SIAM J. CONTROL OPTIM.
Vol. 54, No. 5, pp. 2274–2308
© 2016 Society for Industrial and Applied Mathematics

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

OPEN-LOOP AND CLOSED-LOOP SOLVABILITIES FOR STOCHASTIC LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS*

JINGRUI SUN†, XUN LI†, AND JIONGMIN YONG‡

Abstract. This paper is concerned with a stochastic linear quadratic (LQ) optimal control problem. The notions of open-loop and closed-loop solvabilities are introduced. A simple example shows that these two solvabilities are different. Closed-loop solvability is established by means of solvability of the corresponding Riccati equation, which is implied by the uniform convexity of the quadratic cost functional. Conditions ensuring the convexity of the cost functional are discussed, including the issue of how negative the control weighting matrix-valued function $R(\cdot)$ can be. Finiteness of the LQ problem is characterized by the convergence of the solutions to a family of Riccati equations. Then, a minimizing sequence, whose convergence is equivalent to the open-loop solvability of the problem, is constructed. Finally, some illustrative examples are presented.

Key words. linear quadratic optimal control, stochastic differential equation, Riccati equation, finiteness, open-loop solvability, closed-loop solvability

AMS subject classifications. 49N10, 49N35, 93E20

DOI. 10.1137/15M103532X

1. Introduction.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$ [16, 30]. Consider the following controlled linear stochastic differential equation (SDE) on a finite horizon $[t, T]$:

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases} \tag{1.1}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are given deterministic matrix-valued functions of proper dimensions, and $b(\cdot), \sigma(\cdot)$ are vector-valued $\mathbb{F}$-progressively measurable processes. In the above, $X(\cdot)$, valued in $\mathbb{R}^n$, is the *state process*, and $u(\cdot)$, valued in $\mathbb{R}^m$, is the *control process*. For any $t \in [0, T)$, we introduce the following Hilbert space:

$$\mathcal{U}[t, T] = \Big\{u : [t, T] \times \Omega \to \mathbb{R}^m \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \; \mathbb{E}\int_t^T |u(s)|^2 ds < \infty\Big\}.$$

*Received by the editors August 14, 2015; accepted for publication (in revised form) May 16, 2016; published electronically September 7, 2016.
http://www.siam.org/journals/sicon/54-5/M103532.html
Funding: The first author was partially supported by the National Natural Science Foundation of China (11401556) and the Fundamental Research Funds for the Central Universities (WK 2040000012). The second author was partially supported by Hong Kong RGC under grants 519913, 15209614, and 15224215. The third author was partially supported by NSF DMS-1406776.
†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China (sjr@mail.ustc.edu.cn, malixun@polyu.edu.hk).
‡Department of Mathematics, University of Central Florida, Orlando, FL 32816 (jiongmin.yong@ucf.edu).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Any $u(\cdot) \in \mathcal{U}[t,T]$ is called an *admissible control* (on $[t,T]$). Under some mild conditions on the coefficients, for any *initial pair* $(t,x) \in [0,T) \times \mathbb{R}^n$ and admissible control $u(\cdot) \in \mathcal{U}[t,T]$, (1.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot\,; t,x,u(\cdot))$. Next we introduce the following cost functional:

$$
\begin{aligned}
J(t,x;u(\cdot)) \triangleq \mathbb{E}\Big\{ &\langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\
&+ \int_t^T \bigg[\left\langle \begin{pmatrix} Q(s) & S(s)^\top \\ S(s) & R(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle \\
&+ 2 \left\langle \begin{pmatrix} q(s) \\ \rho(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle \bigg] ds \Big\},
\end{aligned} \tag{1.2}
$$

where G is a symmetric matrix, and $Q(\cdot)$, $S(\cdot)$, $R(\cdot)$ are deterministic matrix-valued functions of proper dimensions with $Q(\cdot)^\top = Q(\cdot)$, $R(\cdot)^\top = R(\cdot)$; g is allowed to be an $\mathcal{F}_T$-measurable random variable, and $q(\cdot), \rho(\cdot)$ are allowed to be vector-valued $\mathbb{F}$-progressively measurable processes. The classical stochastic linear quadratic (LQ) optimal control problem can be stated as follows.

Problem (SLQ). For any given initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, find a $u^*(\cdot) \in \mathcal{U}[t,T]$, such that

$$
J(t,x;u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) \triangleq V(t,x). \tag{1.3}
$$

It is well accepted that any $u^*(\cdot) \in \mathcal{U}[t,T]$ satisfying (1.3) is called an *optimal control* of Problem (SLQ) for the initial pair (t,x), and the corresponding $X^*(\cdot) \equiv X(\cdot\,; t,x,u^*(\cdot))$ is called an *optimal state process*; the pair $(X^*(\cdot), u^*(\cdot))$ is called an *optimal pair*. The function $V(\cdot\,,\cdot)$ is called the *value function* of Problem (SLQ). When $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$, we denote the corresponding Problem (SLQ) by Problem $(\text{SLQ})^0$. The corresponding cost functional and value function are denoted by $J^0(t,x;u(\cdot))$ and $V^0(t,x)$, respectively.

When the stochastic part is absent, with $b(\cdot)$, g, $q(\cdot)$, and $\rho(\cdot)$ being deterministic, Problem (SLQ) is reduced to a deterministic LQ optimal control problem, called Problem (DLQ). Hence, Problem (DLQ) can be regarded as a special case of Problem (SLQ). The history of Problem (DLQ) can be traced back to the work of Bellman, Glicksberg, and Gross [4] in 1958, Kalman [15] in 1960, and Letov [18] in 1961. In the deterministic case, it is well known that $R(s) \geqslant 0$ is necessary for Problem (DLQ) to be finite (meaning that the infimum of the cost functional is finite). When the control weighting matrix $R(s)$ in the cost function is uniformly positive definite, under some mild additional conditions on the other weighting coefficients, the problem can be solved elegantly via the Riccati equation; see [3] for a thorough study of the Riccati equation approach (see also [30]). Stochastic LQ problems were first studied by Wonham [26] in 1968 and were later studied by several researchers (see [11, 5] for examples). In those works, the assumption that $R(s) > 0$ was taken for granted. More precisely, under the *standard* conditions

$$
G \geqslant 0, \quad R(s) \geqslant \delta I, \quad Q(s) - S(s)^\top R(s)^{-1} S(s) \geqslant 0 \quad \text{a.e. } s \in [0,T], \tag{1.4}
$$

for some $\delta > 0$, the corresponding Riccati equation is uniquely solvable and Problem (SLQ) admits a unique optimal control which has a linear state feedback representation (see [30, Chapter 6] or [9]). In 1998, Chen, Li, and Zhou [6] found that Problem (SLQ) might still be solvable even if $R(s)$ is not positive semidefinite. See also some

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

follow-up works of Lim and Zhou [19], Chen and Zhou [9], and Chen and Yong [8], as well as the works of Hu and Zhou [13] and Qian and Zhou [22] on the study of solvability of indefinite Riccati equations (under certain technical conditions). In 2001, Ait Rami, Moore, and Zhou [1] introduced a generalized Riccati equation involving the pseudoinverse of a matrix and an additional algebraic constraint; see also Ait Rami, Zhou, and Moore [2] for stochastic LQ optimal control problems on $[0,\infty)$, and a follow-up work of Wu and Zhou [27]. Recently, based on the work of Yong [28], Huang, Li, and Yong [14] studied a mean-field LQ optimal control problem on $[0,\infty)$. For stochastic LQ optimal control problems with random coefficients, we further refer the reader to the works of Chen and Yong [7], Tang [24, 25], Du [12], and Kohlmann and Tang [17].

Most recently, Sun and Yong [23] established that the existence of open-loop optimal controls is equivalent to the solvability of the corresponding optimality system which is a forward-backward SDE (FBSDE), and the existence of closed-loop optimal strategies is equivalent to the existence of a regular solution to the corresponding Riccati equation. From this point of view, this paper can be regarded as a continuation of [23], in a certain sense. Inspired by a result found in [29], we are able to cook up an example for which open-loop optimal controls exist but the closed-loop optimal strategy does not exist. Because of this, it is necessary to distinguish open-loop and closed-loop solvabilities of Problem (SLQ). Next, having the equivalence between the solvability of the Riccati equation and the closed-loop solvability of Problem (SLQ), it is natural to seek conditions under which the Riccati equation is solvable, and the sought conditions are expected to be more general than (1.4) so that they could include some (although perhaps not all) cases in which $R(\cdot)$ is allowed to be not positive semidefinite. One of our main results in this paper is to establish the equivalence between the *strongly regular solvability* of the Riccati equation (see below for a definition) and the uniform convexity of the cost functional. Note that the uniform convexity condition is much weaker than (1.4) and is different from conditions imposed in [22].

The finiteness of Problem (SLQ) (meaning that the infimum of the cost functional is finite) is another important issue. The notion was introduced in [30] (see also [7, 8]). Some investigations were carried out in [20]. In this paper, due to the perfect structure of Problem $(\text{SLQ})^0$, its finiteness will be characterized by the convergence of the solutions to a family of Riccati equations. As a byproduct, we will construct minimizing sequences of Problem (SLQ) in a very natural way, and the convergence of the sequences will lead to the open-loop solvability of Problem (SLQ).

Among other things, we find several interesting facts which are listed below:

Fact 1. The value function $V(t,x)$ is not necessarily continuous in t even if Problem (SLQ) admits a *continuous* open-loop optimal control at all initial pairs $(t,x)\in[0,T)\times\mathbb{R}^n$.

Fact 2. If Problem $(\text{SLQ})^0$ is finite at t, then it is finite at any $t'>t$.

Fact 3. For Problem (SLQ) with $D(\cdot)=0$, under the assumption that $R(\cdot)$ is uniformly positive definite, without assuming the nonnegativity of $Q(\cdot)$ and G, the finiteness and the unique closed-loop solvability of Problem (SLQ) are equivalent, and they are also equivalent to the uniform convexity of the cost functional.

Fact 4. The existence of a regular solution to the Riccati equation implies the open-loop solvability of Problem (SLQ). However, it may happen that for any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, Problem (SLQ) admits a *continuous* open-loop optimal control, while the Riccati equation does not admit a regular solution. This corrects

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

an incorrect result found in [1] (see section 4 for details).

The rest of the paper is organized as follows. Section 2 collects some preliminary results. In section 3, we study the cost functional from a Hilbert space viewpoint and represent it as a quadratic functional of $u(\cdot)$. Section 4 is devoted to the strongly regular solvability of the Riccati equation under the uniform convexity of the cost functional. In section 5, we discuss the finiteness of Problem (SLQ) as well as the convexity of the cost functional. In section 6, we characterize the open-loop solvability of Problem (SLQ) by means of the convergence of minimizing sequences. Some examples are presented in section 7 to illustrate some relevant results obtained.

2. Preliminaries. We recall that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the space of all $(n\times m)$ matrices, endowed with the inner product $\langle M,N\rangle \mapsto \operatorname{tr}[M^\top N]$ and the norm $|M| = \sqrt{\operatorname{tr}[M^\top M]}$, $\mathbb{S}^n \subseteq \mathbb{R}^{n\times n}$ is the set of all $(n\times n)$ symmetric matrices, and $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ is the set of all $(n\times n)$ positive-definite matrices. When there is no confusion, we shall use $\langle\cdot\,,\cdot\rangle$ for inner products in possibly different Hilbert spaces. Also, $M^\dagger$ stands for the (Moore–Penrose) pseudoinverse of the matrix M [21], and $\mathcal{R}(M)$ stands for the range of the matrix M. Let $T>0$ be a fixed time horizon. For any $t\in[0,T]$ and Euclidean space $\mathbb{H}$, we let $L^p(t,T;\mathbb{H})$ $(1\leqslant p\leqslant\infty)$ be the space of all $\mathbb{H}$-valued functions that are L^p-integrable on $[t,T]$ and $C([t,T];\mathbb{H})$ be the space of all $\mathbb{H}$-valued continuous functions on $[t,T]$. Next, we introduce the following spaces:

$$
\begin{aligned}
L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}) &= \Big\{\xi:\Omega\to\mathbb{H}\ \big|\ \xi \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}|\xi|^2<\infty\Big\},\\
L^2_{\mathbb{F}}(t,T;\mathbb{H}) &= \bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad\qquad \mathbb{E}\int_t^T|\varphi(s)|^2ds<\infty\bigg\},\\
L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H})) &= \bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous,}\\
&\qquad\qquad \mathbb{E}\Big(\sup_{t\leqslant s\leqslant T}|\varphi(s)|^2\Big)<\infty\bigg\},\\
L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H})) &= \bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad\qquad \mathbb{E}\Big(\int_t^T|\varphi(s)|ds\Big)^2<\infty\bigg\}.
\end{aligned}
$$

For an $\mathbb{S}^n$-valued function $F(\cdot)$ on $[t,T]$, we use the notation $F(\cdot)\gg 0$ to indicate that $F(\cdot)$ is uniformly positive definite on $[t,T]$, i.e., there exists a constant $\delta>0$ such that

$$F(s)\geqslant\delta I\quad \text{a.e. } s\in[t,T].$$

The following standard assumptions will be in force throughout this paper.

(H1) The coefficients of the state equation satisfy the following:

$$
\begin{cases}
A(\cdot)\in L^1(0,T;\mathbb{R}^{n\times n}), & B(\cdot)\in L^2(0,T;\mathbb{R}^{n\times m}), \quad b(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\\
C(\cdot)\in L^2(0,T;\mathbb{R}^{n\times n}), & D(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}), \quad \sigma(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).
\end{cases}
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

(H2) The weighting coefficients in the cost functional satisfy the following:

$$\begin{cases} Q(\cdot)\in L^1(0,T;\mathbb{S}^n), & S(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n}), \quad R(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\ q(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), & \rho(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \qquad G\in\mathbb{S}^n, \quad g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \end{cases}$$

By [23, Proposition 2.1], under (H1)–(H2), for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, the state equation (1.1) admits a unique strong solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$, and the cost functional (1.2) is well defined. Then Problem (SLQ) makes sense. It is worth pointing out that in (H2), we do not impose any positive-definiteness/nonnegativeness conditions on the weighting matrix/matrix-valued functions G, $Q(\cdot)$ and $R(\cdot)$. We now introduce the following definition.

DEFINITION 2.1. (i) *Problem* (SLQ) *is said to be finite at initial pair* $(t,x)\in[0,T]\times\mathbb{R}^n$ *if*

$$V(t,x)>-\infty. \tag{2.1}$$

Problem (SLQ) *is said to be finite at* $t\in[0,T]$ *if* (2.1) *holds for all* $x\in\mathbb{R}^n$*, and Problem* (SLQ) *is said to be finite if* (2.1) *holds for all* $(t,x)\in[0,T]\times\mathbb{R}^n$.

(ii) *An element* $u^*(\cdot)\in\mathcal{U}[t,T]$ *is called an open-loop optimal control of Problem* (SLQ) *for the initial pair* $(t,x)\in[0,T]\times\mathbb{R}^n$ *if*

$$J(t,x;u^*(\cdot))\leqslant J(t,x;u(\cdot)) \quad \forall u(\cdot)\in\mathcal{U}[t,T]. \tag{2.2}$$

If an open-loop optimal control (uniquely) exists for $(t,x)\in[0,T]\times\mathbb{R}^n$*, Problem* (SLQ) *is said to be (uniquely) open-loop solvable at* $(t,x)\in[0,T]\times\mathbb{R}^n$*. Problem* (SLQ) *is said to be (uniquely) open-loop solvable at* $t\in[0,T)$ *if for the given* t*,* (2.2) *holds for all* $x\in\mathbb{R}^n$*, and Problem* (SLQ) *is said to be (uniquely) open-loop solvable (on* $[0,T)\times\mathbb{R}^n$*) if it is (uniquely) open-loop solvable at all* $(t,x)\in[0,T)\times\mathbb{R}^n$.

