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A MULTISCALE APPROACH FOR OPTIMAL CONTROL PROBLEMS OF LINEAR PARABOLIC EQUATIONS*

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Abstract. This paper discusses multiscale analysis for optimal control problems of linear parabolic equations with rapidly oscillating coefficients that depend on spatial and temporal variables. There are mainly three new results in the present paper. First, we obtain the convergence results with an explicit convergence rate for the multiscale asymptotic expansions of the solution of the optimal control problem in the case without constraints. Second, for a general bounded Lipschitz polygonal domain, the boundary layer solution is defined and the corresponding convergence results are also derived. Finally, an explicit convergence rate $\varepsilon^{1/2}$ in the presence of constraint is reported.

Key words. optimal control, parabolic equation with rapidly oscillating coefficients, multiscale asymptotic expansion, boundary layer solution

AMS subject classifications. 65F10, 78M05

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1. Introduction.

In this paper we consider the optimal control governed by parabolic equations with rapidly oscillating coefficients. For a control $v^\varepsilon \in \mathcal{U}_{ad}$, the state of the system $y^\varepsilon(v^\varepsilon(x,t))$ is given by the solution of

$$(1.1)\quad \begin{cases} \dfrac{\partial y^\varepsilon(v^\varepsilon)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a^\varepsilon_{ij}(x,t)\dfrac{\partial y^\varepsilon(v^\varepsilon)}{\partial x_j}\Big) = f(x,t) + v^\varepsilon(x,t),\ (x,t)\in\Omega\times(0,T),\\ y^\varepsilon(v^\varepsilon) = g_0(x,t), \quad (x,t)\in\Gamma_0\times(0,T),\\ \sigma_\varepsilon(y^\varepsilon(v^\varepsilon)) \equiv \nu_i a^\varepsilon_{ij}(x,t)\dfrac{\partial y^\varepsilon(v^\varepsilon)}{\partial x_j} = g_1(x,t), \quad (x,t)\in\Gamma_1\times(0,T),\\ y^\varepsilon(v^\varepsilon)|_{t=0} = \phi_0(x), \quad x\in\Omega, \end{cases}$$

where $\Omega \subset R^n$ is a bounded Lipschitz convex polygonal domain or a bounded smooth domain, $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, and Γ_0 and Γ_1 are, respectively, the Dirichlet and Neumann boundaries. $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ is the outward unit normal to Γ_1. Here $v^\varepsilon(x,t)$ is a control function; $f(x,t)$, $g_0(x,t)$, $g_1(x,t)$, and $\phi_0(x)$ are known functions; and $\varepsilon > 0$ is a small parameter which represents the relative size of a periodic cell. Here and below, the Einstein summation convention is used: summation is taken over

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repeated indices. We use the notation of Sobolev spaces given in Lions and Magenes' classical book [21].

Remark 1.1. In this paper, we assume that the coefficients $a^{\varepsilon}_{ij}(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$. According to the relation between spatial and temporal variables, we consider explicitly four specific cases; i.e., $k = 0, 1, 2, 3$; see [5]. For the sake of simplicity, we write $y^{\varepsilon}(v^{\varepsilon})$ instead of $y^{\varepsilon,k}(v^{\varepsilon,k})$ in (1.1). Other functions or functionals are similar in what follows.

Let $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-k}t$, $k = 0, 1, 2, 3$, and assume the following:

(A_1) For $k = 1, 2, 3$, $a_{ij}(\xi, \tau)$ are 1-periodic and τ_0-periodic in ξ, τ, respectively, and for $k = 0$, $a_{ij}(\xi, t)$ is 1-periodic in ξ.

(A_2) $a_{ij}(\xi, \tau) = a_{ji}(\xi, \tau)$, $a_{ij} \in L^{\infty}(R^n_{\xi} \times R_{\tau})$.

(A_3) $a_{ij}(\xi, \tau)\eta_i\eta_j \geq \sigma_0|\eta|^2$, $|\eta|^2 = \eta_i\eta_i$, $\sigma_0 > 0$ is a constant.

(A_4) $f \in L^2(0, T; L^2(\Omega))$, $g_0 \in L^2(0, T; H^{1/2}(\Gamma_0))$, $g_1 \in L^2(0, T; L^2(\Gamma_1))$, $\phi_0 \in H^1(\Omega)$.

The cost function is given by

$$\mathcal{J}_{\varepsilon}(v^{\varepsilon}) = \int_0^T \int_{\Omega} |y^{\varepsilon}(v^{\varepsilon}) - z_d|^2 dxdt + \int_0^T \int_{\Omega_U} \gamma(v^{\varepsilon})^2 dxdt, \tag{1.2}$$

where $\Omega_U \subset \Omega$ is a control domain, $\mathcal{U} = L^2(0, T; L^2(\Omega_U))$, the set of admissible controls $\mathcal{U}_{ad}$ is a closed nonempty convex subset of $\mathcal{U}$, $z_d \in \mathcal{U}_{ad}$ is a given element, $\gamma = \gamma(x, t) \geq \gamma_0 > 0$ is a given function, and $\gamma \in L^{\infty}(\Omega \times (0, T))$.

