the local stops residing between two neighboring express ones, i.e., L_1, L_2, and L_3. For each origin stop, the number of possible destination stops that are on the local line only is approximately $\frac{2\bar{l}}{s} - \frac{2\bar{l}}{Ns}$. Thus, the total number of local-to-local O-D pairs that originate from L_1, L_2, and L_3 is $(N-1)\left(\frac{2\bar{l}}{s} - \frac{2\bar{l}}{Ns}\right) = \frac{2\bar{l}(N-1)^2}{Ns}$.

The trips that satisfy the condition $M \leq M_1$ consist of two groups. (Note that in Figure 10, $M_1 = 2$.) The first group entails those trips that do not pass through an express stop (i.e., trips with origins and destinations that are both located between two neighboring express stops); see the three dots enclosed by the smaller of the dashed boxes in Figure 10. The number of O-D pairs in this group is simply $(N-2) + (N-3) + \cdots + 1 = \frac{(N-1)(N-2)}{2}$. The second group contains the trips that pass through at least one express stop; see in Figure 10 the O-D pairs located between the smaller dashed box and the vertical dashed line labeled $M_1 = 2$. The number of O-D pairs in this second group is $(N-1)^2 \times (M_1 + 1)$.

Thus, the probability that a local-to-local trip belongs to the first group of local-only trips is:

$$p_1 = \min\left\{1, \frac{\frac{(N-1)(N-2)}{2}}{\frac{2\bar{l}(N-1)^2}{Ns}}\right\} = \min\left\{1, \frac{N(N-2)s}{4\bar{l}(N-1)}\right\}. \tag{C4}$$

The average patron cost of trips in this first group is $\frac{s}{2v_w} + \frac{H_L}{2} + \frac{E[l|\text{first group of local-only}]}{v} + \tau \times E[m|\text{first group of local-only}]$, where the last term is computed as follows:

$$E[m|\text{first group of local-only}] = \frac{\sum_{i=1}^{N-2}(i+(i-1)+\cdots+1)}{\frac{(N-2)(N-1)}{2}} = \frac{\sum_{i=1}^{N-2}\left(\frac{i(i+1)}{2}\right)}{\frac{(N-2)(N-1)}{2}} = \frac{\sum_{i=1}^{N-2}\left(\frac{[(i+1)^3-i^3-1]}{6}\right)}{\frac{(N-2)(N-1)}{2}}$$

$$= \frac{1}{3} \times \frac{\sum_{i=1}^{N-2}[(i+1)^3-i^3-1]}{(N-2)(N-1)} = \frac{1}{3} \times \frac{(2^3-1^3-1+3^3-2^3-1+\cdots+(N-2)^3-(N-3)^3-1+(N-1)^3-(N-2)^3-1)}{(N-2)(N-1)}$$

$$= \frac{(N-1)^3-N}{3(N-2)(N-1)} = \frac{N^3-3N^2+2N}{3(N-2)(N-1)} = \frac{N}{3}.$$

The probability that a local-to-local trip belongs to the second group is:

$$p_2 = \min\left\{1-p_1, \frac{(N-1)^2 \times (M_1+1)}{\frac{2\bar{l}(N-1)^2}{Ns}}\right\} = \min\left\{1-p_1, \frac{N(M_1+1)s}{2\bar{l}}\right\}. \tag{C5}$$

The average patron cost of trips in this second group is $\frac{s}{2v_w} + \frac{H_L}{2} + \frac{E[l|\text{second group of local-only}]}{v} + \tau \times E[m|\text{second group of local-only}]$, where the last term is:

$$E[m|\text{second group of local-only}] = E[MN + m'|\text{second group of local-only}]$$

$$= N \times E[M|\text{second group of local-only}] + E[m'|\text{second group of local-only}]$$

$$= N \times \frac{0+M_1}{2} + E[m'|\text{second group of local-only}] = \frac{NM_1}{2} + N.$$

The last equality holds because m' is the sum of the numbers of local stops encountered at both ends of the trip, and because the expected number of local stops at each end is $\frac{N}{2}$.

