CUSTOMER ORDER SCHEDULING TO MINIMIZE TOTAL WEIGHTED COMPLETION TIME

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Abstract In this paper we study the scheduling problem in which each customer order consists of several jobs of different types, which are to be processed on m facilities. Each facility is dedicated to the processing of only one type of jobs. All jobs of an order have to be delivered to the customer at the same time. The objective is to schedule all the orders to minimize the total weighted order completion time. While the problem has been shown to be unary NP-hard, we develop a heuristics to tackle the problem and analyze its worst-case performance.

Keywords: customer order scheduling, linear programming relaxation, approximation algorithm

1. Introduction

In this paper we consider the problem of scheduling customer orders on multiple facilities to minimize the weighted order completion time. We are given n customer orders to be processed on m facilities. Each order may consist of several jobs of different types (families). Each job is processed by a facility dedicated to the processing of its type of jobs. All jobs of an order have to be delivered to the customer at the same time. The objective is to schedule the orders on the facilities so as to minimize the total weighted order completion time.

There are several papers dealing with the multiple facility customer order total weighted order completion time scheduling problem. Wagneur and Sriskandarajah (1993) showed that the problem to minimize the total order completion time is unary NP-hard even when $m = 2$. Unfortunately, as pointed out by Leung et al (2005), their proof is not correct. The complexity status of this two machine problem remains open. However, Leung et al (2002) proved that the total order completion time is NP-hard in the strong sense when $m = 3$. Sung and Yoon (1998) showed that the worst-case performance of the weighted shortest processing time (WSPT) rule for permutation schedules is 2 for $m = 2$. Wang and Cheng (2003) and Leung et al (2005) established some heuristics based on WSPT rule and analyzed their worst-case performance. In this paper, we will provide a new heuristic to tackle the problem and analyze their worst-case error bounds.

We first give a formal description of the problem under study. There is a set of orders $N = \{O_1, \ldots, O_n\}$ to be processed. Each order O_j consists of m jobs $J_{1j}, \ldots, J_{mj}$. A job J_{ij} is to be processed on facility M_i, $i = 1, \ldots, m$. The processing time of J_{ij} is denoted as p_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Associated with each order O_j is a weight w_j. We assume that all p_{ij} and w_j are non-negative integers.

We define the following variables for a given schedule σ:

$C_{ij}(\sigma) =$ the completion time of J_{ij};

$C_j(\sigma) = \max_i\{C_{ij}(\sigma)\}$, the completion time of O_j.

When there is no ambiguity, we simplify $C_{ij}(\sigma)$ and $C_j(\sigma)$ as C_{ij} and C_j, respectively. The objective function that we consider requires the minimization of $\sum w_j C_j$. We denote our problem as **P**. We will also simplify $\sum w_j C_j$ as F and denote the optimal solution as F^*.

We assume that each order cannot contain more that one job to be processed on the same facility. This is not a strict assumption as it seems because if there are more than one job to be processed on the same facility, they can be combined into one job that, once started, has to be processed without interruption until completion.

We note that our search for an optimal solution can be restricted to the class of schedules in which each of the facilities M_i, $i = 1, \ldots, m$, starts at time zero and has no intermediate idle time. Furthermore, it also suffices to consider permutation schedules, i.e., schedules in which all orders are sequenced in the same order on all facilities.

2. Linear Programming Relaxation

In this section we present an approximation algorithm with data independent worst-case performance for **P** based on the linear programming relaxation. Our algorithm is inspired by recent work on the unrelated parallel machine total weighted completion time minimization scheduling problem with release times by Hall et al (1997) and Phillips et al (1998). For the unrelated parallel machine total weighted completion time problem with release times, Phillips et al developed an 8-approximation algorithm for the preemptive version, while Hall et al established a $\frac{16}{3}$-approximation algorithm for the nonpreemptive version.

Similar to that of Hall et al (1997), we introduce an interval-indexed formulation of our problem. We divide the time horizon of potential completion times into the following intervals: $[0, 1]$, $(1, 2]$, $(2, 4], \ldots, (2^{L-1}, 2^L]$, where L is chosen to be the smallest integer such that $2^L \geq \max_i\{\sum_{j=1}^{n} p_{ij}\}$. For conciseness, let $\tau_0 = 0$, and $\tau_l = 2^l$, $l = 1, \ldots, L$, and so the lth interval runs from τ_{l-1} to τ_l, $l = 1, \ldots, L$. Let the decision variable x_{jl} indicate if order O_j is scheduled to complete within the interval $(\tau_{l-1}, \tau_l]$. Consider the following linear programming relaxation LP:

$$\text{minimize} \sum_{j=1}^{n} w_j \sum_{l=1}^{L} \tau_{l-1} x_{jl} \tag{1}$$

subject to

$$\sum_{l=1}^{L} x_{jl} = 1, \quad j = 1, \ldots, n; \tag{2}$$

$$\sum_{k=1}^{l} \sum_{j=1}^{n} p_{ij} x_{jk} \leq \tau_l, \quad i = 1, \ldots, m, \quad l = 1, \ldots, L; \tag{3}$$

$$x_{jl} \geq 0, \quad j = 1, \ldots, n, \quad l = 1, \ldots, L. \tag{4}$$

It is not difficult to see that LP is a relaxation of the integer programming formulation of **P**, and the solution of LP is an obvious lower bound for the problem. So we have the following result.