(iii) *A pair* $(\Theta^*(\cdot),v^*(\cdot))\in L^2(t,T;\mathbb{R}^{m\times n})\times\mathcal{U}[t,T]$ *is called a closed-loop optimal strategy of Problem* (SLQ) *on* $[t,T]$ *if*

$$J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))\leqslant J(t,x;u(\cdot)) \quad \forall x\in\mathbb{R}^n,\ \forall u(\cdot)\in\mathcal{U}[t,T], \tag{2.3}$$

where $X^*(\cdot)$ *is the strong solution to the following closed-loop system:*

$$\begin{cases} dX^*(s)=\Big\{\big[A(s)+B(s)\Theta^*(s)\big]X^*(s)+B(s)v^*(s)+b(s)\Big\}ds\\ \qquad\quad +\Big\{\big[C(s)+D(s)\Theta^*(s)\big]X^*(s)+D(s)v^*(s)+\sigma(s)\Big\}dW(s),\\ X^*(t)=x. \end{cases} \tag{2.4}$$

If a closed-loop optimal strategy (uniquely) exists on $[t,T]$*, Problem* (SLQ) *is said to be (uniquely) closed-loop solvable on* $[t,T]$*. Problem* (SLQ) *is said to be (uniquely) closed-loop solvable if it is (uniquely) closed-loop solvable on any* $[t,T]$.

We emphasize that for given initial time $t\in[0,T)$, an open-loop optimal control is allowed to depend on the initial state x, whereas a closed-loop optimal strategy is required to be independent of the initial state x. One sees that if $(\Theta^*(\cdot),v^*(\cdot))$ is a closed-loop optimal strategy of Problem (SLQ) on $[t,T]$, then the outcome $u^*(\cdot)\equiv\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for the initial pair $(t,X^*(t))$. Hence, the existence of closed-loop optimal strategies implies the existence of open-loop optimal controls. But the existence of open-loop optimal controls does

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

not necessarily imply the existence of a closed-loop optimal strategy (see Example 7.1). Due to the above-indicated situation, unlike in [23] and in classical literature on LQ problems, we distinguish the notions of open-loop and closed-loop solvabilities for Problem (SLQ). Because of the nature of closed-loop strategies, we define the finiteness of Problem (SLQ) only in terms of open-loop controls.

To conclude this section, we present some lemmas which will be used frequently in what follows.

LEMMA 2.2. *Let* (H1)–(H2) *hold, and let* $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$. *Let* $P(\cdot) \in C([0,T];\mathbb{S}^n)$ *be the solution to the following Lyapunov equation:*

$$\begin{cases} \dot{P} + P(A+B\Theta) + (A+B\Theta)^\top P + (C+D\Theta)^\top P(C+D\Theta) \\ \quad + \Theta^\top R\Theta + S^\top\Theta + \Theta^\top S + Q = 0 \quad \text{a.e. } s\in[0,T], \\ P(T) = G. \end{cases} \tag{2.5}$$

Then for any $(t,x)\in[0,T)\times\mathbb{R}^n$ *and* $u(\cdot)\in\mathcal{U}[t,T]$, *we have*

$$\begin{aligned} J^0(t,x;\Theta(\cdot)X(\cdot)+u(\cdot)) = \langle P(t)x,x\rangle + \mathbb{E}\int_t^T \Big\{ &\big\langle (R+D^\top PD)u,u\big\rangle \\ &+ 2\big\langle \big[B^\top P + D^\top PC + S + (R+D^\top PD)\Theta\big]X,u\big\rangle\Big\} ds. \end{aligned}$$

Proof. For any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)$ be the solution of

$$\begin{cases} dX(s) = \big[(A+B\Theta)X + Bu\big]ds + \big[(C+D\Theta)X + Du\big]dW(s), \quad s\in[t,T], \\ X(t) = x. \end{cases}$$

Applying Itô's formula to $s\mapsto\langle P(s)X(s),X(s)\rangle$, we have

$$\begin{aligned} &J^0(t,x;\Theta(\cdot)X(\cdot)+u(\cdot)) \\ &= \mathbb{E}\left\{\langle GX(T),X(T)\rangle + \int_t^T \left\langle \begin{pmatrix} Q & S^\top \\ S & R\end{pmatrix}\begin{pmatrix} X \\ \Theta X+u\end{pmatrix}, \begin{pmatrix} X \\ \Theta X + u\end{pmatrix}\right\rangle ds\right\} \\ &= \langle P(t)x,x\rangle + \mathbb{E}\int_t^T \Big\{\big\langle\big[\dot{P} + P(A+B\Theta) + (A+B\Theta)^\top P + (C+D\Theta)^\top P(C+D\Theta) \\ &\qquad + Q + \Theta^\top S + S^\top\Theta + \Theta^\top R\Theta\big]X,X\big\rangle + \big\langle (R+D^\top PD)u,u\big\rangle \\ &\qquad + 2\big\langle\big[B^\top P + D^\top PC + S + (R+D^\top PD)\Theta\big]X,u\big\rangle\Big\}ds \\ &= \langle P(t)x,x\rangle + \mathbb{E}\int_t^T \Big\{2\big\langle\big[B^\top P + D^\top PC + S + (R+D^\top PD)\Theta\big]X,u\big\rangle \\ &\qquad + \big\langle (R+D^\top PD)u,u\big\rangle\Big\}ds. \end{aligned}$$

This completes the proof. ∎

LEMMA 2.3. *For any* $u(\cdot)\in\mathcal{U}[t,T]$, *let* $X^{(u)}(\cdot)$ *be the solution of*

$$\begin{cases} dX^{(u)}(s) = \big[A(s)X^{(u)}(s) + B(s)u(s)\big]ds \\ \qquad\qquad + \big[C(s)X^{(u)}(s) + D(s)u(s)\big]dW(s), \quad s\in[t,T], \\ X^{(u)}(t) = 0. \end{cases} \tag{2.6}$$

Then for any $\Theta(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$, *there exists a constant* $\gamma>0$ *such that*

$$\mathbb{E}\int_t^T \big|u(s) - \Theta(s)X^{(u)}(s)\big|^2 ds \geqslant \gamma\mathbb{E}\int_t^T |u(s)|^2 ds \quad \forall u(\cdot)\in\mathcal{U}[t,T]. \tag{2.7}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Proof. Let $\Theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$. Define a bounded linear operator $\mathfrak{L} : \mathcal{U}[t, T] \to \mathcal{U}[t, T]$ by

$$\mathfrak{L}u = u - \Theta X^{(u)}.$$

Then $\mathfrak{L}$ is bijective, and its inverse $\mathfrak{L}^{-1}$ is given by

$$\mathfrak{L}^{-1}u = u + \Theta \widetilde{X}^{(u)},$$

where $\widetilde{X}^{(u)}(\cdot)$ is the solution of

$$\begin{cases} d\widetilde{X}^{(u)}(s) = \Big\{\big[A(s) + B(s)\Theta(s)\big]\widetilde{X}^{(u)}(s) + B(s)u(s)\Big\}ds \\ \qquad\qquad\quad + \Big\{\big[C(s) + D(s)\Theta(s)\big]\widetilde{X}^{(u)}(s) + D(s)u(s)\Big\}dW(s), \quad s \in [t, T], \\ \widetilde{X}^{(u)}(t) = 0. \end{cases}$$

By the bounded inverse theorem, $\mathfrak{L}^{-1}$ is bounded with norm $\|\mathfrak{L}^{-1}\| > 0$. Thus,

$$\begin{aligned} \mathbb{E}\int_t^T |u(s)|^2 ds &= \mathbb{E}\int_t^T |(\mathfrak{L}^{-1}\mathfrak{L}u)(s)|^2 ds \leqslant \|\mathfrak{L}^{-1}\|\mathbb{E}\int_t^T |(\mathfrak{L}u)(s)|^2 ds \\ &= \|\mathfrak{L}^{-1}\|\mathbb{E}\int_t^T \big|u(s) - \Theta(s)X^{(u)}(s)\big|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[t, T], \end{aligned}$$

which implies (2.7) with $\gamma = \|\mathfrak{L}^{-1}\|^{-1}$. ∎

3. Representation of the cost functional.

In this section, we will present a representation of the cost functional for Problem (SLQ), from which we will obtain some basic conditions ensuring the convexity of the cost functional. Convexity of the cost functional will play a crucial role in the study of finiteness and open-loop/closed-loop solvability of Problem (SLQ). The following proposition is a summary of some relevant results found in [30].

PROPOSITION 3.1. *Let* (H1)–(H2) *hold. For any* $t \in [0, T)$, *there exist a bounded self-adjoint linear operator* $M_2(t) : \mathcal{U}[t, T] \to \mathcal{U}[t, T]$, *a bounded linear operator* $M_1(t) : \mathbb{R}^n \to \mathcal{U}[t, T]$, *an* $M_0(t) \in \mathbb{S}^n$, *and* $\nu_t(\cdot) \in \mathcal{U}[t, T], y_t \in \mathbb{R}^n, c_t \in \mathbb{R}$ *such that*

$$\begin{aligned} J(t, x; u(\cdot)) &= \langle M_2(t)u, u\rangle + 2\langle M_1(t)x, u\rangle + \langle M_0(t)x, x\rangle \\ &\quad + 2\langle u, \nu_t\rangle + 2\langle x, y_t\rangle + c_t, \\ J^0(t, x; u(\cdot)) &= \langle M_2(t)u, u\rangle + 2\langle M_1(t)x, u\rangle + \langle M_0(t)x, x\rangle \\ &\quad \forall\, (x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[t, T]. \end{aligned} \tag{3.1}$$

Moreover, let $(X_0(\cdot), Y(\cdot), Z(\cdot))$ *be the adapted solution of the following (decoupled) linear FBSDE:*

$$\begin{cases} dX_0(s) = \big[A(s)X_0(s) + B(s)u(s)\big]ds + \big[C(s)X_0(s) + D(s)u(s)\big]dW(s), \\ dY(s) = -\big[A(s)^\top Y(s) + C(s)^\top Z(s) + Q(s)X_0(s) + S(s)^\top u(s)\big]ds + Z(s)dW(s), \\ X_0(t) = 0, \qquad Y(T) = GX_0(T). \end{cases}$$

Then

$$(M_2(t)u(\cdot))(s) = B(s)^\top Y(s) + D(s)^\top Z(s) + S(s)X_0(s) + R(s)u(s), \quad s \in [t, T].$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Let $(\bar{X}_0(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the adapted solution to the following (decoupled) FBSDE:

$$\begin{cases} d\bar{X}_0(s) = A(s)\bar{X}_0(s)ds + C(s)\bar{X}_0(s)dW(s), \\ d\bar{Y}(s) = -\big[A(s)^\top \bar{Y}(s) + C(s)^\top \bar{Z}(s) + Q(s)\bar{X}_0(s)\big]ds + \bar{Z}(s)dW(s), \\ \bar{X}_0(t) = x, \qquad \bar{Y}(T) = G\bar{X}_0(T). \end{cases}$$

Then

$$\begin{cases} (M_1(t)x)(s) = B(s)^\top \bar{Y}(s) + D(s)^\top \bar{Z}(s) + S(s)\bar{X}_0(s), \quad s \in [t, T], \\ M_0(t)x = \mathbb{E}\big[\bar{Y}(t)\big]. \end{cases}$$

Also, let $(\widehat{X}_0(\cdot), \widehat{Y}(\cdot), \widehat{Z}(\cdot))$ be the adapted solution to the following (decoupled) FBSDE:

$$\begin{cases} d\widehat{X}_0(s) = \big[A(s)\widehat{X}_0(s) + b(s)\big]ds + \big[C(s)\widehat{X}_0(s) + \sigma(s)\big]dW(s), \\ d\widehat{Y}(s) = -\big[A(s)^\top \widehat{Y}(s) + C(s)^\top \widehat{Z}(s) + Q(s)\widehat{X}_0(s) + q(s)\big]ds + \widehat{Z}(s)dW(s), \\ \widehat{X}_0(t) = 0, \qquad \widehat{Y}(T) = G\widehat{X}_0(T) + g. \end{cases}$$

Then

$$\nu_t(s) = B(s)^\top \widehat{Y}(s) + D(s)^\top \widehat{Z}(s) + S(s)\widehat{X}_0(s) + \rho(s), \quad s \in [t, T].$$

Finally, $M_0(\cdot)$ solves the following Lyapunov equation on $[0, T]$:

$$\begin{cases} \dot{M}_0(t) + M_0(t)A(t) + A(t)^\top M_0(t) + C(t)^\top M_0(t)C(t) + Q(t) = 0, \\ M_0(T) = G, \end{cases} \tag{3.2}$$

and it admits the following representation:

$$M_0(t) = \mathbb{E}\bigg\{\big[\Phi(T)\Phi(t)^{-1}\big]^\top G\big[\Phi(T)\Phi(t)^{-1}\big] + \int_t^T \big[\Phi(s)\Phi(t)^{-1}\big]^\top Q(s)\big[\Phi(s)\Phi(t)^{-1}\big]ds\bigg\},$$

where $\Phi(\cdot)$ is the solution to the following SDE for an $\mathbb{R}^{n\times n}$-valued process:

$$\begin{cases} d\Phi(s) = A(s)\Phi(s)ds + C(s)\Phi(s)dW(s), \quad s \geqslant 0, \\ \Phi(0) = I. \end{cases}$$

Remark 3.2. The operator $M_2(t)$ also admits the following representation (see [30, Chapter 6]):

$$M_2(t) = \widehat{L}_t^* G \widehat{L}_t + L_t^* Q L_t + S L_t + L_t^* S^\top + R, \tag{3.3}$$

where the operators

$$L_t : \mathcal{U}[t, T] \to L^2_{\mathbb{F}}(t, T; \mathbb{R}^n), \qquad \widehat{L}_t : \mathcal{U}[t, T] \to L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$$

are defined as follows:

$$(L_t u)(\cdot) = \Phi(\cdot)\left\{\int_t^\cdot \Phi(r)^{-1}\big[B(r) - C(r)D(r)\big]u(r)dr + \int_t^\cdot \Phi(r)^{-1}D(r)u(r)dW(r)\right\},$$
$$\widehat{L}_t u = (L_t u)(T),$$

and L_t^* and $\widehat{L}_t^*$ are the adjoint operators of L_t and $\widehat{L}_t$, respectively.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

From the representation of the cost functional, we have the following simple corollary.

Corollary 3.3. *Let* (H1)–(H2) *hold and* $t \in [0,T)$ *be given. For any* $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, *and* $u(\cdot), v(\cdot) \in \mathcal{U}[t,T]$, *the following holds:*

$$
\begin{aligned}
&J(t,x;u(\cdot)+\lambda v(\cdot)) = J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) \\
&\quad + 2\lambda\mathbb{E}\int_t^T \big\langle B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s), v(s)\big\rangle ds,
\end{aligned} \tag{3.4}
$$

where $(X(\cdot),Y(\cdot),Z(\cdot))$ *is the adapted solution to the following (decoupled) linear FBSDE:*

$$
\begin{cases}
dX(s) = \big[A(s)X(s)+B(s)u(s)+b(s)\big]ds \\
\qquad\qquad + \big[C(s)X(s)+D(s)u(s)+\sigma(s)\big]dW(s), \qquad s\in[t,T], \\
dY(s) = -\big[A(s)^\top Y(s)+C(s)^\top Z(s)+Q(s)X(s)+S(s)^\top u(s)+q(s)\big]ds \\
\qquad\qquad\qquad\qquad + Z(s)dW(s), \qquad s\in[t,T], \\
X(t)=x, \qquad Y(T)=GX(T)+g.
\end{cases} \tag{3.5}
$$

Consequently, the map $u(\cdot)\mapsto J(t,x;u(\cdot))$ *is Fréchet differentiable with the Fréchet derivative given by*

$$
\mathcal{D}J(t,x;u(\cdot))(s) = 2\big[B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s)\big],
$$

and (3.4) *can also be written as*

$$
J(t,x;u(\cdot)+\lambda v(\cdot)) = J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) + \lambda\mathbb{E}\int_t^T \langle \mathcal{D}J(t,x;u(\cdot))(s), v(s)\rangle ds.
$$

Proof. From Proposition 3.1, we have

$$
\begin{aligned}
J(t,x;u(\cdot)+\lambda v(\cdot)) &= \langle M_2(t)(u+\lambda v), u+\lambda v\rangle + 2\langle M_1(t)x, u+\lambda v\rangle \\
&\quad + \langle M_0(t)x,x\rangle + 2\langle u+\lambda v,\nu_t\rangle + 2\langle x,y_t\rangle + c_t \\
&= J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) + 2\lambda\langle M_2(t)u+M_1(t)x+\nu_t, v\rangle.
\end{aligned}
$$

From the representation of $M_1(t)$, $M_2(t)$, and ν_t in Proposition 3.1, we see that

$$
\begin{aligned}
&(M_2(t)u)(s) + (M_1(t)x)(s) + \nu_t(s) \\
&= B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s), \quad s\in[t,T],
\end{aligned}
$$

with $(X(\cdot),Y(\cdot),Z(\cdot))$ being the adapted solution to the FBSDE (3.5). The rest of the proof is clear. ∎

Note that if $u(\cdot)$ happens to be an open-loop optimal control of Problem (SLQ), then the following *stationarity condition* holds:

$$
\mathcal{D}J(t,x;u(\cdot)) = 2\big[B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s)\big] = 0, \tag{3.6}
$$

which brings a coupling into the FBSDE (3.5). We call (3.5), together with the stationarity condition (3.6), the *optimality system* for the open-loop optimal control of Problem (SLQ).