Finally, for the trips that take a local-express-local route (see the dots on the right side of the vertical dashed line in Figure 10), their probability of occurrence is simply $1 - p_1 - p_2$. The average patron cost of these local-express-local trips is $\frac{s}{2v_w} + H_L + \frac{H_E}{2} + \frac{E[l|\text{local-express-local}]}{v} + 2C_t + \tau \times E[M + m'|M > M_1]$, where the last term is approximately:

$$E\left[M + m'|M > M_1\right] = E[M|M > M_1] + E[m'|M > M_1]$$

$$= \frac{(M_1+1)+\left(\frac{2\bar{l}}{Ns}-1\right)}{2} + N.$$

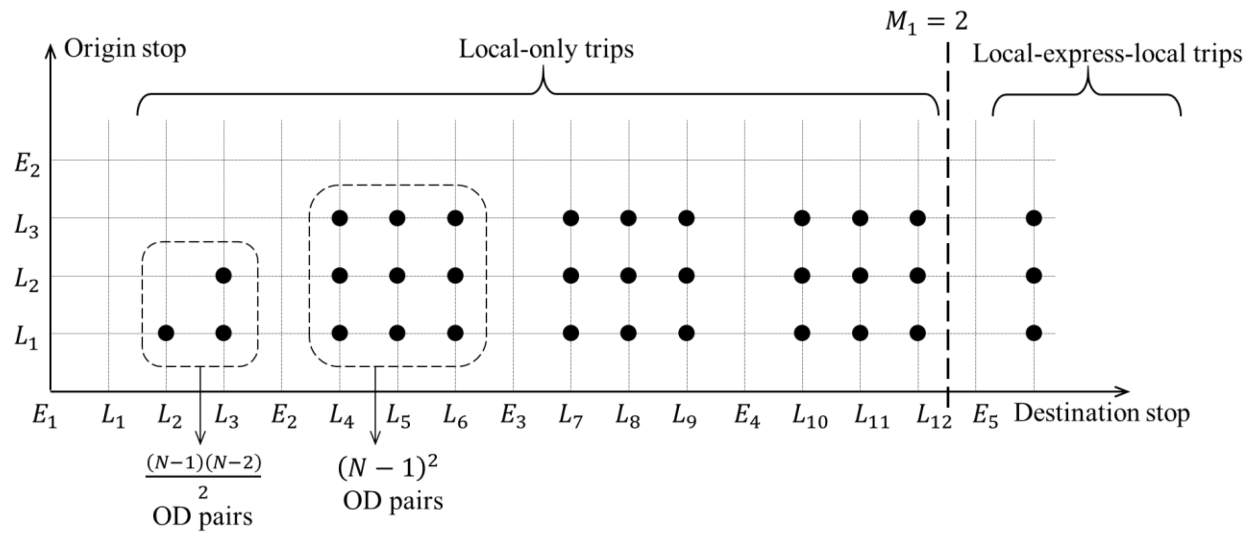


Figure 10. O-D pairs of local-only trips ($N = 4$)

Thus the average patron cost for local-local trips is:

$$C_{P\_EL\_LL} = \frac{s}{2v_w} + \frac{H_L}{2} + \frac{\bar{l}}{v} + p_1 \cdot \frac{N}{3}\tau + p_2 \cdot \frac{(M_1+2)N}{2}\tau + \left(1 - p_1 - p_2\right)\left(\frac{H_E+H_L}{2} + 2C_t + \left(\frac{\frac{2\bar{l}}{Ns}+M_1}{2} + N\right)\tau\right). \tag{C6}$$

This explains how $C_{P\_EL\_LL}$ was calculated in (8).

C.4 Maximum patron flows (Q_L and Q_E) for the express and local routes

We first calculate the cross-sectional passenger flow of the express route, since the passenger flow on this route is spatially uniform over the corridor. Thus it is equal to the passenger-kms generated from a 1-km×1-hr domain of the system (Wirasinghe and Ghoneim, 1981). Note that the total passenger-kms generated from this time-space domain is $\lambda\bar{l}$, and that the passenger-kms served by the express route is $\lambda\bar{l}$ minus that by the local route. We next calculate the latter.