Lemma 1 *For* **P**, *the optimal value of LP is a lower bound for the optimal total weighted completion time* F^*.

Based on linear programming relaxation, we propose the following heuristic.

Algorithm HLP:

(i) Solve LP and let the optimal solution be $\overline{x}_{jl}$, $j = 1, \ldots, n$, $l = 1, \ldots, L$.

(ii) Let $\overline{C}_j = \sum_{l=1}^{L} \tau_{l-1} \overline{x}_{jl}, \quad j = 1, \ldots, n$.

(iii) Schedule the jobs in a nondecreasing order of $\overline{C}_j$. Let F_{H4} the corresponding objective function value.

Theorem 1 $F_{HLP}/F^* \leq \frac{16}{3}$.

Proof First, observe that the optimal solution of LP is precisely $\sum_{j=1}^n w_j \overline{C}_j$, and so $F^* \geq \sum_{j=1}^n w_j \overline{C}_j$ from Lemma 2.

Define a series α_h, $h = 1, \ldots, \infty$, as follows:

$$\alpha_1 = \frac{1}{4}, \tag{5}$$

$$\alpha_h = 1 - \sum_{g=1}^{h-1} \alpha_g - \frac{1-\alpha_1}{1+\sum_{g=1}^{h-1}\alpha_g}, \quad h = 2, \ldots, \infty. \tag{6}$$

It is not difficult to show that $\alpha_h > 0$, $\sum_{g=1}^h \alpha_g \leq \frac{1}{2}$, and

$$\lim_{h\to\infty} \sum_{g=1}^h \alpha_g = \frac{1}{2}. \tag{7}$$

Now assume, without loss of generality, that all orders are indexed in a nondecreasing order of $\overline{C}_j$, i.e., $\overline{C}_{j-1} \leq \overline{C}_j$, $j = 1, \ldots, n$. For any O_j, suppose that $\tau_{u-1} < \overline{C}_j \leq \tau_u$ for some u. We consider the following three cases.

Case 1 $\overline{C}_j < \frac{5\tau_{u-1}}{4}$.

For any $k = 1, \ldots, j$, we have

$$\frac{5}{4}\tau_{u-1} > \overline{C}_j \geq \overline{C}_k = \sum_{l=1}^{L} \tau_{l-1}\overline{x}_{kl} > \tau_u \sum_{l=u+1}^{L} \overline{x}_{kl} = \tau_u \left(1 - \sum_{l=1}^{u} \overline{x}_{kl}\right),$$

and so

$$\sum_{l=1}^{u} \overline{x}_{kl} > \frac{3}{8}.$$

Hence, we have

$$C_j = \max_i \left\{\sum_{k=1}^{j} p_{ik}\right\} = \max_i \left\{\sum_{k=1}^{j} p_{ik}\right\} \sum_{l=1}^{u} \overline{x}_{kl} / \sum_{l=1}^{u} \overline{x}_{kl} \leq \max_i \left\{\sum_{l=1}^{u}\sum_{k=1}^{n} p_{ik}\overline{x}_{kl}\right\} / \sum_{l=1}^{u} \overline{x}_{kl}.$$

From (3), we have

$$C_j < \frac{\tau_u}{\sum_{l=1}^{u} \overline{x}_{kl}} < \frac{16}{3}\tau_u < \frac{16}{3}\overline{C}_j.$$

Case 2 $\frac{5\tau_{u-1}}{4} \leq \overline{C}_j < \frac{3\tau_{u-1}}{2}$.

From (7), we know that there exists some h such that

$$\left(1+\sum_{g=1}^{h-1}\alpha_g\right)\tau_{u-1} < \overline{C}_j \le \left(1+\sum_{g=1}^{h}\alpha_g\right)\tau_{u-1}. \qquad (8)$$

For any $k = 1, \ldots, j.$, we have

$$\tau_{u-1}\left(1+\sum_{g=1}^{h}\alpha_g\right) \ge \overline{C}_j \ge \overline{C}_k = \sum_{l=1}^{L}\tau_{l-1}\overline{x}_{kl} > \tau_u \sum_{l=u+1}^{L}\overline{x}_{kl} = \tau_u\left(1-\sum_{l=1}^{u}\overline{x}_{kl}\right),$$

and so

$$\sum_{l=1}^{u}\overline{x}_{kl} > \frac{1-\sum_{g=1}^{h}\alpha_g}{2}.$$

Hence, we have

$$C_j = \max_i\left\{\sum_{k=1}^{j}p_{ik}\right\} < \frac{\tau_u}{\sum_{l=1}^{u}\overline{x}_{kl}} < \frac{2\tau_u}{1-\sum_{g=1}^{h}\alpha_g}.$$