The following concerns the convexity of the cost functional, whose proof is straightforward by making use of the representation (3.1) of the cost functional.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

COROLLARY 3.4. *Let* (H1)–(H2) *hold, and let* $t \in [0, T)$ *be given. Then the following are equivalent:*

(i) $u(\cdot) \mapsto J(t, x; u(\cdot))$ *is convex for some* $x \in \mathbb{R}^n$.
(ii) $u(\cdot) \mapsto J(t, x; u(\cdot))$ *is convex for any* $x \in \mathbb{R}^n$.
(iii) $u(\cdot) \mapsto J^0(t, x; u(\cdot))$ *is convex for some* $x \in \mathbb{R}^n$.
(iv) $u(\cdot) \mapsto J^0(t, x; u(\cdot))$ *is convex for any* $x \in \mathbb{R}^n$.
(v) $J^0(t, 0; u(\cdot)) \geqslant 0$ *for all* $u(\cdot) \in \mathcal{U}[t, T]$.
(vi) $M_2(t) \geqslant 0$.

Similar to the above, we have that $u(\cdot) \mapsto J(t, x; u(\cdot))$ is uniformly convex if and only if

$$J^0(t, 0; u(\cdot)) \geqslant \lambda \mathbb{E} \int_t^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[t, T], \tag{3.7}$$

for some $\lambda > 0$. This is also equivalent to the following:

$$M_2(t) \geqslant \lambda I \tag{3.8}$$

for some $\lambda > 0$. Further, it is obvious that if the standard conditions (1.4) hold, then

$$M_2(t) = \widehat{L}_t^* G \widehat{L}_t + L_t^* \big(Q - S^\top R^{-1} S\big) L_t + \big(L_t^* S^\top R^{-\frac{1}{2}} + R^{\frac{1}{2}}\big)\big(R^{-\frac{1}{2}} S L_t + R^{\frac{1}{2}}\big) \geqslant 0,$$

which means that the functional $u(\cdot) \mapsto J^0(t, 0, u(\cdot))$ is convex. The following result tells us that under (1.4), one actually has the uniform convexity of the cost functional.

PROPOSITION 3.5. *Let* (H1)–(H2) *and* (1.4) *hold. Then for any* $t \in [0, T)$*, the map* $u(\cdot) \mapsto J^0(t, 0; u(\cdot))$ *is uniformly convex.*

Proof. For any $u(\cdot) \in \mathcal{U}[t, T]$, let $X^{(u)}(\cdot)$ be the solution of

$$\begin{cases} dX^{(u)}(s) = \big[A(s)X^{(u)}(s) + B(s)u(s)\big]ds + \big[C(s)X^{(u)}(s) + D(s)u(s)\big]dW(s), \\ X^{(u)}(t) = 0. \end{cases}$$

Then by Lemma 2.3 (taking $\Theta(\cdot) = -R(\cdot)^{-1}S(\cdot)$), we have

$$\begin{aligned} & J^0(t, 0; u(\cdot)) \\ & = \mathbb{E}\bigg\{\big\langle GX^{(u)}(T), X^{(u)}(T)\big\rangle + \int_t^T \Big[\big\langle QX^{(u)}, X^{(u)}\big\rangle + 2\big\langle SX^{(u)}, u\big\rangle + \langle Ru, u\rangle\Big] ds\bigg\} \\ & \geqslant \mathbb{E}\int_t^T \Big[\big\langle QX^{(u)}, X^{(u)}\big\rangle + 2\big\langle SX^{(u)}, u\big\rangle + \langle Ru, u\rangle\Big] ds \\ & = \mathbb{E}\int_t^T \Big[\big\langle \big(Q - S^\top R^{-1} S\big) X^{(u)}, X^{(u)}\big\rangle + \big\langle R\big(u + R^{-1} S X^{(u)}\big), u + R^{-1} S X^{(u)}\big\rangle\Big] ds \\ & \geqslant \delta \mathbb{E}\int_t^T \big|u + R^{-1} S X^{(u)}\big|^2 ds \\ & \geqslant \delta\gamma \mathbb{E}\int_t^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[t, T], \end{aligned}$$

for some $\gamma > 0$. This completes the proof. ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

4. Solvabilities of Problem (SLQ), uniform convexity of the cost functional, and the Riccati equation. We begin with a simple result concerning the open-loop solvability of Problem (SLQ).

PROPOSITION 4.1. *Let* (H1)–(H2) *hold. Suppose the map* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is uniformly convex. Then Problem* (SLQ) *is uniquely open-loop solvable, and there exists a constant* $\alpha \in \mathbb{R}$ *such that*

$$V^0(t,x) \geqslant \alpha|x|^2 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{4.1}$$

Note that in the above, the constant α does not have to be nonnegative.

Proof. First, by the uniform convexity of $u(\cdot) \mapsto J^0(0,0;u(\cdot))$, we may assume that

$$J^0(0,0;u(\cdot)) \geqslant \lambda\,\mathbb{E}\int_0^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[0,T], \tag{4.2}$$

for some $\lambda > 0$. Now, for any $t \in [0,T)$ and any $u(\cdot) \in \mathcal{U}[t,T]$, we define the *zero-extension* of $u(\cdot)$ as follows:

$$[\,0I_{[0,t)} \oplus u(\cdot)](s) = \begin{cases} 0, & s \in [0,t), \\ u(s), & s \in [t,T]. \end{cases} \tag{4.3}$$

Then $v(\cdot) \equiv 0I_{[0,t)} \oplus u(\cdot) \in \mathcal{U}[0,T]$, and due to the initial state being 0, the solution $X(s)$ of

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)v(s)\big]ds + \big[C(s)X(s) + D(s)v(s)\big]dW(s), \quad s \in [0,T], \\ X(0) = 0 \end{cases}$$

satisfies $X(s) = 0,\ s \in [0,t]$. Hence,

$$J^0(t,0;u(\cdot)) = J^0(0,0;0I_{[0,t)} \oplus u(\cdot)) \geqslant \lambda\mathbb{E}\int_0^T \big|[0I_{[0,t)} \oplus u(\cdot)](s)\big|^2 ds = \lambda\mathbb{E}\int_t^T |u(s)|^2 ds.$$

Thus, $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is uniformly convex for any given $(t,x) \in [0,T) \times \mathbb{R}^n$. By Corollary 3.3, we have for any $u(\cdot) \in \mathcal{U}[t,T]$

$$\begin{aligned} &J(t,x;u(\cdot)) = J(t,x;0) + J^0(t,0;u(\cdot)) + \mathbb{E}\int_t^T \big\langle \mathcal{D}J(t,x;0)(s), u(s)\big\rangle ds \\ &\geqslant J(t,x;0) + J^0(t,0;u(\cdot)) - \frac{\lambda}{2}\mathbb{E}\int_t^T |u(s)|^2 ds - \frac{1}{2\lambda}\mathbb{E}\int_t^T |\mathcal{D}J(t,x;0)(s)|^2 ds \\ &\geqslant \frac{\lambda}{2}\mathbb{E}\int_t^T |u(s)|^2 ds + J(t,x;0) - \frac{1}{2\lambda}\mathbb{E}\int_t^T |\mathcal{D}J(t,x;0)(s)|^2 ds. \end{aligned} \tag{4.4}$$

Consequently, by a standard argument involving minimizing sequence and locally weak compactness of Hilbert spaces, we see that for any given initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, Problem (SLQ) admits a unique open-loop optimal control. Moreover, when $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$, (4.4) implies that

$$V^0(t,x) \geqslant J^0(t,x;0) - \frac{1}{2\lambda}\mathbb{E}\int_t^T |\mathcal{D}J^0(t,x;0)(s)|^2 ds. \tag{4.5}$$

Note that the functions on the right-hand side of (4.5) are quadratic in x and continuous in t. Inequality (4.1) follows immediately. ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Now, we introduce the following Riccati equation associated with Problem (SLQ):

$$
\begin{cases}
\dot{P}(s)+P(s)A(s)+A(s)^\top P(s)+C(s)^\top P(s)C(s)+Q(s)\\
\quad -\big[P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top\big]\big[R(s)+D(s)^\top P(s)D(s)\big]^\dagger\\
\qquad \cdot\big[B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s)\big]=0 \quad \text{a.e. } s\in[0,T],\\
P(T)=G.
\end{cases} \tag{4.6}
$$

A solution $P(\cdot)\in C([0,T];\mathbb{S}^n)$ of (4.6) is said to be *regular* if

$$
\mathcal{R}\big(B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s)\big)\subseteq\mathcal{R}\big(R(s)+D(s)^\top P(s)D(s)\big) \quad \text{a.e. } s\in[0,T], \tag{4.7}
$$

$$
\big(R+D^\top PD\big)^\dagger\big(B^\top P+D^\top PC+S\big)\in L^2(0,T;\mathbb{R}^{m\times n}), \tag{4.8}
$$

$$
R(s)+D(s)^\top P(s)D(s)\geqslant 0 \quad \text{a.e. } s\in[0,T]. \tag{4.9}
$$

A solution $P(\cdot)$ of (4.6) is said to be *strongly regular* if

$$
R(s)+D(s)^\top P(s)D(s)\geqslant \lambda I \quad \text{a.e. } s\in[0,T], \tag{4.10}
$$

for some $\lambda>0$. The Riccati equation (4.6) is said to be (*strongly*) *regularly solvable*, if it admits a (strongly) regular solution. Clearly, condition (4.10) implies (4.7)–(4.9). Thus, a strongly regular solution $P(\cdot)$ must be regular. Moreover, it was shown in [23] that if a regular solution of (4.6) exists, it must be unique.

In [1], it was shown that for Problem $(\text{SLQ})^0$, the existence of a continuous open-loop optimal control for any initial pair $(t,x)\in[0,T]\times\mathbb{R}^n$ is equivalent to the solvability of the corresponding Riccati equation (4.6) with constraints (4.7) and (4.9). More precisely, their result can be stated as follows (in terms of our notation and equation numbers).

THEOREM 4.2. *Suppose that $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are continuous and $S(\cdot)=0$. Then Problem* $(\text{SLQ})^0$ *has a continuous open-loop optimal control for any initial pair $(t,x)\in[0,T]\times\mathbb{R}^n$ if and only if the Riccati equation* (4.6) *has a solution $P(\cdot)$ such that* (4.7) *and* (4.9) *hold.*

This result is incorrect. We will present two counterexamples in section 7.

Instead of Theorem 4.2, in [23], the following were proved, which establishes the equivalence between the closed-loop solvability of Problem (SLQ) and the regular solvability of the Riccati equation (4.6).

THEOREM 4.3. *Let* (H1)–(H2) *hold. Then Problem* (SLQ) *is closed-loop solvable on $[0,T]$ if and only if the Riccati equation* (4.6) *admits a regular solution $P(\cdot)\in C([0,T];\mathbb{S}^n)$ and the adapted solution $(\eta(\cdot),\zeta(\cdot))$ of the backward stochastic differential equation (BSDE)*

$$
\begin{cases}
d\eta(s)=-\Big\{\big[A^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger B^\top\big]\eta\\
\qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]\zeta\\
\qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]P\sigma\\
\qquad\qquad -(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger\rho+Pb+q\Big\}ds\\
\qquad\qquad\qquad +\zeta dW(s), \quad s\in[0,T],\\
\eta(T)=g,
\end{cases} \tag{4.11}
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

satisfies

$$\begin{cases} B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \in \mathcal{R}(R + D^\top PD) \quad \text{a.e. a.s.} \\ (R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m). \end{cases} \tag{4.12}$$

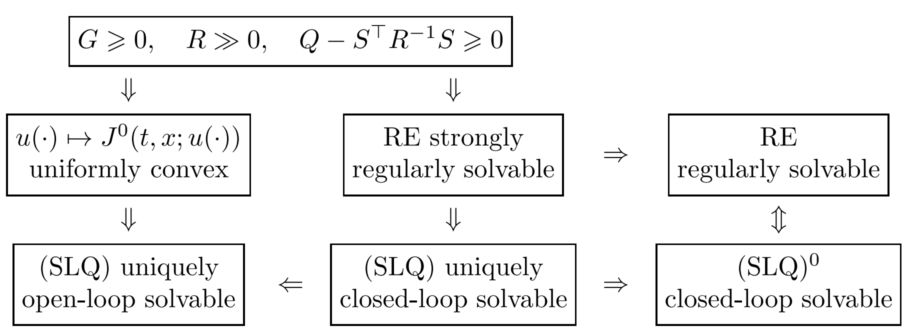
In this case, Problem (SLQ) *is closed-loop solvable on any* $[t, T]$*, and the closed-loop optimal strategy* $(\Theta^*(\cdot), v^*(\cdot))$ *admits the representation*

$$\begin{cases} \Theta^* = -(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) \\ \qquad\quad + \big[I - (R + D^\top PD)^\dagger(R + D^\top PD)\big]\Pi, \\ v^* \ = -(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \\ \qquad\quad + \big[I - (R + D^\top PD)^\dagger(R + D^\top PD)\big]\nu \end{cases} \tag{4.13}$$

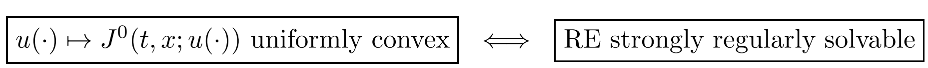
for some $\Pi(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$*, and the value function is given by*

$$\begin{aligned} V(t,x) = \mathbb{E}\bigg\{ &\langle P(t)x, x\rangle + 2\langle\eta(t), x\rangle + \int_t^T \Big[\langle P\sigma,\sigma\rangle + 2\langle\eta, b\rangle + 2\langle\zeta,\sigma\rangle \\ &- \big\langle (R{+}D^\top PD)^\dagger(B^\top\eta{+}D^\top\zeta{+}D^\top P\sigma{+}\rho), B^\top\eta{+}D^\top\zeta{+}D^\top P\sigma{+}\rho\big\rangle\Big] ds\bigg\}. \end{aligned} \tag{4.14}$$

From the above theorem, we see that the existence of a strongly regular solution to the Riccati equation (4.6) implies the unique closed-loop solvability of Problem (SLQ), which, by the remark right after Definition 2.1, implies the unique open-loop solvability of Problem (SLQ). Particularly, when $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$, the adapted solution of (4.11) is $(0, 0)$, and (4.12) holds automatically. Thus, the existence of a regular solution to the Riccati equation (4.6) is equivalent to the closed-loop solvability of Problem $(\text{SLQ})^0$, which implies the open-loop solvability of Problem $(\text{SLQ})^0$. However, as we mentioned earlier, the inverse is false (see section 7 for further details). On the other hand, it is known that under the standard conditions (1.4), the Riccati equation (4.6) admits a unique positive semidefinite solution $P(\cdot)$, and Problem (SLQ) admits a unique open-loop optimal control which has a state feedback form, represented via the solution of the Riccati equation (see [30, 9]). To summarize, we have the following diagram:



where "RE" stands for the Riccati equation (4.6). It is clear that the uniform convexity of the map $u(\cdot) \mapsto J^0(t, x; u(\cdot))$ does not imply the standard conditions (1.4), which will be even clearer by the results of section 5 below. Therefore, we desire to establish the following:



Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

This is our next goal. To achieve this, we first present the following proposition, which plays a key technical role in this section.