The optimal control problem is to find $u^{\varepsilon} \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}_{\varepsilon}(u^{\varepsilon}) = \inf_{v^{\varepsilon} \in \mathcal{U}_{ad}} \mathcal{J}_{\varepsilon}(v^{\varepsilon}). \tag{1.3}$$

By setting $\mathcal{A}_{\varepsilon} w \equiv -\frac{\partial}{\partial x_i}(a^{\varepsilon}_{ij}(x,t)\frac{\partial w}{\partial x_j})$, we have the following lemma.

LEMMA 1.1 (see [23, Thm. 2.1, p. 114]). *If conditions* (A$_2$)–(A$_4$) *are satisfied, then the optimal control* $u^{\varepsilon}(x,t)$ *of* (1.1)–(1.3) *is characterized through the unique solution* $\{y^{\varepsilon}(u^{\varepsilon}), \psi^{\varepsilon}(u^{\varepsilon}), u^{\varepsilon}(x,t)\}$ *of the optimality system given by*

$$\begin{cases} \frac{\partial}{\partial t} y^{\varepsilon}(u^{\varepsilon}) + \mathcal{A}_{\varepsilon} y^{\varepsilon}(u^{\varepsilon}) = f(x,t) + u^{\varepsilon}(x,t), \quad (x,t) \in \Omega \times (0,T), \\ y^{\varepsilon}(u^{\varepsilon}) = g_0(x,t), \quad (x,t) \in \Gamma_0 \times (0,T), \\ \sigma_{\varepsilon}(y^{\varepsilon}(u^{\varepsilon})) \equiv \nu_i a^{\varepsilon}_{ij}(x,t) \dfrac{\partial y^{\varepsilon}(u^{\varepsilon})}{\partial x_j} = g_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T), \\ y^{\varepsilon}(u^{\varepsilon})|_{t=0} = \phi_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

$$\begin{cases} -\frac{\partial}{\partial t} \psi^{\varepsilon}(u^{\varepsilon}) + \mathcal{A}_{\varepsilon} \psi^{\varepsilon}(u^{\varepsilon}) = y^{\varepsilon}(u^{\varepsilon}) - z_d(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \psi^{\varepsilon}(u^{\varepsilon}) = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ \sigma_{\varepsilon}(\psi^{\varepsilon}(u^{\varepsilon})) \equiv \nu_i a^{\varepsilon}_{ij}(x,t) \dfrac{\partial \psi^{\varepsilon}(u^{\varepsilon})}{\partial x_j} = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ \psi^{\varepsilon}(u^{\varepsilon})|_{t=T} = 0, \quad x \in \Omega, \end{cases} \tag{1.5}$$

$$\int_0^T \int_{\Omega_U} (\psi^{\varepsilon} + \gamma u^{\varepsilon})(v^{\varepsilon} - u^{\varepsilon}) dxdt \geq 0 \quad \forall v^{\varepsilon} \in \mathcal{U}_{ad},\ u^{\varepsilon} \in \mathcal{U}_{ad}, \tag{1.6}$$

where $u^\varepsilon \in L^2(0,T;L^2(\Omega_U))$, $y^\varepsilon(u^\varepsilon)$, $\psi^\varepsilon(u^\varepsilon) \in L^2(0,T;V)$, *and* $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. *Specifically, in the case without any constraints, i.e.,* $\Omega_U = \Omega, \mathcal{U}_{ad} = L^2(0,T;L^2(\Omega))$, *we have*

$$u^\varepsilon = -\gamma^{-1}\psi^\varepsilon(u^\varepsilon), \quad (x,t) \in \Omega \times (0,T). \tag{1.7}$$

The optimal control problem (1.1)–(1.3) arises in composite media, that is, media with a large number of heterogeneities (inclusions or holes). In such cases, the direct accurate numerical computation of the solution is difficult because it requires a very fine mesh. We recall that the homogenization method gives the overall behavior by incorporating the fluctuations due to the heterogeneities.

Lions [22] first studied the homogenization method for the control problem and the observation on the boundary for parabolic equations with rapidly oscillating coefficients, which are independent of time t, and derived convergence results in the periodic case. Fabre, Puel, and Zuazua [13] investigated the homogenization method for approximate controllability of the semilinear case with Dirichlet boundary conditions by means of a fixed point technique. Kesavan and Saint Jean Paulin [16] obtained homogenization results in nonperiodic cases in the framework of H-convergence and extended them to optimal control systems governed by elliptic boundary value problems in perforated domains (see [17]). Donato and Nabil [12] presented the homogenization and correctors for an approximate controllability problem of the linear heat equation with rapidly oscillating coefficients in a periodically perforated domain. Conca, Osses, and Saint Jean Paulin [11] studied a limit control for a semilinear elliptic equation with a uniformly Lipschitz nonlinearity and rapidly oscillating coefficients in a perforated domain with the control distributed on a compact subset interior to the domain (also see [25]). The periodic unfolding method presented by Cioranescu, Damlanian, and Griso (cf. [9]) is a significant breakthrough in this field. It has a great number of applications; see [10]. In addition, there are many other results, e.g., see [3, 4, 26].