It consists of the passenger-kms contributed by the local-route portion of travel for the trips that occur: i) between an express and a local stop (with probability $\frac{2(N-1)}{N^2}$), denoted $\tilde{Q}_{L\_EL}$; and ii) between two local stops (with probability $\frac{(N-1)^2}{N^2}$), denoted $\tilde{Q}_{L\_LL}$.

The $\tilde{Q}_{L\_EL}$ consists of the passenger-kms contributed by local-only trips, and by the local-route part of travel of the express-local (or local-express) trips. The former is $\frac{2(N-1)}{N^2} \times p_0 \times \lambda \times \frac{N(M_0+1)s}{2}$, and the latter $\frac{2(N-1)}{N^2} \times (1-p_0) \times \lambda \times \frac{Ns}{2}$.

The $\tilde{Q}_{L\_LL}$ consists of the passenger-kms contributed by: i) the local-only trips whose origin and destination stops are both located between two neighboring express stops (i.e., the first group of local-only trips in Section C.3), $\frac{(N-1)^2}{N^2} \times p_1 \times \lambda \times \frac{Ns}{3}$; ii) the local-only trips that contain at least one express stop (i.e., the second group of local-only trips in Section C.3), $\frac{(N-1)^2}{N^2} \times p_2 \times \lambda \times \frac{N(M_1+2)s}{3}$; and iii) the local-route portion of travel of the local-express-local trips, $\frac{(N-1)^2}{N^2} \times (1-p_1-p_2) \times \lambda \times Ns$.

Hence, we have:

$$Q_E = \lambda\left(\bar{l} - \frac{2(N-1)}{N^2} \cdot \left(\frac{Ns}{2} + p_0 \cdot \frac{NM_0 s}{2}\right) - \left(\frac{N-1}{N}\right)^2 \cdot \left(Ns - p_1 \cdot \frac{2Ns}{3} + p_2 \cdot \frac{N(M_1-1)s}{3}\right)\right). \tag{C7}$$

For the local route, the patron flow attains the maximum near an express stop because a significant portion of patrons will make transfers at express stops. Since all the express stops are the same, the maximum flow can be calculated as the total number of express stops visited by all the local-route passengers divided by the number of express stops. We next calculate the number of express stops visited by the local-route passengers for various types of trips.

For a local-express or express-local trip, the patron will choose to travel by the local route only if $M \le M_0$, where M is the number of express stops the trip spans over (see Appendix C.2). Since M is uniformly distributed between 0 and M_0, the average number of express stops experienced is $\frac{M_0}{2} + \frac{1}{2}$. Note that the added $\frac{1}{2}$ accounts for the destination or origin express stop. Thus the number of express stops visited by this portion of local trips is $\lambda p_0 \frac{2(N-1)}{N^2} \cdot \frac{(M_0+1)}{2}$ per km per hour.

If $M > M_0$, the patron will take the local transit to the nearest downstream express stop and transfer to the express transit. Thus the number of express stops he visits via the local route is $\frac{1}{2}$. The number of express stops visited by this portion of local trips is $\lambda(1-p_0)\frac{2(N-1)}{N^2}\cdot\frac{1}{2}$ per km per hour.

Similarly, for a local-express-local trip, the patron will choose to take the local route only if $M \leq M_1$ (see Appendix C.3). The number of express stops visited by these local-only trips is $\lambda p_2 \frac{(N-1)^2}{N^2}\cdot\frac{M_1+1}{2}$ per km per hour. On the other hand, the patrons who make two transfers between express and local will totally visit $\lambda(1-p_1-p_2)\frac{(N-1)^2}{N^2}\cdot 1$ express stops.

Hence, the maximum patron flow on the local route is:

$$Q_L = \frac{\lambda p_0 \frac{2(N-1)}{N^2}\cdot\frac{(M_0+1)}{2} + \lambda(1-p_0)\frac{2(N-1)}{N^2}\cdot\frac{1}{2} + \lambda p_2 \frac{(N-1)^2}{N^2}\cdot\frac{M_1+1}{2} + \lambda(1-p_1-p_2)\frac{(N-1)^2}{N^2}\cdot 1}{1/Ns} = \frac{\lambda s(N-1)}{N}\Big(1 + p_0 M_0 + \frac{p_2(N-1)(M_1-1)}{2} + (1-p_1)(N-1)\Big). \quad \text{(C8)}$$

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