PROPOSITION 4.4. *Let* (H1)–(H2) *and* (4.2) *hold. Then for any* $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$*, the solution* $P(\cdot) \in C([0,T];\mathbb{S}^n)$ *to the Lyapunov equation*

$$\begin{cases} \dot{P} + P(A+B\Theta) + (A+B\Theta)^\top P + (C+D\Theta)^\top P(C+D\Theta) \\ \quad + \Theta^\top R\Theta + S^\top\Theta + \Theta^\top S + Q = 0 \quad \text{a.e. } s \in [0,T], \\ P(T) = G \end{cases} \tag{4.15}$$

satisfies

$$R(t) + D(t)^\top P(t)D(t) \geqslant \lambda I \text{ a.e. } t \in [0,T], \text{ and } P(t) \geqslant \alpha I \ \forall t \in [0,T], \tag{4.16}$$

where $\alpha \in \mathbb{R}$ *is the constant appearing in* (4.1).

Proof. Let $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$, and let $P(\cdot)$ be the solution to (4.15). For any $u(\cdot) \in \mathcal{U}[0,T]$, let $X_0(\cdot)$ be the solution of

$$\begin{cases} dX_0(s) = \big[(A+B\Theta)X_0 + Bu\big]ds + \big[(C+D\Theta)X_0 + Du\big]dW(s), \quad s \in [0,T], \\ X_0(0) = 0. \end{cases}$$

By (4.2) and Lemma 2.2, we have

$$\begin{aligned} &\lambda\mathbb{E}\int_0^T |\Theta(s)X_0(s) + u(s)|^2 ds \leqslant J^0(0,0;\Theta(\cdot)X_0(\cdot) + u(\cdot)) \\ &= \mathbb{E}\int_0^T \Big\{2\big\langle\big[B^\top P + D^\top PC + S + (R + D^\top PD)\Theta\big]X_0, u\big\rangle + \big\langle(R + D^\top PD)u, u\big\rangle\Big\}ds. \end{aligned}$$

Hence, for any $u(\cdot) \in \mathcal{U}[0,T]$, the following holds:

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\{2\big\langle\big[B^\top P + D^\top PC + S + (R + D^\top PD - \lambda I)\Theta\big]X_0, u\big\rangle \\ &\qquad + \big\langle(R + D^\top PD - \lambda I)u, u\big\rangle\Big\}ds = \lambda\mathbb{E}\int_0^T |\Theta(s)X_0(s)|^2 ds \geqslant 0. \end{aligned} \tag{4.17}$$

Now, fix any $u_0 \in \mathbb{R}^m$, and take $u(s) = u_0\mathbf{1}_{[t,t+h]}(s)$, with $0 \leqslant t < t+h \leqslant T$. Then

$$\begin{cases} d\big[\mathbb{E}X_0(s)\big] = \Big\{\big[A(s) + B(s)\Theta(s)\big]\mathbb{E}X_0(s) + B(s)u_0\mathbf{1}_{[t,t+h]}(s)\Big\}ds, \quad s \in [0,T], \\ \mathbb{E}X_0(0) = 0. \end{cases}$$

Hence,

$$\mathbb{E}X_0(s) = \begin{cases} 0, & s \in [0,t], \\ \Phi(s)\displaystyle\int_t^{s\wedge(t+h)} \Phi(r)^{-1}B(r)u_0 dr, & s \in [t,T], \end{cases}$$

where $\Phi(\cdot)$ is the solution of the following $\mathbb{R}^{n\times n}$-valued ordinary differential equation:

$$\begin{cases} \dot{\Phi}(s) = \big[A(s) + B(s)\Theta(s)\big]\Phi(s), \quad s \in [0,T], \\ \Phi(0) = I. \end{cases}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Consequently, (4.17) becomes

$$\int_t^{t+h}\Big\{2\Big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD-\lambda I)\Theta\big]\Phi(s)\int_t^s\Phi(r)^{-1}B(r)u_0dr,u_0\Big\rangle \\ +\big\langle(R+D^\top PD-\lambda I)u_0,u_0\big\rangle\Big\}ds\geqslant 0.$$

Dividing both sides of the above by h and letting $h\to 0$, we obtain

$$\big\langle\big[R(t)+D(t)^\top P(t)D(t)-\lambda I\big]u_0,u_0\big\rangle\geqslant 0\quad \text{a.e. } t\in[0,T],\ \forall u_0\in\mathbb{R}^m.$$

The first inequality in (4.16) follows. To prove the second, for any $(t,x)\in[0,T]\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)$ be the solution of

$$\begin{cases} dX(s)=\big[(A+B\Theta)X+Bu\big]ds+\big[(C+D\Theta)X+Du\big]dW(s), & s\in[t,T],\\ X(t)=x. \end{cases}$$

By Proposition 4.1 and Lemma 2.2, we have

$$\begin{aligned}\alpha|x|^2&\leqslant V^0(t,x)\leqslant J^0(t,x;\Theta(\cdot)X(\cdot)+u(\cdot))\\ &=\langle P(t)x,x\rangle+\mathbb{E}\int_t^T\Big\{2\big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD)\Theta\big]X,u\big\rangle\\ &\qquad\qquad+\big\langle(R+D^\top PD)u,u\big\rangle\Big\}ds.\end{aligned}$$

In particular, by taking $u(\cdot)=0$ in the above, we obtain

$$\langle P(t)x,x\rangle\geqslant\alpha|x|^2\quad\forall(t,x)\in[0,T]\times\mathbb{R}^n,$$

and the second inequality therefore follows. ∎

Now we state the main result of this section.

THEOREM 4.5. *Let* (H1)–(H2) *hold. Then the following statements are equivalent:*

(i) *The map $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ is uniformly convex; i.e., there exists a $\lambda>0$ such that* (4.2) *holds.*

(ii) *The Riccati equation* (4.6) *admits a strongly regular solution $P(\cdot)$.*

Proof. (i) ⇒ (ii) Let P_0 be the solution of

$$\begin{cases}\dot P_0+P_0A+A^\top P_0+C^\top P_0C+Q=0 \quad \text{a.e. } s\in[0,T],\\ P_0(T)=G.\end{cases}\tag{4.18}$$

Applying Proposition 4.4 with $\Theta=0$, we obtain that

$$R(t)+D(t)^\top P_0(t)D(t)\geqslant\lambda I,\quad P_0(t)\geqslant\alpha I,\quad \text{a.e. } t\in[0,T].$$

Next, inductively, for $i=0,1,2,\dots$, we set

$$\begin{cases}\Theta_i=-(R+D^\top P_iD)^{-1}(B^\top P_i+D^\top P_iC+S),\\ A_i=A+B\Theta_i,\quad C_i=C+D\Theta_i,\end{cases}\tag{4.19}$$

and let P_{i+1} be the solution of

$$\begin{cases}\dot P_{i+1}+P_{i+1}A_i+A_i^\top P_{i+1}+C_i^\top P_{i+1}C_i+\Theta_i^\top R\Theta_i+S^\top\Theta_i+\Theta_i^\top S+Q=0,\\ P_{i+1}(T)=G.\end{cases}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

By Proposition 4.4, we see that

$$R(t)+D(t)^\top P_{i+1}(t)D(t)\geqslant \lambda I,\quad P_{i+1}(t)\geqslant \alpha I \qquad \text{a.e. } s\in[0,T],\quad i=0,1,2,\ldots. \tag{4.20}$$

We now claim that $\{P_i\}_{i=1}^\infty$ converges uniformly in $C([0,T];\mathbb{S}^n)$. To show this, let

$$\Delta_i\triangleq P_i-P_{i+1},\quad \Lambda_i\triangleq\Theta_{i-1}-\Theta_i,\quad i\geqslant 1.$$

Then for $i\geqslant 1$, we have

$$\begin{aligned}-\dot\Delta_i&=P_iA_{i-1}+A_{i-1}^\top P_i+C_{i-1}^\top P_iC_{i-1}+\Theta_{i-1}^\top R\Theta_{i-1}+S^\top\Theta_{i-1}+\Theta_{i-1}^\top S\\&\quad-P_{i+1}A_i-A_i^\top P_{i+1}-C_i^\top P_{i+1}C_i-\Theta_i^\top R\Theta_i-S^\top\Theta_i-\Theta_i^\top S\\&=\Delta_iA_i+A_i^\top\Delta_i+C_i^\top\Delta_iC_i+P_i(A_{i-1}-A_i)+(A_{i-1}-A_i)^\top P_i\\&\quad+C_{i-1}^\top P_iC_{i-1}-C_i^\top P_iC_i+\Theta_{i-1}^\top R\Theta_{i-1}-\Theta_i^\top R\Theta_i+S^\top\Lambda_i+\Lambda_i^\top S.\end{aligned}\tag{4.21}$$

By (4.19), we have the following:

$$\begin{cases}A_{i-1}-A_i=B\Lambda_i,\qquad C_{i-1}-C_i=D\Lambda_i,\\ C_{i-1}^\top P_iC_{i-1}-C_i^\top P_iC_i=\Lambda_i^\top D^\top P_iD\Lambda_i+C_i^\top P_iD\Lambda_i+\Lambda_i^\top D^\top P_iC_i,\\ \Theta_{i-1}^\top R\Theta_{i-1}-\Theta_i^\top R\Theta_i=\Lambda_i^\top R\Lambda_i+\Lambda_i^\top R\Theta_i+\Theta_i^\top R\Lambda_i.\end{cases}\tag{4.22}$$

Note that

$$B^\top P_i+D^\top P_iC_i+R\Theta_i+S=B^\top P_i+D^\top P_iC+S+(R+D^\top P_iD)\Theta_i=0.$$

Thus, plugging (4.22) into (4.21) yields

$$\begin{aligned}&-(\dot\Delta_i+\Delta_iA_i+A_i^\top\Delta_i+C_i^\top\Delta_iC_i)\\&=P_iB\Lambda_i+\Lambda_i^\top B^\top P_i+\Lambda_i^\top D^\top P_iD\Lambda_i+C_i^\top P_iD\Lambda_i+\Lambda_i^\top D^\top P_iC_i\\&\quad+\Lambda_i^\top R\Lambda_i+\Lambda_i^\top R\Theta_i+\Theta_i^\top R\Lambda_i+S^\top\Lambda_i+\Lambda_i^\top S\\&=\Lambda_i^\top(R+D^\top P_iD)\Lambda_i+(P_iB+C_i^\top P_iD+\Theta_i^\top R+S^\top)\Lambda_i\\&\quad+\Lambda_i^\top(B^\top P_i+D^\top P_iC_i+R\Theta_i+S)\\&=\Lambda_i^\top(R+D^\top P_iD)\Lambda_i\geqslant 0,\end{aligned}\tag{4.23}$$

which, together with $\Delta_i(T)=0$, implies $\Delta_i(\cdot)\geqslant 0$. Also, noting (4.20), we obtain

$$P_1(s)\geqslant P_i(s)\geqslant P_{i+1}(s)\geqslant\alpha I\quad \forall s\in[0,T],\ \forall i\geqslant 1.$$

Therefore, the sequence $\{P_i\}_{i=1}^\infty$ is uniformly bounded. Consequently, there exists a constant $K>0$ such that (noting (4.20))

$$\begin{cases}|P_i(s)|,\ |R_i(s)|\leqslant K,\\ |\Theta_i(s)|\leqslant K\big(|B(s)|+|C(s)|+|S(s)|\big),\\ |A_i(s)|\leqslant|A(s)|+K|B(s)|\big(|B(s)|+|C(s)|+|S(s)|\big),\\ |C_i(s)|\leqslant|C(s)|+K\big(|B(s)|+|C(s)|+|S(s)|\big),\end{cases}\quad \text{a.e. } s\in[0,T],\ \forall\, i\geqslant 0,\tag{4.24}$$

where $R_i(s)\triangleq R(s)+D^\top(s)P_i(s)D(s)$. Observe that

$$\Lambda_i=R_i^{-1}D^\top\Delta_{i-1}DR_{i-1}^{-1}(B^\top P_i+D^\top P_iC+S)-R_{i-1}^{-1}(B^\top\Delta_{i-1}+D^\top\Delta_{i-1}C).\tag{4.25}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Thus, noting (4.24), one has

$$\begin{aligned}|\Lambda_i(s)^\top R_i(s)\Lambda_i(s)| &\leqslant \Big(|\Theta_i(s)|+|\Theta_{i-1}(s)|\Big)\,|R_i(s)|\,|\Theta_{i-1}(s)-\Theta_i(s)|\\ &\leqslant K\Big(|B(s)|+|C(s)|+|S(s)|\Big)^2|\Delta_{i-1}(s)|.\end{aligned}\tag{4.26}$$

Equation (4.23), together with $\Delta_i(T)=0$, implies that

$$\Delta_i(s)=\int_s^T \big(\Delta_i A_i + A_i^\top \Delta_i + C_i^\top \Delta_i C_i + \Lambda_i^\top R_i \Lambda_i\big)dr.$$

Making use of (4.26) and still noting (4.24), we get

$$|\Delta_i(s)| \leqslant \int_s^T \varphi(r)\Big[|\Delta_i(r)|+|\Delta_{i-1}(r)|\Big]dr \quad \forall s\in[0,T],\ \forall i\geqslant 1,$$

where $\varphi(\cdot)$ is a nonnegative integrable function independent of $\Delta_i(\cdot)$. By Gronwall's inequality,

$$|\Delta_i(s)| \leqslant e^{\int_0^T \varphi(r)dr}\int_s^T \varphi(r)|\Delta_{i-1}(r)|dr \equiv c\int_s^T \varphi(r)|\Delta_{i-1}(r)|dr.$$

Set $a \triangleq \max_{0\leqslant s\leqslant T}|\Delta_0(s)|$. By induction, we deduce that

$$|\Delta_i(s)| \leqslant a\frac{c^i}{i!}\bigg(\int_s^T \varphi(r)dr\bigg)^i \quad \forall s\in[0,T],$$

which implies the uniform convergence of $\{P_i\}_{i=1}^\infty$. We denote P the limit of $\{P_i\}_{i=1}^\infty$; then (noting (4.20))

$$R(s)+D(s)^\top P(s)D(s)=\lim_{i\to\infty} R(s)+D(s)^\top P_i(s)D(s)\geqslant \lambda I \quad \text{a.e. } s\in[0,T],$$

and as $i\to\infty$,

$$\begin{cases}\Theta_i \to -(R+D^\top PD)^{-1}(B^\top P + D^\top PC + S)\equiv \Theta \quad \text{in } L^2,\\ A_i\to A+B\Theta \quad \text{in } L^1, \qquad C_i\to C+D\Theta \quad \text{in } L^2.\end{cases}$$

Therefore, $P(\cdot)$ satisfies the following equation:

$$\begin{cases}\dot P + P(A+B\Theta)+(A+B\Theta)^\top P + (C+D\Theta)^\top P(C+D\Theta)\\ \quad +\Theta^\top R\Theta + S^\top\Theta+\Theta^\top S + Q = 0 \quad \text{a.e. } s\in[0,T],\\ P(T)=G,\end{cases}$$

which is equivalent to (4.6).