The numerical results have shown that the accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small (see [6, 8]). Hence one hopes to seek the multiscale asymptotic methods and the associated numerical algorithms in a number of engineering applications. Lions [20] presented the asymptotic expansions for a type of optimal control of a stiff state equation with small parameter ε and derived the convergence results. Cao [7] investigated the optimal control on the boundary for second order elliptic equation with rapidly oscillating coefficients and obtained the multiscale asymptotic expansions of the solution for the problem in the case without any constraints, and the homogenized result in the case with constraints. However, an explicit convergence rate in the presence of constraints was not derived in [7].

Compared with elliptic equations, to the best of our knowledge there are few results in the literature on multiscale asymptotic methods for optimal control problem like (1.1)–(1.3). The new contributions obtained in this paper are the determination of the convergence rate for the approximate solutions and the definition of boundary layer solutions.

The remainder of this paper is organized as follows. In section 2, the multiscale expansion of the solution for the optimal control problem (1.1)–(1.3) is presented. The boundary layer solutions in the cases without constraints are defined in section 3. In section 4, the convergence results and their proofs in the case without constraints are provided. Finally, an explicit convergence rate in the presence of constraints is derived. By C we shall denote a positive constant independent of ε.

2. Multiscale asymptotic expansions and regularity of the optimal control.

2.1. Multiscale asymptotic expansions. In this section, we first present the multiscale asymptotic expansions of the solution for the optimal control problem (1.4)–(1.6). Following the idea of [1], the multiscale asymptotic expansions of the solutions $y^\varepsilon(u^\varepsilon(x,t))$ and $\psi^\varepsilon(u^\varepsilon(x,t))$ of problems (1.4) and (1.5) can be defined as follows:

$$y_s^\varepsilon(x,t) = \begin{cases} y^0(u^0(x,t)) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial y^0(u^0(x,t))}{\partial x_{\alpha_1}}, \quad s=1, \\ y^0(u^0(x,t)) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial y^0(u^0(x,t))}{\partial x_{\alpha_1}} \\ \qquad + \varepsilon^2 N_{\alpha_1\alpha_2}(\xi,\tau)\dfrac{\partial^2 y^0(u^0(x,t))}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, \quad s=2, \end{cases} \tag{2.1}$$

$$\psi_s^\varepsilon(x,t) = \begin{cases} \psi^0(u^0(x,t)) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial \psi^0(u^0(x,t))}{\partial x_{\alpha_1}}, \quad s=1, \\ \psi^0(u^0(x,t)) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial \psi^0(u^0(x,t))}{\partial x_{\alpha_1}} \\ \qquad + \varepsilon^2 N_{\alpha_1\alpha_2}(\xi,\tau)\dfrac{\partial^2 \psi^0(u^0(x,t))}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, \quad s=2. \end{cases} \tag{2.2}$$

Remark 2.1. In (2.1) and (2.2), $y_1^\varepsilon(x,t)$, $\psi_1^\varepsilon(x,t)$ and $y_2^\varepsilon(x,t)$, $\psi_2^\varepsilon(x,t)$ are called the first order and the second order multiscale asymptotic solutions associated with $y^\varepsilon(u^\varepsilon)$, $\psi^\varepsilon(u^\varepsilon)$, respectively.

Remark 2.2. Since the definitions of cell functions $N_{\alpha_1}(\xi,\tau)$, $N_{\alpha_1\alpha_2}(\xi,\tau)$, α_1, $\alpha_2 = 1,\dots,n$ are tediously long, we omit them in this paper. The details for them can be found in [1]. For cell functions, we have the following lemma.

LEMMA 2.1 (see [1]). *Let $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-k}t$, $k = 0,1,2,3$. It can be proved that N_{α_1}, $N_{\alpha_1\alpha_2} \in H^{1,1}(Q\times(0,\tau_*))$, $\alpha_1,\alpha_2 = 1,2,\dots,n$, where $Q=(0,1)^n$, $\tau_* = \tau_0$ for $k=1,2,3$; $\tau_* = T$ for $k=0$.*