From (8), we have $\tau_{u-1} < \overline{C}_j/(1+\sum_{g=1}^{h-1}\alpha_g)$, and so

$$C_j < \frac{4\overline{C}_j}{(1-\sum_{g=1}^{h}\alpha_g)(1+\sum_{g=1}^{h-1}\alpha_g)}.$$

From (6), we have

$$\left(1-\sum_{g=1}^{h}\alpha_g\right)\left(1+\sum_{g=1}^{h-1}\alpha_g\right) = \left(\alpha_h + \frac{1-\alpha_1}{1+\sum_{g=1}^{h-1}\alpha_g}\right)\left(1+\sum_{g=1}^{h}\alpha_g\right) > 1-\alpha_1.$$

Hence,

$$C_j < \frac{4\overline{C}_j}{1-\alpha_1} < \frac{16}{3}\overline{C}_j.$$

Case 3 $\overline{C}_j \ge \frac{3\tau_{u-1}}{2}$.

Following an argument similar to that of Case 1, for any $k = 1, \ldots, j$, we have

$$\tau_u \ge \overline{C}_j \ge \overline{C}_k = \sum_{l=1}^{L}\tau_{l-1}\overline{x}_{kl} > \tau_{u+1}\sum_{l=u+2}^{L}\overline{x}_{kl} = \tau_{u+1}\left(1-\sum_{l=1}^{u+1}\overline{x}_{kl}\right),$$

and so

$$\sum_{l=1}^{u+1}\overline{x}_{kl} > \frac{1}{2}.$$

Hence, we have

$$C_j = \max_i \left\{ \sum_{k=1}^{j} p_{ik} \right\} < \frac{\tau_{u+1}}{\sum_{l=1}^{u+1} \overline{x}_{kl}} < \frac{16}{3} \overline{C}_j.$$

The proof is complete.

We notice the rapid development in research on the unrelated parallel machine total weighted completion time scheduling problem with release times in the past few years (Schulz and Skutella 1997, Skutella 1999, and Afrati et al 1999, for instance). It will be interesting to investigate whether the new results in this area can be extended to our problem.

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References

Afrati, F., E. Bampis, C. Chekuri, D. Karger, C. Kenyon, S. Khanna, I. Milis, M. Queyranne, M. Skutella, C. Stain, and M. Sviridenko (1999). Approximation schemes for minimizing average weighted completion time with release dates, *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science*, to appear.

Hall, L. A., Schulz, A. S., Shmoys, D. B., and Wein, J., 1997. Scheduling to minimize average completion time: off-line and on-line approximation algorithms, *Mathematics of Operations Research*, 22, 513-544.

Leung, J. Y-T., H. Li, and M. Pinedo (2002). Order scheduling in an environment with dedicated resources in parallel, *Journal of Scheduling*, to appear.

Leung, J. Y-T., H. Li, and M. Pinedo (2005). Order scheduling models: an overview, in G. Kendall, E. Burke, S. Petrovic, and M. Gendreau (eds.): *Multidisciplinary Scheduling*, 37-56.

Leung, J. Y-T., H. Li, M. Pinedo, C. Sriskandarajah (2005). Open shops with jobs overlap - revisited, *European Journal of Operational Research*, 163, 569C57.

Phillips, C., Stein, C., and Wein, J., 1998. Minimizing average completion time in the presence of release dates, *Mathematical Programming*, 82, 199-233.

Schulz, A. S., and M. Skutella (1997). Scheduling-LPs bear probabilities: randomized approximations for min-sum criteria, in R. Burkard, and G. Woeginger (eds.): *Algorithms - ESA '97, Lecture Notes in Computer Science*, 1284, 416-429.

Skutella, M. S. (1999). Convex quadratic programming relaxations for network scheduling problems, in J. Nesetril (eds.): *Algorithms - ESA '99, Lecture Notes in Computer Science*, 1643, 127-138.

Sung, C. S., and S. H. Yoon (1998). Minimizing total weighted completion time at a pre-assembly stage composed of two feeding machines, *International Journal of Production Economics*, 54, 247-255.

Wagneur, E., and C. Sriskandarajah (1993). Open shops with jobs overlap, *European Journal of Operational Research*, 71, 366-378.

Wang, G., and T. C. E. Cheng, Customer order scheduling to minimize total weighted completion time, in *Proceedings of the 1st Multidisciplinary Comference on Scheduling Theory and Applications, 409-416.*