(ii) $\Rightarrow$ (i) Let $P(\cdot)$ be the strongly regular solution of (4.6). Then there exists a $\lambda>0$ such that

$$R(s)+D(s)^\top P(s)D(s)\geqslant \lambda I \quad \text{a.e. } s\in[0,T]. \tag{4.27}$$

Set

$$\Theta \triangleq -(R+D^\top PD)^{-1}(B^\top P + D^\top PC+S)\in L^2(0,T;\mathbb{R}^{m\times n}).$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

For any $u(\cdot)\in\mathcal{U}[0,T]$, let $X^{(u)}(\cdot)$ be the solution of

$$\begin{cases} dX^{(u)}(s)=\big[A(s)X^{(u)}(s)+B(s)u(s)\big]ds+\big[C(s)X^{(u)}(s)+D(s)u(s)\big]dW(s),\\ X(0)=0.\end{cases}$$

Applying Itô's formula to $s\mapsto\langle P(s)X^{(u)}(s),X^{(u)}(s)\rangle$, we have

$$\begin{aligned} J^0(0,0;u(\cdot)) &= \mathbb{E}\left\{\big\langle GX^{(u)}(T),X^{(u)}(T)\big\rangle+\int_0^T\left\langle\begin{pmatrix}Q & S^\top\\ S & R\end{pmatrix}\begin{pmatrix}X^{(u)}\\ u\end{pmatrix},\begin{pmatrix}X^{(u)}\\ u\end{pmatrix}\right\rangle ds\right\}\\ &= \mathbb{E}\int_0^T\Big[\big\langle\big(\dot P+PA+A^\top P+C^\top PC+Q\big)X^{(u)},X^{(u)}\big\rangle\\ &\qquad +2\big\langle\big(B^\top P+D^\top PC+S\big)X^{(u)},u\big\rangle+\big\langle\big(R+D^\top PD\big)u,u\big\rangle\Big]ds\\ &= \mathbb{E}\int_0^T\Big[\big\langle\Theta^\top\big(R+D^\top PD\big)\Theta X^{(u)},X^{(u)}\big\rangle-2\big\langle\big(R+D^\top PD\big)\Theta X^{(u)},u\big\rangle\\ &\qquad +\big\langle\big(R+D^\top PD\big)u,u\big\rangle\Big]ds\\ &= \mathbb{E}\int_0^T\big\langle\big(R+D^\top PD\big)\big(u-\Theta X^{(u)}\big),u-\Theta X^{(u)}\big\rangle ds. \end{aligned}$$

Noting (4.27) and making use of Lemma 2.3, we obtain that

$$\begin{aligned} J^0(0,0;u(\cdot)) &= \mathbb{E}\int_0^T\big\langle\big(R+D^\top PD\big)\big(u-\Theta X^{(u)}\big),u-\Theta X^{(u)}\big\rangle ds\\ &\geqslant \lambda\gamma\mathbb{E}\int_0^T|u(s)|^2ds\quad\forall u(\cdot)\in\mathcal{U}[0,T], \end{aligned}$$

for some $\gamma>0$. Then (i) holds. ∎

Remark 4.6. From the first part of the proof of Theorem 4.5, we see that if (4.2) holds, then the strongly regular solution of (4.6) satisfies (4.10) with the same constant $\lambda>0$.

Combining Theorems 4.3 and 4.5, we obtain the following corollary.

COROLLARY 4.7. *Let* (H1)–(H2) *and* (4.2) *hold. Then Problem* (SLQ) *is uniquely open-loop solvable at any* $(t,x)\in[0,T)\times\mathbb{R}^n$ *with the open-loop optimal control* $u^*(\cdot)$ *being of a state feedback form:*

$$\begin{aligned} u^*(\cdot)=&-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)X^*\\ &-(R+D^\top PD)^{-1}(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho), \end{aligned}\tag{4.28}$$

where $P(\cdot)$ *is the unique strongly regular solution of* (4.6) *with* $(\eta(\cdot),\zeta(\cdot))$ *being the adapted solution of* (4.11) *and* $X^*(\cdot)$ *being the solution of the following closed-loop system:*

$$\begin{cases} dX^*(s)=\Big\{\big[A-B(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\big]X^*\\ \qquad\qquad -B(R+D^\top PD)^{-1}(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)+b\Big\}ds\\ \qquad +\Big\{\big[C-D(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\big]X^*\\ \qquad\qquad -D(R+D^\top PD)^{-1}(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)+\sigma\Big\}dW(s),\quad s\in[t,T],\\ X^*(t)=x. \end{cases}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Proof. By Theorem 4.5, the Riccati equation (4.6) admits a unique strongly regular solution $P(\cdot) \in C([0,T];\mathbb{S}^n)$. Hence, the adapted solution $(\eta(\cdot),\zeta(\cdot))$ of (4.11) satisfies (4.12) automatically. Now applying Theorem 4.3 and noting the remark right after Definition 2.1, we get the desired result. ∎

Remark 4.8. Under the assumptions of Corollary 4.7, when $b(\cdot)$, $\sigma(\cdot)$, g, $q(\cdot)$, $\rho(\cdot) = 0$, the adapted solution of (4.11) is $(\eta(\cdot),\zeta(\cdot)) \equiv (0,0)$. Thus, for Problem (SLQ)^0, the unique optimal control $u^*(\cdot)$ at initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$ is given by

$$u^*(\cdot) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)X^*,$$

with $P(\cdot)$ being the unique strongly regular solution of (4.6) and $X^*(\cdot)$ being the solution of the following closed-loop system:

$$\begin{cases} dX^*(s) = \big[A - B(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* ds \\ \qquad\qquad + \big[C - D(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* dW(s), \quad s \in [t,T], \\ X^*(t) = x. \end{cases}$$

Moreover, by (4.14), the value function of Problem (SLQ)^0 is given by

$$V^0(t,x) = \langle P(t)x, x\rangle, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

5. Finiteness of Problem (SLQ) and convexity of cost functional. We have seen that the uniform convexity of the cost functional implies the open-loop and closed-loop solvabilities of Problem (SLQ). We expect that the finiteness of Problem (SLQ) should be closely related to the convexity of the cost functional. A main purpose of this section is to make this clear. Other relevant issues will also be discussed. First, we introduce the following:

$$\Lambda(s,P(\cdot)) = \begin{pmatrix} \dot P(s)+P(s)A(s)+A(s)^\top P(s)+C(s)^\top P(s)C(s)+Q(s) & P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top \\ B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s) & R(s)+D(s)^\top P(s)D(s) \end{pmatrix}$$

for any $P(\cdot) \in AC(t,T;\mathbb{S}^n)$ which is the set of all absolutely continuous functions $P : [t,T] \to \mathbb{S}^n$. Let

$$\mathcal{P}[t,T] = \Big\{ P(\cdot) \in AC(t,T;\mathbb{S}^n) \;\Big|\; P(T) \leqslant G,\ \Lambda(s,P(\cdot)) \geqslant 0 \text{ a.e. } s \in [t,T] \Big\}.$$

We have the following result.

Proposition 5.1. *Let* (H1)–(H2) *hold, and let* $t \in [0,T)$ *be given. Among the following statements,*

(i) *Problem* (SLQ) *is finite at* t,

(ii) *Problem* (SLQ)^0 *is finite at* t,

(iii) *there exists a* $P(t) \in \mathbb{S}^n$ *such that*

$$V^0(t,x) = \langle P(t)x, x\rangle \quad \forall x \in \mathbb{R}^n, \tag{5.1}$$

(iv) *the map* $u(\cdot) \mapsto J(t,x;u(\cdot))$ *is convex for any* $x \in \mathbb{R}^n$,

(v) $\mathcal{P}[t,T] \neq \emptyset$,

the following implications hold:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)}; \quad \text{(v)} \Rightarrow \text{(ii)}.$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Proof. (i) ⇒ (ii) By Proposition 3.1, for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t,T]$, we have

$$\begin{aligned} V(t,x) + V(t,-x) &\leqslant J(t,x;u(\cdot)) + J(t,-x;-u(\cdot)) \\ &= 2\Big[\big\langle M_2(t)u,u\big\rangle + 2\big\langle M_1(t)x,u\big\rangle + \big\langle M_0(t)x,x\big\rangle + c_t\Big] \\ &= 2\Big[J^0(t,x;u(\cdot)) + c_t\Big], \end{aligned}$$

which implies (ii).

(ii) ⇒ (iii) This part of the proof can be shown by a simple adoption of the well-known result in the deterministic case (see [10, 3]).

(iii) ⇒ (iv) By Corollary 3.4, if $u(\cdot) \mapsto J(t,x;u(\cdot))$ is not convex, then for some $u(\cdot) \in \mathcal{U}[t,T]$, $J^0(t,0;u(\cdot)) < 0$. By Corollary 3.3, we have for all $\lambda \in \mathbb{R}^n$

$$J^0(t,x;\lambda u(\cdot)) = J^0(t,x;0) + \lambda^2 J^0(t,0;u(\cdot)) + \lambda \mathbb{E}\int_t^T \big\langle \mathcal{D}J^0(t,x;0)(s),u(s)\big\rangle ds.$$

Letting $\lambda \to \infty$, we obtain

$$V^0(t,x) \leqslant \lim_{\lambda\to\infty} J^0(t,x;\lambda u(\cdot)) = -\infty,$$

which is a contradiction.

(v) ⇒ (ii) For any $(t,x) \in [0,T)\times\mathbb{R}^n$, $u(\cdot) \in \mathcal{U}[t,T]$, and any $P(\cdot) \in AC(t,T;\mathbb{S}^n)$, one has

$$\begin{aligned} &\mathbb{E}\langle P(T)X(T),X(T)\rangle - \langle P(t)x,x\rangle \\ &= \mathbb{E}\int_t^T \Big\{\big\langle\big(\dot P + PA + A^\top P + C^\top PC\big)X,X\big\rangle \\ &\qquad + 2\big\langle\big(B^\top P + D^\top PC\big)X,u\big\rangle + \big\langle D^\top PDu,u\big\rangle\Big\}ds. \end{aligned}$$

Hence, if $\mathcal{P}[t,T] \neq \emptyset$, then by taking $P(\cdot) \in \mathcal{P}[t,T]$, one has

$$\begin{aligned} J^0(t,x;u(\cdot)) &= \mathbb{E}\bigg\{\langle GX(T),X(T)\rangle + \int_t^T \bigg\langle \begin{pmatrix} Q & S^\top \\ S & R\end{pmatrix}\begin{pmatrix} X \\ u\end{pmatrix},\begin{pmatrix} X \\ u\end{pmatrix}\bigg\rangle ds\bigg\} \\ &= \langle P(t)x,x\rangle + \mathbb{E}\bigg\{\big\langle [G-P(T)]X(T),X(T)\big\rangle \\ &\qquad + \int_t^T \bigg\langle \Lambda(s,P(\cdot))\begin{pmatrix} X(s) \\ u(s)\end{pmatrix},\begin{pmatrix} X(s) \\ u(s)\end{pmatrix}\bigg\rangle ds\bigg\} \\ &\geqslant \langle P(t)x,x\rangle \quad \forall u(\cdot) \in \mathcal{U}[t,T]. \end{aligned}$$

This implies that the corresponding Problem (SLQ)^0 is finite at t. ∎

It is worth pointing out that the convexity of the map $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is not sufficient for the finiteness of Problem (SLQ)^0. We present the following example (see also [20] for an example of a quadratic functional in Hilbert space).

Example 5.2. Consider the one-dimensional controlled SDE

$$\begin{cases} dX(s) = u(s)ds + X(s)dW(s), \quad s \in [t,1], \\ X(t) = x, \end{cases} \tag{5.2}$$

and the cost functional

$$J^0(t,x;u(\cdot)) = \mathbb{E}\left[-X(1)^2 + \int_t^1 e^{1-s}u(s)^2 ds\right]. \tag{5.3}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

We claim that

$$J^0(0,0;u(\cdot)) \geqslant 0 \quad \forall u(\cdot)\in\mathcal{U}[0,T], \tag{5.4}$$

which, by Corollary 3.4, is equivalent to the convexity of $u(\cdot)\mapsto J^0(0,x;u(\cdot))$, but

$$V^0(0,x) = -\infty \quad \forall x\neq 0. \tag{5.5}$$

To show the above, let $u(\cdot)\in\mathcal{U}[0,T]$ and $X(\cdot)\equiv X(\cdot\,;0,x,u(\cdot))$ be the solution of (5.2) with $t=0$. By the variation of constants formula,

$$X(s) = xe^{W(s)-\frac{1}{2}s} + e^{W(s)-\frac{1}{2}s}\int_0^s e^{\frac{1}{2}r-W(r)}u(r)dr, \quad s\in[0,1].$$

Taking $x=0$ and noting that $e^{2[W(1)-W(r)]-(1-r)}$ is independent of $\mathcal{F}_r$, we have

$$\begin{aligned}\mathbb{E}\big[X(1)^2\big] &= \mathbb{E}\left[\int_0^1 e^{W(1)-W(r)-\frac{1}{2}(1-r)}u(r)dr\right]^2 \leqslant \mathbb{E}\int_0^1 e^{2[W(1)-W(r)]-(1-r)}u(r)^2dr\\ &= \int_0^1 \mathbb{E}e^{2[W(1)-W(r)]-(1-r)}\mathbb{E}[u(r)^2]dr = \mathbb{E}\int_0^1 e^{1-r}u(r)^2dr,\end{aligned}$$

and hence

$$J^0(0,0;u(\cdot)) = \mathbb{E}\left[-X(1)^2 + \int_0^1 e^{1-s}u(s)^2ds\right] \geqslant 0 \quad \forall u(\cdot)\in\mathcal{U}[0,T].$$

On the other hand, taking $x\neq 0$ and $u(s)=\lambda e^{W(s)-\frac{1}{2}s},\ \lambda\in\mathbb{R}$, we have

$$X(1) = (x+\lambda)e^{W(1)-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}J^0(0,x;u(\cdot)) &= \mathbb{E}\left[-X(1)^2 + \int_0^1 e^{1-s}u(s)^2ds\right]\\ &= -\mathbb{E}\Big[(x+\lambda)^2e^{2W(1)-1}\Big] + \lambda^2\mathbb{E}\int_0^1 e^{1-s}e^{2W(s)-s}ds\\ &= -(x+\lambda)^2e + \lambda^2 e = -(x^2+2\lambda x)e.\end{aligned}$$

Letting $|\lambda|\to\infty$, with $\lambda x>0$, in the above, we obtain $V^0(0,x)=-\infty$. This proves our claim.

The above example tells us that, besides the convexity of $u(\cdot)\mapsto J^0(t,x;u(\cdot))$, one needs some additional condition(s) in order to get the finiteness of Problem (SLQ)^0 at t. To find such a condition, let us make some observations. Suppose $u(\cdot)\mapsto J^0(0,x;u(\cdot))$ is convex, which, by Corollary 3.4, is equivalent to the following:

$$J^0(0,0;u(\cdot)) \geqslant 0 \quad \forall u(\cdot)\in\mathcal{U}[0,T]. \tag{5.6}$$

Then for any $\varepsilon>0$, consider state equation (1.1) (with $b(\cdot),\sigma(\cdot)=0$) and the following cost functional:

$$\begin{aligned}J^0_\varepsilon(t,x;u(\cdot)) &\triangleq \mathbb{E}\left\{\langle GX(T),X(T)\rangle + \int_t^T\left\langle\begin{pmatrix}Q & S^\top\\ S & R+\varepsilon I\end{pmatrix}\begin{pmatrix}X\\ u\end{pmatrix},\begin{pmatrix}X\\ u\end{pmatrix}\right\rangle ds\right\}\\ &= J^0(t,x;u(\cdot)) + \varepsilon\mathbb{E}\int_t^T|u(s)|^2ds.\end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Denote the corresponding optimal control problem and value function by Problem $(\text{SLQ})^0_\varepsilon$ and $V^0_\varepsilon(\cdot\,,\cdot)$, respectively. By Corollary 3.4 and the convexity of $u(\cdot)\mapsto J^0(0,0;u(\cdot))$, one has

$$J^0_\varepsilon(0,0;u(\cdot)) = J^0(0,0;u(\cdot)) + \varepsilon\mathbb{E}\int_0^T |u(s)|^2 ds \geqslant \varepsilon\mathbb{E}\int_0^T |u(s)|^2 ds \quad \forall u(\cdot)\in\mathcal{U}[0,T],$$

i.e., $u(\cdot)\mapsto J^0_\varepsilon(0,0;u(\cdot))$ is uniformly convex. Hence, it follows from Theorem 4.5 that the Riccati equation

$$\begin{cases} \dot P_\varepsilon + P_\varepsilon A + A^\top P_\varepsilon + C^\top P_\varepsilon C + Q \\ -(P_\varepsilon B + C^\top P_\varepsilon D + S^\top)(R+\varepsilon I + D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon + D^\top P_\varepsilon C + S) = 0, \\ P_\varepsilon(T) = G \end{cases} \tag{5.7}$$

admits a unique strongly regular solution $P_\varepsilon(\cdot)\in C([0,T];\mathbb{S}^n)$ such that (noting Remark 4.6)

$$R(t) + \varepsilon I + D(t)^\top P_\varepsilon(t) D(t) \geqslant \varepsilon I \quad \text{a.e. } t\in[0,T]. \tag{5.8}$$

Now, we are ready to state and prove the following result, which is a characterization of the finiteness of Problem $(\text{SLQ})^0$.