The homogenized control systems associated with (1.4)–(1.6) can be written as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}y^0(u^0) + \mathcal{A}_0 y^0(u^0) = f(x,t) + u^0(x,t), \quad (x,t)\in\Omega\times(0,T), \\ y^0(u^0) = g_0(x,t), \quad (x,t)\in\Gamma_0\times(0,T), \\ \hat\sigma(y^0(u^0)) \equiv \nu_i \hat a_{ij}\dfrac{\partial y^0(u^0)}{\partial x_j} = g_1(x,t), \quad (x,t)\in\Gamma_1\times(0,T), \\ y^0(u^0)|_{t=0} = \phi_0(x), \quad x\in\Omega, \end{cases} \tag{2.3}$$

$$\begin{cases} -\dfrac{\partial}{\partial t}\psi^0(u^0) + \mathcal{A}_0 \psi^0(u^0) = y^0(u^0) - z_d(x,t), \quad (x,t)\in\Omega\times(0,T), \\ \psi^0(u^0) = 0, \quad (x,t)\in\Gamma_0\times(0,T), \\ \hat\sigma(\psi^0(u^0)) \equiv \nu_i \hat a_{ij}\dfrac{\partial \psi^0(u^0)}{\partial x_j} = 0, \quad (x,t)\in\Gamma_1\times(0,T), \\ \psi^0(u^0)|_{t=T} = 0, \quad x\in\Omega, \end{cases} \tag{2.4}$$

$$\int_0^T \int_{\Omega_U} (\psi^0(u^0) + \gamma u^0)(v - u^0) dx dt \geq 0 \quad \forall v \in \mathcal{U}_{ad},\ u^0 \in \mathcal{U}_{ad}, \tag{2.5}$$

where the homogenized operator $\mathcal{A}_0 \equiv -\frac{\partial}{\partial x_i}(\hat{a}_{ij}\frac{\partial}{\partial x_j})$, and the explicit formulas for the homogenized coefficients $\hat{a}_{ij}$ are given in [1].

Remark 2.3. It can be proved that the homogenized operator $\mathcal{A}_0$ is a real symmetric and positive-definite elliptic operator (cf. [1, 5]). Furthermore, following along the lines of the proof of Lemma 1.1, we can prove that there is a unique solution $\{y^0(u^0),\ \psi^0(u^0),\ u^0(x,t)\}$ of the homogenized optimal control system (2.3)–(2.5), where $u^0 \in L^2(0,T;L^2(\Omega_U))$, $y^0(u^0), p^0(u^0) \in L^2(0,T;V)$, and $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.

2.2. Regularity of the optimal control. In order to give the multiscale asymptotic expansions (2.1) and (2.2), the regularity of $y^0(u^0), \psi^0(u^0) \in H^{2,1}(\Omega \times (0,T))$ is required. This regularity of the optimal control is not generally true. However, we can obtain the higher order regularity of the state function $y^0(u^0)$ and the adjoint function $\psi^0(u^0)$ of the homogenized optimal control system (2.3)–(2.5) in the following specific cases.

In the case without any constraints, i.e., $\mathcal{U}_{ad} = L^2(0,T;L^2(\Omega))$, inequality (2.5) reduces to

$$u^0(x,t) = -\gamma^{-1}(x,t)\psi^0(u^0), \quad (x,t) \in \Omega \times (0,T). \tag{2.6}$$

Since the homogenized coefficients $\hat{a}_{ij}$ are constants or sufficiently regular, if we suppose that $\partial\Omega \in C^2$, $\partial\Omega$ is a pure Dirichlet boundary, $f, z_d \in L^2(\Omega \times (0,T))$, $\gamma \in H^{2,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{3}{2},\frac{3}{4}}(\partial\Omega \times (0,T))$, and $\phi_0 \in H^1(\Omega)$, then we may verify that $\psi^0(u^0) \in H^{2,1}(\Omega \times (0,T))$, $u^0 \in H^{2,1}(\Omega \times (0,T))$, and $y^0(u^0) \in H^{2,1}(\Omega \times (0,T))$ (see [23, p. 119]). Furthermore, if $f, z_d \in H^{2,1}(\Omega \times (0,T))$, $\gamma \in H^{2,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{7}{2},\frac{7}{4}}(\partial\Omega \times (0,T))$, $\phi_0 \in H^3(\Omega)$, and the compatibility relations are satisfied (see (2.20) of [21, p. 11]), then we use Theorem 6.2 of [21, p. 37], and obtain $y^0(u^0), \psi^0(u^0) \in H^{4,2}(\Omega \times (0,T))$, where the compatibility relations may be expressed by

$$\begin{aligned} &g_0(x,t)|_{t=0} = \phi_0(x), \\ &\partial_t g_0(x,t)|_{t=0} = f(x,t)|_{t=0} - \mathcal{A}_0(0)\phi_0(x), \\ &g_0(x,t)|_{t=T} = z_d(x,t)|_{t=T}, \end{aligned} \tag{2.7}$$

where $\mathcal{A}_0(0) = \mathcal{A}_0|_{t=0}$ and the homogenized operator $\mathcal{A}_0$ has been defined as in (2.3). Before giving (2.7), we used the facts $u^0(x,t)|_{(x\in\partial\Omega,t=0)} = 0$ and $y^0(u^0)|_{(x\in\partial\Omega,t=T)} = g_0(x,T)$. They can be satisfied because of the boundary conditions of (2.3)–(2.4) and $u^0, y^0(u^0), \psi^0(u^0) \in H^{2,1}(\Omega \times (0,T))$.