THEOREM 5.3. *Let* (H1)–(H2) *and* (5.6) *hold. For any $\varepsilon>0$, let $P_\varepsilon(\cdot)$ be the unique strongly regular solution of the Riccati equation* (5.7). *Then Problem* $(\text{SLQ})^0$ *is finite if and only if $\{P_\varepsilon(0)\}_{\varepsilon>0}$ is bounded from below. In this case, the limit*

$$\lim_{\varepsilon\to 0} P_\varepsilon(t) = P(t) \quad \forall t\in[0,T] \tag{5.9}$$

exists, and (5.1) *holds. Moreover,*

$$R(t) + D(t)^\top P(t) D(t) \geqslant 0 \quad \text{a.e. } t\in[0,T], \tag{5.10}$$

and

$$N(t) \leqslant P(t) \leqslant M_0(t) \quad \forall t\in[0,T], \tag{5.11}$$

where $M_0(\cdot)$ is the solution to the Lyapunov equation (3.2), *and*

$$N(t) = \big[\Phi_A(t)^\top\big]^{-1}\left\{P(0) - \int_0^t \Phi_A(s)^\top\Big[C(s)^\top M_0(s)C(s) + Q(s)\Big]\Phi_A(s)ds\right\}\Phi_A(t)^{-1},$$

with $\Phi_A(\cdot)$ being the solution to the following:

$$\begin{cases} \dot\Phi_A(s) = A(s)\Phi_A(s), \quad s\geqslant 0, \\ \Phi_A(0) = I. \end{cases} \tag{5.12}$$

In particular, if Problem $(\text{SLQ})^0$ *is finite at $t=0$, then it is finite.*

Proof. Necessity. Suppose Problem $(\text{SLQ})^0$ is finite, and let $P:[0,T]\to\mathbb{S}^n$ such that (5.1) holds. For any $\varepsilon_2>\varepsilon_1>0$, we have

$$J^0_{\varepsilon_2}(t,x;u(\cdot)) \geqslant J^0_{\varepsilon_1}(t,x;u(\cdot)) \geqslant J^0(t,x;u(\cdot)) \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n,\ \forall u(\cdot)\in\mathcal{U}[t,T].$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Hence (noting Remark 4.8),

$$\begin{aligned}\langle P_{\varepsilon_2}(t)x,x\rangle &= V^0_{\varepsilon_2}(t,x)=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J^0_{\varepsilon_2}(t,x;u(\cdot))\geqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]}J^0_{\varepsilon_1}(t,x;u(\cdot))\\ &= V^0_{\varepsilon_1}(t,x)=\langle P_{\varepsilon_1}(t)x,x\rangle \geqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]}J^0(t,x;u(\cdot))=V^0(t,x)\\ &=\langle P(t)x,x\rangle \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n.\end{aligned}$$

Thus $\{P_\varepsilon(t)\}_{\varepsilon>0}$ is a nondecreasing sequence with lower bound $P(t)$ and therefore has a limit $\bar{P}(t)$ with

$$\bar{P}(t)\equiv\lim_{\varepsilon\to0}P_\varepsilon(t)\geqslant P(t)\quad \forall t\in[0,T]. \tag{5.13}$$

On the other hand, for any $\delta>0$, we can find a $u^\delta(\cdot)\in\mathcal{U}[t,T]$, such that

$$V^0_\varepsilon(t,x)\leqslant J^0(t,x;u^\delta(\cdot))+\varepsilon\mathbb{E}\int_t^T|u^\delta(s)|^2ds\leqslant V^0(t,x)+\delta+\varepsilon\mathbb{E}\int_t^T|u^\delta(s)|^2ds.$$

Letting $\varepsilon\to0$, we obtain that

$$\langle\bar{P}(t)x,x\rangle=\lim_{\varepsilon\to0}\langle P_\varepsilon(t)x,x\rangle=\lim_{\varepsilon\to0}V^0_\varepsilon(t,x)\leqslant V^0(t,x)+\delta=\langle P(t)x,x\rangle+\delta,$$

from which we see that

$$\langle\bar{P}(t)x,x\rangle\leqslant\langle P(t)x,x\rangle\quad\forall(t,x)\in[0,T]\times\mathbb{R}^n. \tag{5.14}$$

Combining (5.13)–(5.14), we obtain (5.9) with $P(\cdot)$ satisfying (5.1). Moreover, letting $\varepsilon\to0$ in (5.8), we obtain (5.10).

Sufficiency. Suppose there exists a $\beta\in\mathbb{R}$ such that

$$P_\varepsilon(0)\geqslant\beta I\quad\forall\varepsilon>0;$$

then for any $x\in\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[0,T]$, we have

$$J^0(0,x;u(\cdot))+\varepsilon\mathbb{E}\int_0^T|u(s)|^2ds\geqslant V^0_\varepsilon(0,x)=\langle P_\varepsilon(0)x,x\rangle\geqslant\beta|x|^2\quad\forall\varepsilon>0.$$

Letting $\varepsilon\to0$ in the above, we obtain

$$J^0(0,x;u(\cdot))\geqslant\beta|x|^2\quad\forall x\in\mathbb{R}^n,\ \forall u(\cdot)\in\mathcal{U}[0,T],$$

which implies the finiteness of Problem (SLQ)^0 at $t=0$.

Now, let $P(0)\in\mathbb{S}^n$ such that $V^0(0,x)=\langle P(0)x,x\rangle$ for all $x\in\mathbb{R}^n$. Then

$$P(0)\leqslant P_\varepsilon(0)\quad\forall\varepsilon>0. \tag{5.15}$$

Also, by Remark 4.8 and Proposition 3.1,

$$\langle P_\varepsilon(t)x,x\rangle=V^0_\varepsilon(t,x)\leqslant J^0_\varepsilon(t,x;0)=J^0(t,x;0)=\langle M_0(t)x,x\rangle\quad\forall x\in\mathbb{R}^n.$$

This leads to

$$P_\varepsilon(t)\leqslant M_0(t),\quad t\in[0,T],\ \forall\varepsilon>0. \tag{5.16}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

On the other hand, let $\Phi_A(\cdot)$ be the solution of (5.12), and set

$$\Pi_\varepsilon \triangleq (P_\varepsilon B + C^\top P_\varepsilon D + S^\top)(R_\varepsilon + D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon + D^\top P_\varepsilon C + S) \geqslant 0.$$

Then, combining (5.15)–(5.16), we have

$$\begin{aligned}\Phi_A(t)^\top P_\varepsilon(t)\Phi_A(t) &= P_\varepsilon(0) + \int_0^t \Phi_A(s)^\top \Big[\Pi_\varepsilon(s) - C(s)^\top P_\varepsilon(s)C(s) - Q(s)\Big]\Phi_A(s)ds \\ &\geqslant P(0) - \int_0^t \Phi_A(s)^\top \Big[C(s)^\top M_0(s)C(s) + Q(s)\Big]\Phi_A(s)ds.\end{aligned}$$

Thus,

$$\begin{aligned}P_\varepsilon(t) &\geqslant \big[\Phi_A(t)^\top\big]^{-1}\bigg\{P(0) - \int_0^t \Phi_A(s)^\top \Big[C(s)^\top M_0(s)C(s) + Q(s)\Big]\Phi_A(s)ds\bigg\}\Phi_A(t)^{-1} \\ &\equiv N(t), \quad t \in [0,T].\end{aligned}$$

Then, using the same argument as in the previous paragraph, we can show that

$$J^0(t,x;u(\cdot)) \geqslant \langle N(t)x, x\rangle \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n,\ \forall u(\cdot) \in \mathcal{U}[t,T], \tag{5.17}$$

which implies the finiteness of Problem $(\text{SLQ})^0$. Moreover, let $P : [0,T] \to \mathbb{S}^n$ such that (5.1) holds; then

$$\begin{aligned}\langle N(t)x, x\rangle &\leqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) = \langle P(t)x, x\rangle \\ &\leqslant J^0(t,x;0) = \langle M_0(t)x, x\rangle \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n,\end{aligned}$$

and (5.11) follows.

Finally, if Problem $(\text{SLQ})^0$ is finite at $t = 0$, then (5.15) holds, and the finiteness of Problem $(\text{SLQ})^0$ therefore follows. ∎

The following is another sufficient condition for the finiteness of Problem $(\text{SLQ})^0$, which is a corollary of Theorem 5.3 and Proposition 5.1, (v) ⇒ (ii).

COROLLARY 5.4. *Let* (H1)–(H2) *hold. If there exists a* $\Delta(\cdot) \in L^1(0,T;\mathbb{S}^n_+)$ *such that*

$$R + D^\top PD \geqslant \big(B^\top P + D^\top PC + S\big)\Delta^{-1}\big(PB + C^\top PD + S^\top\big) \quad \text{a.e. } s \in [0,T], \tag{5.18}$$

where $P(\cdot)$ *is the solution of the following Lyapunov equation:*

$$\begin{cases}\dot P(s) + P(s)A(s) + A(s)^\top P(s) + C(s)^\top P(s)C(s) + Q(s) = \Delta(s) \quad \text{a.e. } s \in [0,T], \\ P(T) \leqslant G.\end{cases}$$

Then Problem $(\text{SLQ})^0$ *is finite.*

Proof. Under our condition, one has for a.e. $s \in [0,T]$

$$\Lambda(s,P(\cdot)) = \begin{pmatrix}\Delta(s) & P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top \\ B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s) & R(s)+D(s)^\top P(s)D(s)\end{pmatrix} \geqslant 0.$$

Hence, $P(\cdot) \in \mathcal{P}[0,T]$, and Problem $(\text{SLQ})^0$ is finite at $t = 0$. Then the finiteness of Problem $(\text{SLQ})^0$ follows from Theorem 5.3 immediately. ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

We now return to the study of convexity of the map $u(\cdot) \mapsto J^0(t,0;u(\cdot))$. First, from the representation of $M_2(t)$ (see (3.3)), we see that $M_2(t) \geqslant 0$ if and only if

$$R(\cdot) \geqslant -\Big(\widehat{L}_t^* G \widehat{L}_t + L_t^* Q L_t + S L_t + L_t^* S^\top\Big), \tag{5.19}$$

with the right-hand side possibly being nonpositive. Thus, unlike the well-known situation for the deterministic LQ problems (for which $R(\cdot) \geqslant 0$ is necessary for $M_2(t) \geqslant 0$ [30]), $R(\cdot)$ does not have to be positive semidefinite. Actually, as shown by examples in [6, 30], $R(\cdot)$ could even be negative definite to some extent. Let us now take a closer look at this issue.

Note that when $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is convex, for any $\varepsilon > 0$, the unique strongly regular solution $P_\varepsilon(\cdot)$ to the Riccati equation (5.7) satisfies (5.8) and (5.16). Hence,

$$R(t) + D(t)^\top M_0(t) D(t) \geqslant 0 \quad \text{a.e. } t \in [0,T], \tag{5.20}$$

or, equivalently,

$$\begin{aligned} R(t) \,+\, D(t)^\top \mathbb{E}\bigg\{ & \big[\Phi(T)\Phi(t)^{-1}\big]^\top G \big[\Phi(T)\Phi(t)^{-1}\big] \\ & + \int_t^T \big[\Phi(s)\Phi(t)^{-1}\big]^\top Q(s) \big[\Phi(s)\Phi(t)^{-1}\big] ds \bigg\} D(t) \geqslant 0 \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{5.21}$$

This is another necessary condition for the finiteness of Problem $(\text{SLQ})^0$, which is easier to check. From (5.21), we see that if $R(\cdot)$ happens to be negative definite, then in order for $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ to be convex, it is necessary that $D(\cdot)$ be injective, and either G or $Q(\cdot)$ (or both) has to be positive enough to compensate. Note that $D(s)$ was assumed to be invertible in [22]. Therefore, in some sense, our result justifies the assumption of [22].

The following gives a little improvement when more restrictive conditions are assumed.

Proposition 5.5. *Let* (H1)–(H2) *hold. Suppose that*

$$B(\cdot) = 0, \qquad C(\cdot) = 0, \qquad S(\cdot) = 0. \tag{5.22}$$

Then the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is convex if and only if* (5.21) *holds. In this case, Problem* $(\text{SLQ})^0$ *is closed-loop solvable.*

Proof. It suffices to prove the sufficiency. Note that in the current case, the corresponding Riccati equation becomes

$$\begin{cases} \dot{P}(s) + P(s)A(s) + A(s)^\top P(s) + Q(s) = 0 \quad \text{a.e. } s \in [0,T], \\ P(T) = G, \end{cases}$$

whose solution is $M_0(\cdot)$. If (5.21) holds, then it is easy to verify that $M_0(\cdot)$ is regular. Consequently, by Theorem 4.3, Problem $(\text{SLQ})^0$ is closed-loop solvable, and hence $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is convex. ∎

Note that in the case of (5.22), we have

$$\begin{aligned} M_0(t) = & \big[\Phi_A(T)\Phi_A(t)^{-1}\big]^\top G \big[\Phi_A(T)\Phi_A(t)^{-1}\big] \\ & + \int_t^T \big[\Phi_A(s)\Phi_A(t)^{-1}\big]^\top Q(s) \big[\Phi_A(s)\Phi_A(t)^{-1}\big] ds, \end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

with $\Phi_A(\cdot)$ being the solution of (5.12). Hence, (5.21) can also be written as

$$R(t) + D(t)^\top \Big\{ \big[\Phi_A(T)\Phi_A(t)^{-1}\big]^\top G \big[\Phi_A(T)\Phi_A(t)^{-1}\big] + \int_t^T \big[\Phi_A(s)\Phi_A(t)^{-1}\big]^\top Q(s) \big[\Phi_A(s)\Phi_A(t)^{-1}\big] ds \Big\} D(t) \geqslant 0 \quad \text{a.e. } t \in [0,T].$$

From Remarks 4.6 and 4.8, we see that if the uniformly convex condition (4.2) holds, then Problem $(\text{SLQ})^0$ is finite and

$$R(s) + D(s)^\top P(s) D(s) \geqslant \lambda I \quad \text{a.e. } s \in [0,T], \tag{5.23}$$

where $P : [0,T] \to \mathbb{S}^n$ is the function such that (5.1) holds. The following result shows that the converse is also true.

THEOREM 5.6. *Let* (H1)–(H2) *hold. Suppose Problem* $(\text{SLQ})^0$ *is finite, and let* $P : [0,T] \to \mathbb{S}^n$ *such that* (5.1) *holds. If* (5.23) *holds for some* $\lambda > 0$*, then* $P(\cdot)$ *solves the Riccati equation* (4.6)*. Consequently, the map* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is uniformly convex.*

Proof. For any $\varepsilon > 0$, let $P_\varepsilon(\cdot)$ be the unique strongly regular solution of (5.7). By Theorem 5.3,

$$P_\varepsilon(t) \searrow P(t) \quad \text{as } \varepsilon \searrow 0, \ \forall t \in [0,T].$$

Note that $P_\varepsilon(\cdot) \leqslant M_0(\cdot)$ for all $\varepsilon > 0$ and by (5.11), $P(\cdot)$ is bounded. Thus, $\{P_\varepsilon(t)\}_{\varepsilon > 0}$ is uniformly bounded. Also, we have

$$R(s) + D(s)^\top P_\varepsilon(s) D(s) \geqslant R(s) + D(s)^\top P(s) D(s) \geqslant \lambda I \quad \text{a.e. } s \in [0,T], \ \forall \varepsilon > 0.$$

Then it follows from the dominated convergence theorem that

$$\begin{aligned} &P_\varepsilon A + A^\top P_\varepsilon + C^\top P_\varepsilon C + Q \\ &- (P_\varepsilon B + C^\top P_\varepsilon D + S^\top)(R + \varepsilon I + D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon + D^\top P_\varepsilon C + S) \equiv \Lambda_\varepsilon \end{aligned}$$

converges to

$$\begin{aligned} &PA + A^\top P + C^\top P C + Q \\ &- (PB + C^\top P D + S^\top)(R + D^\top P D)^{-1}(B^\top P + D^\top P C + S) \equiv \Lambda \end{aligned}$$

in L^1 as $\varepsilon \to 0$. Therefore,

$$P(t) = \lim_{\varepsilon \to 0} P_\varepsilon(t) = G + \lim_{\varepsilon \to 0} \int_t^T \Lambda_\varepsilon(s) ds = G + \int_t^T \Lambda(s) ds,$$

which, together with (5.23), implies that $P(\cdot)$ is a strongly regular solution of (4.6). Consequently, by Theorem 4.5, $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is uniformly convex. ∎

We now look at the following case:

$$D(\cdot) = 0, \qquad R(\cdot) \gg 0. \tag{5.24}$$

Note that although $D(\cdot) = 0$, since $C(\cdot)$ is not necessarily zero, our state equation is still an SDE. For such a case, the above results can be restated as follows.