On the other hand, we consider two specific cases with constraints as follows:

$$\mathcal{U}_{ad} = \{v| \quad v \geq 0 \quad \text{almost everywhere in } \Omega\} \tag{2.8}$$

and

$$\begin{aligned} \mathcal{U}_{ad} = \{v| \quad &\zeta_0(x,t) \leq v \leq \zeta_1(x,t) \quad \text{almost everywhere in } \Omega, \\ &\zeta_0,\ \zeta_1 \in L^\infty(\Omega \times (0,T))\} \end{aligned} \tag{2.9}$$

In the case of (2.8), we find that

$$u^0(x,t) = \gamma^{-1}(x,t)\max(0, -\psi^0(u^0)), \tag{2.10}$$

and the homogenized optimal control problem (2.3)–(2.5) can be expressed by

$$
(2.11)\quad \begin{cases} \frac{\partial}{\partial t} y^0(u^0) + \mathcal{A}_0 y^0(u^0) = f(x,t) + \max(0, -p^0(u^0)), \quad (x,t) \in \Omega \times (0,T), \\ -\frac{\partial}{\partial t}\psi^0(u^0) + \mathcal{A}_0 \psi^0(u^0) = y^0(u^0) - z_d(x,t), \quad (x,t) \in \Omega \times (0,T), \\ y^0(u^0) = g_0(x,t), \quad \psi^0(u^0) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ y^0(u^0)|_{t=0} = \phi_0(x), \quad \psi^0(u^0)|_{t=T} = 0, \quad x \in \Omega. \end{cases}
$$

If we suppose that $\partial\Omega \in C^2$, $\partial\Omega$ is a pure Dirichlet boundary, and $z_d \in L^2(\Omega \times (0,T))$, then we may verify that $\psi^0(u^0) \in H^{2,1}(\Omega \times (0,T))$ (see [23, p. 119]). If $\gamma \in H^{1,1}(\Omega \times (0,T))$, then we can prove that $u^0 \in H^{1,1}(\Omega \times (0,T))$ (see [23, p. 37 or 120]). Furthermore, if a pure Dirichlet boundary $\partial\Omega \in C^4$, $f \in L^2(0,T;H^1(\Omega))$, $z_d \in H^{2,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{5}{2},\frac{5}{4}}(\Omega \times (0,T))$, $\phi_0 \in H^2(\Omega)$, and the same compatibility relations (2.7) are satisfied, then we obtain $y^0(u^0) \in H^{3,1}(\Omega \times (0,T))$ and $\psi^0(u^0) \in H^{4,2}(\Omega \times (0,T))$.

In the case of (2.9), the homogenized optimal control problem (2.3)–(2.5) can be expressed by

$$
(2.12)\quad \begin{cases} \frac{\partial}{\partial t} y^0(u^0) + \mathcal{A}_0 y^0(u^0) = f(x,t) + \Phi(\zeta_0,\zeta_1)p^0(u^0), \quad (x,t) \in \Omega \times (0,T), \\ -\frac{\partial}{\partial t}\psi^0(u^0) + \mathcal{A}_0 \psi^0(u^0) = y^0(u^0) - z_d(x,t), \quad (x,t) \in \Omega \times (0,T), \\ u^0 = \Phi(\zeta_0,\zeta_1)p^0(u^0), \quad (x,t) \in \Omega \times (0,T), \\ y^0(u^0) = g_0(x,t), \quad \psi^0(u^0) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ y^0(u^0)|_{t=0} = \phi_0(x), \quad \psi^0(u^0)|_{t=T} = 0, \quad x \in \Omega, \end{cases}
$$

where

$$
\Phi(\zeta_0,\zeta_1)p^0(u^0) = \begin{cases} -p^0(u^0) & \text{if} \quad \zeta_0 \le -p^0(u^0) \le \zeta_1, \\ \zeta_0 & \text{if} \quad -p^0(u^0) < \zeta_0, \\ \zeta_1 & \text{if} \quad -p^0(u^0) > \zeta_1. \end{cases}
$$

If $\zeta_i \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;H^1(\Omega))$, $i = 0,1$, then one proves $\Phi(\zeta_0,\zeta_1)p^0(u^0) \in L^2(0,T;H^1(\Omega))$ (see [23, p. 54]). Therefore we can also obtain the above regularity results.