THEOREM 5.7. *Let* (H1)–(H2) *and* (5.24) *hold. Then the following statements are equivalent:*

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

(i) *Problem* (SLQ) *is finite at* $t = 0$*;*
(ii) *Problem* $(\text{SLQ})^0$ *is finite at* $t = 0$*;*
(iii) *the map* $u(\cdot) \mapsto J^0(0, 0; u(\cdot))$ *is uniformly convex;*
(iv) *the Riccati equation*

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad - (PB + S^\top)R^{-1}(B^\top P + S) = 0 \quad \text{a.e. } s \in [0, T], \\ P(T) = G \end{cases} \tag{5.25}$$

admits a unique solution $P(\cdot) \in C([0, T]; \mathbb{S}^n)$*;*
(v) *Problem* (SLQ) *is uniquely closed-loop solvable;*
(vi) *Problem* (SLQ) *is uniquely open-loop solvable.*

Proof. (i) $\Rightarrow$ (ii) This part of the proof follows from Proposition 5.1.

(ii) $\Rightarrow$ (iii) By Theorem 5.3, Problem $(\text{SLQ})^0$ is finite. Since $D(\cdot) = 0, R(\cdot) \gg 0$, (5.23) holds for some $\lambda > 0$, and the result follows from Theorem 5.6.

(iii) $\Leftrightarrow$ (iv) In the case of (5.24), the corresponding Riccati equation becomes (5.25). If $P(\cdot) \in C([0, T]; \mathbb{S}^n)$ is a solution of (5.25), then it is automatically strongly regular. Thus, by Theorem 4.5, we obtain the equivalence of (iii) and (iv).

(iii) $\Rightarrow$ (vi) This part of the proof follows from Proposition 4.1, and (iv) $\Rightarrow$ (v) follows from Theorem 4.3.

Finally, (v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) are trivial. ∎

An interesting point of the above is that under condition (5.24), finiteness of Problem (SLQ) implies the closed-loop solvability of Problem (SLQ). In the deterministic case, such a fact was first revealed in [31] for two-person zero-sum differential games and was proved in [29] for deterministic LQ problems by means of Fredholm operators.

6. Minimizing sequences and open-loop solvabilities.

In section 4, we showed that under the uniform convexity condition (4.2), Problem (SLQ) is open-loop solvable and the open-loop optimal control has a linear state feedback representation. In this section, we study the open-loop solvability of Problem (SLQ) without the uniform convexity condition.

First we construct a minimizing sequence for Problem (SLQ) when it is finite.

THEOREM 6.1. *Let* (H1)–(H2) *hold. Suppose Problem* (SLQ) *is finite. For any* $\varepsilon > 0$*, let* $P_\varepsilon(\cdot)$ *be the unique strongly regular solution to the Riccati equation* (5.7)*. Further, let* $(\eta_\varepsilon(\cdot), \zeta_\varepsilon(\cdot))$ *and* $X_\varepsilon(\cdot) \equiv X_\varepsilon(\cdot\,; t, x)$ *be the (adapted) solutions to the following BSDE and closed-loop system, respectively:*

$$\begin{cases} d\eta_\varepsilon(s) = -\Big[(A + B\Theta_\varepsilon)^\top \eta_\varepsilon + (C + D\Theta_\varepsilon)^\top \zeta_\varepsilon \\ \qquad\qquad + (C + D\Theta_\varepsilon)^\top P_\varepsilon \sigma - \Theta_\varepsilon^\top \rho + P_\varepsilon b + q\Big] ds + \zeta_\varepsilon dW(s), \quad s \in [0, T], \\ \eta_\varepsilon(T) = g, \end{cases}$$

$$\begin{cases} dX_\varepsilon(s) = \Big[(A + B\Theta_\varepsilon)X_\varepsilon + Bv_\varepsilon + b\Big] ds \\ \qquad\qquad + \Big[(C + D\Theta_\varepsilon)X_\varepsilon + Dv_\varepsilon + \sigma\Big] dW(s), \quad s \in [t, T], \\ X_\varepsilon(t) = x, \end{cases}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

where

$$\begin{cases} \Theta_\varepsilon = -\big(R+\varepsilon I + D^\top P_\varepsilon D\big)^{-1}\big(B^\top P_\varepsilon + D^\top P_\varepsilon C + S\big), \\ v_\varepsilon = -\big(R+\varepsilon I + D^\top P_\varepsilon D\big)^{-1}\big(B^\top \eta_\varepsilon + D^\top \zeta_\varepsilon + D^\top P_\varepsilon \sigma + \rho\big). \end{cases} \tag{6.1}$$

Then

$$u_\varepsilon(\cdot) \triangleq \Theta_\varepsilon(\cdot)X_\varepsilon(\cdot) + v_\varepsilon(\cdot), \quad \varepsilon > 0, \tag{6.2}$$

is a minimizing sequence of $u(\cdot) \mapsto J(t,x;u(\cdot))$*:*

$$\lim_{\varepsilon\to 0} J(t,x;u_\varepsilon(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) = V(t,x). \tag{6.3}$$

Proof. For any $\varepsilon > 0$, consider state equation (1.1) and the following cost functional:

$$J_\varepsilon(t,x;u(\cdot)) \triangleq J(t,x;u(\cdot)) + \varepsilon\mathbb{E}\int_t^T |u(s)|^2 ds. \tag{6.4}$$

Denote the above problem by Problem $(\text{SLQ})_\varepsilon$ and the corresponding value function by $V_\varepsilon(\cdot\,,\cdot)$. By Corollary 4.7, $u_\varepsilon(\cdot)$ defined by (6.2) is the unique optimal control of Problem $(\text{SLQ})_\varepsilon$ at $(t,x) \in [0,T)\times\mathbb{R}^n$. Note that

$$\begin{aligned} \varepsilon\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds &= J_\varepsilon(t,x;u_\varepsilon(\cdot)) - J(t,x;u_\varepsilon(\cdot)) = V_\varepsilon(t,x) - J(t,x;u_\varepsilon(\cdot)) \\ &\leqslant V_\varepsilon(t,x) - V(t,x) \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0} J(t,x;u_\varepsilon(\cdot)) = \lim_{\varepsilon\to 0}\left[V_\varepsilon(t,x) - \varepsilon\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds\right] = V(t,x).$$

The proof is completed. ∎

Using the minimizing sequence constructed in Theorem 6.1, the open-loop solvability of Problem (SLQ) can be characterized as follows.

THEOREM 6.2. *Let* (H1)–(H2) *hold. Suppose* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is convex. Let* $(t,x) \in [0,T)\times\mathbb{R}^n$ *and* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *be the sequence defined by* (6.2)*. Then the following statements are equivalent:*

(i) *Problem* (SLQ) *is open-loop solvable at* (t,x)*;*
(ii) *the sequence* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *admits a weakly convergent subsequence;*
(iii) *the sequence* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *admits a strongly convergent subsequence.*

In this case, the weak (strong) limit of any weakly (strongly) convergent subsequence of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *is an open-loop optimal control of Problem* (SLQ) *at* (t,x)*.*

To prove Theorem 6.2, we need the following lemma whose proof is straightforward.

LEMMA 6.3. *Let* $\mathcal{H}$ *be a Hilbert space with norm* $\|\cdot\|$ *and* $\theta, \theta_n \in \mathcal{H}$, $n = 1,2,\ldots.$

(i) *If* $\theta_n \to \theta$ *weakly, then* $\|\theta\| \leqslant \liminf_{n\to\infty}\|\theta_n\|$.
(ii) $\theta_n \to \theta$ *strongly if and only if*

$$\|\theta_n\| \to \|\theta\| \quad \textit{and} \quad \theta_n \to \theta \ \textit{weakly.}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Proof of Theorem 6.2. (i) ⇒ (ii) and (i) ⇒ (iii) Let $v^*(\cdot)$ be an open-loop optimal control of Problem (SLQ) at (t,x). By Corollary 4.7, for any $\varepsilon > 0$, $u_\varepsilon(\cdot)$ defined by (6.2) is the unique optimal control of Problem $(\text{SLQ})_\varepsilon$ at (t,x) and

$$V_\varepsilon(t,x) = J_\varepsilon(t,x;u_\varepsilon(\cdot)) \geqslant V(t,x) + \varepsilon\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds. \tag{6.5}$$

Also, we have

$$V_\varepsilon(t,x) \leqslant J_\varepsilon(t,x;v^*(\cdot)) = V(t,x) + \varepsilon\mathbb{E}\int_t^T |v^*(s)|^2 ds. \tag{6.6}$$

Combining (6.5)–(6.6), we have

$$\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds \leqslant \frac{V_\varepsilon(t,x) - V(t,x)}{\varepsilon} \leqslant \mathbb{E}\int_t^T |v^*(s)|^2 ds \quad \forall\varepsilon > 0. \tag{6.7}$$

Thus, $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in the Hilbert space $\mathcal{U}[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ and hence admits a weakly convergent subsequence $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$. Let $u^*(\cdot)$ be the weak limit of $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$. Since $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex and continuous, it is hence sequentially weakly lower semicontinuous. Thus (noting (6.3)),

$$V(t,x) \leqslant J(t,x;u^*(\cdot)) \leqslant \liminf_{k\to\infty} J(t,x;u_{\varepsilon_k}(\cdot)) = V(t,x),$$

which implies that $u^*(\cdot)$ is also an open-loop optimal control of Problem (SLQ) at (t,x). Now replacing $v^*(\cdot)$ with $u^*(\cdot)$ in (6.7), we have

$$\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds \leqslant \mathbb{E}\int_t^T |u^*(s)|^2 ds \quad \forall\varepsilon > 0. \tag{6.8}$$

Also, by Lemma 6.3(i),

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds \leqslant \liminf_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}(s)|^2 ds. \tag{6.9}$$

Combining (6.8)–(6.9), we see that

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds = \lim_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}(s)|^2 ds.$$

Then it follows from Lemma 6.3(ii) that $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ converges to $u^*(\cdot)$ strongly.

(iii) ⇒ (ii) This part of the proof is obvious.

(ii) ⇒ (i) Let $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ be a weakly convergent subsequence of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ with weak limit $u^*(\cdot)$. Then $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ is bounded in $\mathcal{U}[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. For any $u(\cdot) \in \mathcal{U}[t,T]$, we have

$$J(t,x;u(\cdot)) + \varepsilon_k\mathbb{E}\int_t^T |u(s)|^2 ds \geqslant V_{\varepsilon_k}(t,x) = J(t,x;u_{\varepsilon_k}(\cdot)) + \varepsilon_k\mathbb{E}\int_t^T |u_{\varepsilon_k}(s)|^2 ds.$$

Note that $u(\cdot) \mapsto J(t,x;u(\cdot))$ is sequentially weakly lower semicontinuous. Letting $k \to \infty$ in the above, we obtain

$$J(t,x;u^*(\cdot)) \leqslant \liminf_{k\to\infty} J(t,x;u_{\varepsilon_k}(\cdot)) \leqslant J(t,x;u(\cdot)) \quad \forall u(\cdot) \in \mathcal{U}[t,T].$$

Hence, $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) at (t,x). ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

From the proof of Theorem 6.2, we see that the open-loop solvability of Problem (SLQ) at (t, x) is also equivalent to the L^2-boundedness of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$. In particular, the open-loop solvability of Problem $(\text{SLQ})^0$ at (t, x) is equivalent to the L^2-boundedness of $\{\Theta_\varepsilon(\cdot)X_\varepsilon(\cdot)\}_{\varepsilon>0}$ with $X_\varepsilon(\cdot)$ being the solution of

$$\begin{cases} dX_\varepsilon(s) = (A + B\Theta_\varepsilon)X_\varepsilon ds + (C + D\Theta_\varepsilon)X_\varepsilon dW(s), \quad s \in [t, T], \\ X_\varepsilon(t) = x. \end{cases}$$

Since the $L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$-norm of $X_\varepsilon(\cdot)$ is dominated by the L^2-norm of $\Theta_\varepsilon(\cdot)$, we conjecture that the L^2-boundedness of $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ will lead to the open-loop solvability of Problem $(\text{SLQ})^0$ at (t, x). Actually, we have the following result.

PROPOSITION 6.4. *Let* (H1)–(H2) *hold. Suppose* $u(\cdot) \mapsto J^0(0, 0; u(\cdot))$ *is convex, and let* $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ *be the sequence defined by* (6.1). *If*

$$\sup_{\varepsilon>0} \int_0^T |\Theta_\varepsilon(s)|^2 ds < \infty, \tag{6.10}$$

then the Riccati equation (4.6) *admits a regular solution* $P(\cdot) \in C([0, T]; \mathbb{S}^n)$. *Consequently, Problem* $(\text{SLQ})^0$ *is closed-loop solvable.*

Proof. For any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, let $X_\varepsilon(\cdot)$ be the solution of

$$\begin{cases} dX_\varepsilon(s) = \big[A(s) + B(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)ds + \big[C(s) + D(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)dW(s), \\ X_\varepsilon(0) = x. \end{cases}$$

By Itô's formula, we have

$$\begin{aligned} \mathbb{E}|X_\varepsilon(t)|^2 &= |x|^2 + \mathbb{E}\int_0^t \Big[|(C + D\Theta_\varepsilon)X_\varepsilon|^2 + 2\langle (A + B\Theta_\varepsilon)X_\varepsilon, X_\varepsilon\rangle\Big] ds \\ &\leqslant |x|^2 + \int_0^t \Big(|C + D\Theta_\varepsilon|^2 + 2|A + B\Theta_\varepsilon|\Big)\mathbb{E}|X_\varepsilon|^2 ds \quad \forall t \in [0, T]. \end{aligned}$$

Thus, by Gronwall's inequality,

$$\begin{aligned} \mathbb{E}|X_\varepsilon(t)|^2 &\leqslant |x|^2 \exp\left\{\int_0^T \Big[|C(s) + D(s)\Theta_\varepsilon(s)|^2 + 2|A(s) + B(s)\Theta_\varepsilon(s)|\Big] ds\right\} \\ &\leqslant |x|^2 \exp\left\{K\left(1 + \int_0^T |\Theta_\varepsilon(s)|^2 ds\right)\right\} \quad \forall t \in [0, T], \end{aligned}$$

where $K > 0$ is some constant depending only on $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$. Hence,

$$\begin{aligned} \mathbb{E}\int_0^T |\Theta_\varepsilon(s)X_\varepsilon(s)|^2 ds &\leqslant \int_0^T |\Theta_\varepsilon(s)|^2 \mathbb{E}|X_\varepsilon(s)|^2 ds \\ &\leqslant |x|^2 \exp\left\{K\left(1 + \int_0^T |\Theta_\varepsilon(s)|^2 ds\right)\right\} \int_0^T |\Theta_\varepsilon(s)|^2 ds, \end{aligned}$$

which, together with (6.10), implies the L^2-boundedness of $\{\Theta_\varepsilon(s)X_\varepsilon(s)\}_{\varepsilon>0}$. Thus, by Theorem 6.2, Problem $(\text{SLQ})^0$ is open-loop solvable at $t = 0$, and by Theorem 5.3,