3. Boundary layer solutions in the case without constraints. If $\Omega \subset R^n$ is a bounded convex polygonal domain, then generally speaking, the assumptions $y^0(u^0)$, $\psi^0(u^0) \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$, are invalid. In addition, the multiscale asymptotic solutions $y_s^\varepsilon(x,t)$, $\psi_s^\varepsilon(x,t)$, $s = 1,2$, as defined in (2.1) and (2.2) do not satisfy the boundary conditions of the function $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ of the original optimal control problem (1.4)–(1.6). To overcome these difficulties, we need to define the boundary layer solutions. To begin, we introduce the following notation: Let $\overline{\Omega}_0 = \bigcup_{z \in \widehat{T}_\varepsilon} \varepsilon(z + \overline{Q}) \subset \Omega$ as illustrated in Figure 1, where the index set $\widehat{T}_\varepsilon = \{z = (z_1, \ldots, z_n) \in Z^n,\ \varepsilon(z+Q) \subset \Omega\}$, and the unit cube $Q = (0,1)^n$. The boundary layer $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$ is as shown in Figure 2, where $dist(\partial\Omega_0, \partial\Omega) \ge \varepsilon$.

Remark 3.1. In this paper, suppose that a whole domain Ω is a periodic structure in all coordinate axes. If a periodic parameter ε is sufficiently small, then it implies that we can choose an interior subdomain $\Omega_0 \subset\subset \Omega$ such that $dist(\partial\Omega_0, \partial\Omega) \ge \varepsilon$.

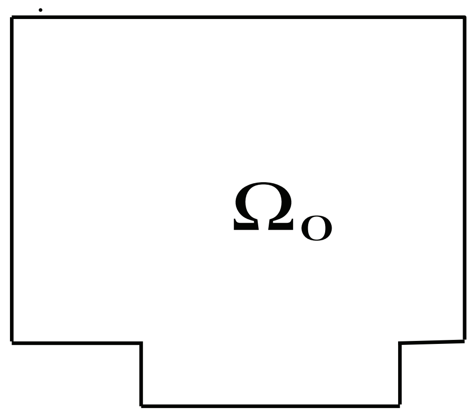

FIG. 1. *Interior subdomain* Ω_0.

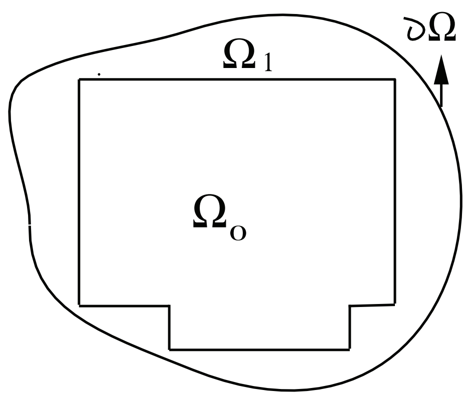

FIG. 2. *The boundary layer* Ω_1.

We define the boundary layer solutions given by

$$\text{(3.1)}\quad \begin{cases} \frac{\partial}{\partial t} y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) + \mathcal{A}_\varepsilon y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = f(x,t) + u_s^{\varepsilon,b}(x,t), \quad (x,t) \in \Omega_1 \times (0,T), \\ y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = g_0(x,t), \quad (x,t) \in \Gamma_0 \times (0,T), \\ y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = y_s^{\varepsilon}(x,t), \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(y_s^{\varepsilon,b}(u_s^{\varepsilon,b})) = g_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T), \\ y_s^{\varepsilon,b}(u_s^{\varepsilon,b})|_{t=0} = \phi_0(x), \quad x \in \Omega_1, \end{cases}$$

$$\text{(3.2)}\quad \begin{cases} -\frac{\partial}{\partial t} \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) + \mathcal{A}_\varepsilon \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - z_d(x,t), \quad (x,t) \in \Omega_1 \times (0,T), \\ \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) = \psi_s^{\varepsilon}(x,t), \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b})) = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b})|_{t=T} = 0, \quad x \in \Omega_1, \end{cases}$$

$$\text{(3.3)}\quad u_s^{\varepsilon,b}(x,t) = -\gamma^{-1}(x,t)\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}),$$

where the operators $\mathcal{A}_\varepsilon$, σ_ε are as defined in section 1. One can prove that following proposition.

PROPOSITION 3.1. *For any fixed $s = 1, 2$, there is a unique solution $\{y_s^{\varepsilon,b}(u_s^{\varepsilon,b}), \psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}), u_s^{\varepsilon,b}\}$ of the optimal control system (3.1)–(3.3), and $y_s^{\varepsilon,b}(u_s^{\varepsilon,b})$, $\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b})$, $u_s^{\varepsilon,b} \in L^2(0, T; V)$, where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.*

Proof. Given $\Omega_0 \subset\subset \Omega$, $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, for $(x, t) \in \Omega_0 \times (0, T)$, $y_s^\varepsilon(x, t)$, $\psi_s^\varepsilon(x, t)$, $s = 1, 2$, are as defined in (2.1) and (2.2), respectively. We first define the following initial-boundary value problem:

$$
\begin{cases}
-\dfrac{\partial\theta_s^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon\theta_s^\varepsilon = 0, \quad (x,t) \in \Omega_1 \times (0,T),\\
\theta_s^\varepsilon(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,T),\\
\theta_s^\varepsilon(x,t) = \psi_s^\varepsilon(x,t), \quad (x,t) \in \Gamma^* \times (0,T),\\
\sigma_\varepsilon(\theta_s^\varepsilon) \equiv \nu_i a_{ij}^\varepsilon(x,t)\dfrac{\partial\theta_s^\varepsilon}{\partial x_j} = 0, \quad (x,t) \in \Gamma_1 \times (0,T),\\
\theta_s^\varepsilon(x,t)|_{t=T} = 0, \quad x \in \Omega_1.
\end{cases} \tag{3.4}
$$