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Problem $(\text{SLQ})^0$ is finite. Now let $P : [0,T] \to \mathbb{S}^n$ such that (5.1) holds. Then by Theorem 5.3,

$$R + D^\top PD \geqslant 0 \quad \text{a.e.} \tag{6.11}$$

Let $\{\Theta_{\varepsilon_k}(\cdot)\}$ be a weakly convergent subsequence of $\{\Theta_\varepsilon(\cdot)\}$ with weak limit $\Theta(\cdot)$. Since

$$R + \varepsilon I + D^\top P_\varepsilon D \to R + D^\top PD \quad \text{as } \varepsilon \to 0$$

and $\{R(\cdot) + \varepsilon I + D(\cdot)^\top P_\varepsilon(\cdot) D(\cdot)\}_{0<\varepsilon\leqslant 1}$ is uniformly bounded, we have

$$B^\top P_{\varepsilon_k} + D^\top P_{\varepsilon_k} C + S = -(R + \varepsilon_k I + D^\top P_{\varepsilon_k} D)\Theta_{\varepsilon_k} \to -(R + D^\top PD)\Theta$$

weakly in L^2. Also, note that

$$B^\top P_{\varepsilon_k} + D^\top P_{\varepsilon_k} C + S \to B^\top P + D^\top PC + S$$

strongly in L^2. Thus,

$$-(R + D^\top PD)\Theta = B^\top P + D^\top PC + S.$$

This implies

$$\mathcal{R}\big(B^\top P + D^\top PC + S\big) \subseteq \mathcal{R}\big(R + D^\top PD\big) \quad \text{a.e.} \tag{6.12}$$

Since

$$(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = -\big(R + D^\top PD\big)^\dagger \big(R + D^\top PD\big)\Theta$$

and $\big(R + D^\top PD\big)^\dagger \big(R + D^\top PD\big)$ is an orthogonal projection, we have

$$(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \in L^2(0,T;\mathbb{R}^{m\times n}), \tag{6.13}$$

and

$$\Theta = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\Pi$$

for some $\Pi(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$. Finally, letting $k \to \infty$, we have

$$\begin{aligned}
P(t) = \lim_{k\to\infty} P_{\varepsilon_k}(t) &= G + \lim_{k\to\infty} \int_t^T \Big[P_{\varepsilon_k} A + A^\top P_{\varepsilon_k} + C^\top P_{\varepsilon_k} C + Q \\
&\qquad\qquad + (P_{\varepsilon_k} B + C^\top P_{\varepsilon_k} D + S^\top)\Theta_{\varepsilon_k}\Big] ds \\
&= G + \int_t^T \Big[PA + A^\top P + C^\top PC + Q + (PB + C^\top PD + S^\top)\Theta\Big] ds \\
&= G + \int_t^T \Big[PA + A^\top P + C^\top PC + Q \\
&\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)\Big] ds,
\end{aligned}$$

which, together with (6.11)–(6.13), implies $P(\cdot)$ is a regular solution of (4.6). ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

7. Examples. In this section we present two examples to illustrate some results we obtained. In the first example, the stochastic LQ problem admits a *continuous* open-loop optimal control at all $(t, x) \in [0, T) \times \mathbb{R}^n$, and hence it is open-loop solvable, while the value function is *not* continuous in t; the corresponding Riccati equation has a unique solution $P(\cdot)$, which does *not* satisfy the range condition (4.7) and therefore is not regular. Thus, the problem is *not* closed-loop solvable on any $[t, T]$. This example also tells us that the necessity part of Theorem 4.2 (a result from [1]) does not hold.

Example 7.1. Consider the following Problem $(\mathrm{SLQ})^0$ with one-dimensional state equation:

$$\begin{cases} dX(s) = \big[u_1(s) + u_2(s)\big]ds + \big[u_1(s) - u_2(s)\big]dW(s), \quad s \in [t, 1], \\ X(t) = x \end{cases} \tag{7.1}$$

and cost functional

$$J^0(t, x; u(\cdot)) = \mathbb{E}X(1)^2. \tag{7.2}$$

In this example, $u(\cdot) = (u_1(\cdot), u_2(\cdot))^\top$ and

$$\begin{aligned} &A = 0, \quad B = (1, 1), \quad C = 0, \qquad D = (1, -1), \\ &G = 1, \quad Q = 0, \qquad S = (0, 0)^\top, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The corresponding Riccati equation reads

$$\begin{cases} \dot{P} = P^2(1, 1)\begin{pmatrix} P & -P \\ -P & P \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \dfrac{P}{4}(1, 1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0, \\ P(1) = 1. \end{cases} \tag{7.3}$$

Obviously, (7.3) has a unique solution $P(\cdot) \equiv 1$, and

$$\begin{aligned} \mathcal{R}\big(B(s)^\top P(s) + D(s)^\top P(s)C(s) + S(s)\big) &= \mathcal{R}\big((1, 1)^\top\big) = \big\{(a, a)^\top : a \in \mathbb{R}\big\}, \\ \mathcal{R}\big(R(s) + D(s)^\top P(s)D(s)\big) = \mathcal{R}\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right) &= \big\{(a, -a)^\top : a \in \mathbb{R}\big\}. \end{aligned}$$

Thus, the range condition (4.7) does not hold, and hence $P(\cdot)$ is not regular. By Theorem 4.3, the problem is not closed-loop solvable on any $[t, 1]$.

Now for any $\varepsilon > 0$, consider state equation (7.1) and the cost functional

$$J_\varepsilon^0(t, x; u(\cdot)) = \mathbb{E}\left[X(1)^2 + \varepsilon \int_t^T |u(s)|^2 ds\right]. \tag{7.4}$$

The Riccati equation for the above problem reads

$$\begin{cases} \dot{P}_\varepsilon = P_\varepsilon^2(1, 1)\begin{pmatrix} \varepsilon + P_\varepsilon & -P_\varepsilon \\ -P_\varepsilon & \varepsilon + P_\varepsilon \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \dfrac{2}{\varepsilon}P_\varepsilon^2, \\ P_\varepsilon(1) = 1, \end{cases} \tag{7.5}$$

whose solution is given by

$$P_\varepsilon(t) = \frac{\varepsilon}{\varepsilon + 2 - 2t}, \qquad t \in [0, 1]. \tag{7.6}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Letting $\varepsilon \to 0$, we have

$$P_0(t) \triangleq \lim_{\varepsilon\to 0} P_\varepsilon(t) = \begin{cases} 0, & 0 \leqslant t < 1, \\ 1, & t = 1. \end{cases} \tag{7.7}$$

Thus, by Theorem 5.3, the original Problem (SLQ)^0 is finite with value function

$$V^0(t,x) = 0, \quad 0 \leqslant t < 1; \qquad V^0(1,x) = x^2 \quad \forall x \in \mathbb{R}. \tag{7.8}$$

Next, set

$$\Theta_\varepsilon \triangleq -(R + \varepsilon I + D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon + D^\top P_\varepsilon C + S) = -\frac{P_\varepsilon}{\varepsilon}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then the solution of

$$\begin{cases} dX_\varepsilon(s) = \big[A(s) + B(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)ds + \big[C(s) + D(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)dW(s) \\ \qquad\quad = -\dfrac{2P_\varepsilon}{\varepsilon}X_\varepsilon(s)ds, \qquad s \in [t,T], \\ X_\varepsilon(t) = x \end{cases}$$

is given by

$$X_\varepsilon(s) = x\exp\left\{-\int_t^s \frac{2P_\varepsilon(r)}{\varepsilon}dr\right\} = \frac{\varepsilon + 2 - 2s}{\varepsilon + 2 - 2t}x, \qquad t \leqslant s \leqslant 1, \tag{7.9}$$

and hence

$$u_\varepsilon(s) \triangleq \Theta_\varepsilon(s)X_\varepsilon(s) = -\left(\frac{x}{\varepsilon + 2 - 2t}, \frac{x}{\varepsilon + 2 - 2t}\right)^\top, \qquad t \leqslant s \leqslant 1. \tag{7.10}$$

Note that for $t \in [0,1)$,

$$u_\varepsilon(\cdot) \to -\left(\frac{x}{2 - 2t}, \frac{x}{2 - 2t}\right)^\top \quad \text{in } L^2 \text{ as } \varepsilon \to 0.$$

Thus, by Theorem 6.2, the original Problem (SLQ)^0 is open-loop solvable at any $(t,x) \in [0,T) \times \mathbb{R}$ with an open-loop optimal control

$$u^*_{(t,x)}(s) = -\left(\frac{x}{2 - 2t}, \frac{x}{2 - 2t}\right)^\top, \qquad t \leqslant s \leqslant 1, \tag{7.11}$$

which is continuous in $s \in [t,1]$.

The following example shows that the sufficiency part of Theorem 4.2 does not hold either.

Example 7.2. Consider the following deterministic one-dimensional state equation:

$$\begin{cases} dX(s) = u(s)ds, \quad s \in [t,1], \\ X(t) = x, \end{cases}$$

and cost functional

$$J(t,x;u(\cdot)) = X(1)^2 + \int_t^1 s^2 u(s)^2 ds.$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

The Riccati equation for the above problem reads

$$(7.12)\qquad \begin{cases} \dot{P}(t) = \dfrac{P(t)^2}{t^2}\mathbf{1}_{(0,1]}(t) \quad \text{a.e. } t \in [0,1], \\ P(1) = 1. \end{cases}$$

It is easy to see that $P(t) = t$ is the unique solution of (7.12), satisfying (4.7) and (4.9). Now, we claim that this problem does not admit an open-loop optimal control for initial pair $(0, x)$ with $x \neq 0$. In fact, if for some $x \neq 0$ there exists an open-loop optimal control $u^*(\cdot) \in \mathcal{U}[0,T]$, then, by the maximum principle, the solution $(X^*(\cdot), Y^*(\cdot))$ of the (decoupled) forward-backward differential equation

$$(7.13)\qquad \begin{cases} \dot{X}^*(s) = u^*(s), \quad \dot{Y}^*(s) = 0, \qquad s \in [0,1], \\ X^*(0) = x, \quad Y^*(1) = X^*(1) \end{cases}$$

must satisfy

$$(7.14)\qquad Y^*(s) + s^2 u^*(s) = 0 \qquad \text{a.e. } s \in [0,1].$$

Observe that the solution $(X^*(\cdot), Y^*(\cdot))$ of (7.13) is given by

$$X^*(s) = x + \int_0^s u^*(r)dr, \quad Y^*(s) \equiv X^*(1), \qquad s \in [0,1].$$

Hence,

$$u^*(s) = \frac{X^*(1)}{s^2} \qquad \text{a.e. } s \in (0,1].$$

Noting that $u^*(\cdot)$ is square-integrable, we must have $X^*(1) = 0$, and hence $u^*(\cdot) = 0$. Consequently,

$$x = X^*(1) - \int_0^1 u^*(r)dr = 0,$$

which is a contradiction.

8. Conclusion. In this paper, we have studied the open-loop and closed-loop solvabilities for a general class of stochastic LQ problems with deterministic coefficients. It is observed that these two solvabilities are essentially different. A crucial result that makes our approach work is the equivalence of the strongly regular solvability of the Riccati equation and the uniform convexity of the cost functional. Such a result brings new insights into the internal structure of the LQ problem and explains the fundamental reason the weighting matrices in the cost functional could be indefinite. We expect that a theory could also be established for problems with random coefficients. For that case, the Riccati equation (4.6) would become a nonlinear BSDE whose solvability is very challenging. We hope to report some relevant results along this line in our future publications.

REFERENCES

[1] M. Ait Rami, J. B. Moore, and X. Y. Zhou, *Indefinite stochastic linear quadratic control and generalized differential Riccati equation*, SIAM J. Control Optim., 40 (2002), pp. 1296–1311, doi:10.1137/S0363012900371083.

[2] M. Ait Rami, X. Y. Zhou, and J. B. Moore, *Well-posedness and attainability of indefinite stochastic linear quadratic control in infinite time horizon*, Systems Control Lett., 41 (2000), pp. 123–133.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[3] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice–Hall, Englewood Cliffs, NJ, 1989.

[4] R. Bellman, I. Glicksberg, and O. Gross, *Some Aspects of the Mathematical Theory of Control Processes*, RAND Corporation, Santa Monica, CA, 1958.

[5] A. Bensoussan, *Lectures on Stochastic Control, Part* I, in Nonlinear Filtering and Stochastic Control, Lecture Notes in Math. 972, Springer-Verlag, Berlin, 1982, pp. 1–39.

[6] S. Chen, X. Li, and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim., 36 (1998), pp. 1685–1702, doi:10.1137/S0363012996310478.

[7] S. Chen and J. Yong, *Stochastic linear quadratic optimal control problems with random coefficients*, Chinese Ann. Math. Ser. B, 21 (2000), pp. 323–338.

[8] S. Chen and J. Yong, *Stochastic linear quadratic optimal control problems*, Appl. Math. Optim., 43 (2001), pp. 21–45.

[9] S. Chen and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs.* II, SIAM J. Control Optim., 39 (2000), pp. 1065–1081, doi:10.1137/S0363012998346578.

[10] D. Clements, B. D. O. Anderson, and P. J. Moylan, *Matrix inequality solution to linear-quadratic singular control problems*, IEEE Trans. Automat. Control, 22 (1977), pp. 55–57.

[11] M. H. A. Davis, *Linear Estimation and Stochastic Control*, Chapman and Hall, London, 1977.

[12] K. Du, *Solvability conditions for indefinite linear quadratic optimal stochastic control problems and associated stochastic Riccati equations*, SIAM J. Control Optim., 53 (2015), pp. 3673–3689, doi:10.1137/140956051.

[13] Y. Hu and X. Y. Zhou, *Indefinite stochastic Riccati equations*, SIAM J. Control Optim., 42 (2003), pp. 123–137, doi:10.1137/S0363012901391330.

[14] J. Huang, X. Li, and J. Yong, *A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon*, Math. Control Relat. Fields, 5 (2015), pp. 97–139.

[15] R. E. Kalman, *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana, 5 (1960), pp. 102–119.

[16] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer-Verlag, New York, 1991.

[17] M. Kohlmann and S. Tang, *Minimization of risk and linear quadratic optimal control theory*, SIAM J. Control Optim., 42 (2003), pp. 1118–1142, doi:10.1137/S0363012900372465.

[18] A. M. Letov, *The analytical design of control systems*, Automat. Remote Control, 22 (1961), pp. 363–372.

[19] A. E. B. Lim and X. Y. Zhou, *Stochastic optimal LQR control with integral quadratic constraints and indefinite control weights*, IEEE Trans. Automat. Control, 44 (1999), pp. 359–369.

[20] L. Mou and J. Yong, *Two-person zero-sum linear quadratic stochastic differential games by a Hilbert space method*, J. Ind. Manag. Optim., 2 (2006), pp. 95–117.

[21] R. Penrose, *A generalized inverse of matrices*, Proc. Cambridge Philos. Soc., 52 (1955), pp. 17–19.

[22] Z. Qian and X. Y. Zhou, *Existence of solutions to a class of indefinite stochastic Riccati equations*, SIAM J. Control Optim., 51 (2013), pp. 221–229, doi:10.1137/120873777.

[23] J. Sun and J. Yong, *Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points*, SIAM J. Control Optim., 52 (2014), pp. 4082–4121, doi:10.1137/140953642.

[24] S. Tang, *General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations*, SIAM J. Control Optim., 42 (2003), pp. 53–75, doi:10.1137/S0363012901387550.

[25] S. Tang, *Dynamic programming for general linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., 53 (2015), pp. 1082–1106, doi:10.1137/140979940.

[26] W. M. Wonham, *On a matrix Riccati equation of stochastic control*, SIAM J. Control, 6 (1968), pp. 681–697, doi:10.1137/0306044.

[27] H. Wu and X. Y. Zhou, *Stochastic frequency characteristics*, SIAM J. Control Optim., 40 (2001), pp. 557–576, doi:10.1137/S0363012900373756.

[28] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*, SIAM J. Control Optim., 51 (2013), pp. 2809–2838, doi:10.1137/120892477.

[29] J. Yong, *Differential Games: A Concise Introduction*, World Scientific, Hackensack, NJ, 2015.

[30] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

[31] P. Zhang, *Some results on two-person zero-sum linear quadratic differential games*, SIAM J. Control Optim., 43 (2005), pp. 2157–2165, doi:10.1137/S036301290342560X.

Downloaded 07/19/22 to 158.132.161.181 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.