We change t into $T - t$ and can verify that there is a unique solution $\theta_s^\varepsilon(x, t)$ of problem (3.4). Subtracting (3.4) from (3.2) yields

$$
\begin{cases}
-\dfrac{\partial}{\partial t}(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon) + \mathcal{A}_\varepsilon(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon) = y_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - z_d,\\
\qquad (x,t) \in \Omega_1 \times (0,T),\\
(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon) = 0, \quad (x,t) \in \Gamma_0 \times (0,T),\\
(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon) = 0, \quad (x,t) \in \Gamma^* \times (0,T),\\
\sigma_\varepsilon(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon) = 0, \quad (x,t) \in \Gamma_1 \times (0,T),\\
(\psi_s^{\varepsilon,b}(u_s^{\varepsilon,b}) - \theta_s^\varepsilon)|_{t=T} = 0, \quad x \in \Omega_1.
\end{cases} \tag{3.5}
$$

Combining (3.1) and (3.5), following along the lines of the proofs of Lemma 1.1, one can show that there is a unique solution $\{y_s^{\varepsilon,b}, \psi_s^{\varepsilon,b} - \theta_s^\varepsilon, u_s^{\varepsilon,b}\}$ for the optimal control system (3.1), (3.5), and (3.3). Therefore the proof of Proposition 3.1 is complete.

If $\Omega \subset R^n$ is a bounded Lipschitz polygonal domain in the case without any constraints, then we define the multiscale asymptotic expansions of the solutions for the optimal control systems (1.4)–(1.6) as follows:

$$
Y_s^\varepsilon(x,t) = \begin{cases} y_s^\varepsilon(x,t), \quad (x,t) \in \overline{\Omega}_0 \times [0,T),\\ y_s^{\varepsilon,b}(x,t), \quad (x,t) \in \Omega_1 \times (0,T), \end{cases} \tag{3.6}
$$

$$
\Psi_s^\varepsilon(x,t) = \begin{cases} \psi_s^\varepsilon(x,t), \quad (x,t) \in \overline{\Omega}_0 \times (0,T),\\ \psi_s^{\varepsilon,b}(x,t), \quad (x,t) \in \Omega_1 \times (0,T), \end{cases} \tag{3.7}
$$

$$
u_s^\varepsilon(x,t) = -\gamma^{-1}\psi_s^\varepsilon(x,t), \quad U_s^\varepsilon(x,t) = -\gamma^{-1}\Psi_s^\varepsilon(x,t), \tag{3.8}
$$

4. Convergence in the case without constraints.

In the previous section, the multiscale asymptotic expansions of the solutions for the optimal control problem (1.4)–(1.6) in the case without any constraints are given. In this section, we obtain the convergence theorems for the multiscale asymptotic expansions.

We recall (1.2) and define the cost function of the homogenized control problem (2.3)–(2.5) without any constraints as follows:

$$
\widehat{\mathcal{J}}(v) = \int_0^T \int_\Omega |y^0(v) - z_d|^2 dxdt + \int_0^T \int_{\Omega_U} \gamma v^2 dxdt, \quad \gamma = \gamma(x,t) \geq \gamma_0 > 0, \tag{4.1}
$$

where $\Omega_U = \Omega$, and $u^0 \in \mathcal{U}_{ad} \subset L^2(0,T;L^2(\Omega))$ is the unique weak solution of the homogenized optimal control problem, i.e.,

$$\widehat{\mathcal{J}}(u^0) = \inf \widehat{\mathcal{J}}(v) \quad \forall v \in \mathcal{U}_{ad}, \quad u^0 \in \mathcal{U}_{ad}. \tag{4.2}$$

To begin, we prove the following proposition.

PROPOSITION 4.1. *Under assumptions* (A_1)–(A_4)*, it follows that*

$$\|y^\varepsilon - y^0\|_{L^2(0,T;L^2(\Omega))} \to 0, \quad \varepsilon \to 0, \tag{4.3}$$

$$\|u^\varepsilon - u^0\|_{L^2(0,T;L^2(\Omega))} \to 0, \quad \varepsilon \to 0, \tag{4.4}$$

$$\|\psi^\varepsilon - \psi^0\|_{L^2(0,T;L^2(\Omega))} \to 0, \quad \varepsilon \to 0. \tag{4.5}$$

Proof. We prove (4.3) and (4.4). The proof of (4.5) is similar.

For simplicity, we set $y^\varepsilon(u^\varepsilon) = y^\varepsilon$, $y^0(u^0) = y^0$. Since

$$\mathcal{J}_\varepsilon(v) \geq \int_0^T \int_\Omega \gamma v^2(x,t)dxdt, \quad \gamma \in L^\infty(\Omega \times (0,T)),$$

we infer that $\|v\|_{L^2(0,T;L^2(\Omega))} \to +\infty$ implies $\mathcal{J}_\varepsilon(v) \to +\infty$. Hence we always assume that $\|u^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C$ without loss of generality.

Thanks to (A_3), we have

$$\|y^\varepsilon\|_{L^2(0,T;V)} \leq C$$

and

$$\left\|\frac{\partial y^\varepsilon}{\partial t}\right\|_{L^2(0,T;V')} \leq C,$$

where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and V' is the dual of V.

We thus extract a subsequence, still denoted by u^ε, y^ε, such that

$$\begin{cases} u^\varepsilon \longrightarrow \tilde{u}^0, \quad \varepsilon \to 0 \quad \tilde{u}^0 \in \mathcal{U}_{ad}, \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \text{ weakly}, \\ y^\varepsilon \longrightarrow \tilde{y}^0, \quad \varepsilon \to 0 \quad \text{in} \quad L^2(0,T;V) \text{ weakly}, \\ \dfrac{\partial y^\varepsilon}{\partial t} \longrightarrow \dfrac{\partial \tilde{y}^0}{\partial t}, \quad \varepsilon \to 0 \quad \text{in} \quad L^2(0,T;V') \text{ weakly}. \end{cases} \tag{4.6}$$

From (4.6), using Theorem 3.58 of [8, p. 61], we obtain

$$y^\varepsilon \longrightarrow \tilde{y}^0, \quad \varepsilon \to 0, \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \text{ strongly}, \tag{4.7}$$

and consequently,

$$\varliminf_{\varepsilon \to 0} \mathcal{J}_\varepsilon(u^\varepsilon) \geq \int_0^T \int_\Omega |\tilde{y}^0 - z_d|^2 dxdt + \int_0^T \int_\Omega \gamma(\tilde{u}^0)^2 dxdt \equiv A. \tag{4.8}$$

For all $v \in \mathcal{U}_{ad}$, from (1.2) and (4.1), by using the convergence result of the homogenization method for parabolic equations (see (1.36) of [5, p. 241]), we get

$$\mathcal{J}_\varepsilon(v) \to \widehat{\mathcal{J}}(v). \tag{4.9}$$

We thus obtain $\mathcal{J}_\varepsilon(u^\varepsilon) \le \widehat{\mathcal{J}}(v)\ \forall v \in \mathcal{U}_{ad}$, and

$$A \le \underline{\lim}_{\varepsilon\to 0}\mathcal{J}_\varepsilon(u^\varepsilon) \le \widehat{\mathcal{J}}(v) \quad \forall v \in \mathcal{U}_{ad}. \tag{4.10}$$

We now show that $\tilde{y}^0 = y^0(\tilde{u}^0)$. Setting $\varphi^\varepsilon = y^\varepsilon(u^\varepsilon) - y^\varepsilon(\tilde{u}^0)$, recalling (1.1) and (1.4), we get

$$\begin{cases} \dfrac{\partial \varphi^\varepsilon}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}^\varepsilon(x,t)\dfrac{\partial \varphi^\varepsilon}{\partial x_j}\Big) = u^\varepsilon - \tilde{u}^0, \quad (x,t) \in \Omega\times(0,T),\\ \varphi^\varepsilon = 0, \quad (x,t)\in \Gamma_0\times(0,T),\\ \sigma_\varepsilon(\varphi^\varepsilon) = 0, \quad (x,t)\in\Gamma_1\times(0,T),\\ \varphi^\varepsilon|_{t=0} = 0, \quad x\in\Omega. \end{cases} \tag{4.11}$$

Thanks to $u^\varepsilon \longrightarrow \tilde{u}^0$, $\varepsilon \to 0$ in $L^2(0,T;L^2(\Omega))$ weakly, by using the convergence result of the homogenization method for parabolic equations (see Theorem 11.4 of [8, p. 211]), we have

$$\varphi^\varepsilon \longrightarrow 0 \quad \text{in} \quad L^2(0,T;V) \quad \text{weakly},$$

where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. We thus deduce $\tilde{y}^0 = y^0(\tilde{u}^0)$ so that $A = \widehat{\mathcal{J}}(\tilde{u}^0)$.

Hence (4.2) and $A = \widehat{\mathcal{J}}(\tilde{u}^0) \le \widehat{\mathcal{J}}(u^0)$ prove that $\tilde{u}^0 = u^0$, $\tilde{y}^0 = y^0(u^0) = y^0$. Equation (4.3) follows from (4.7).

On the other hand, $J_\varepsilon(u^\varepsilon) \le \widehat{\mathcal{J}}(v)\ \forall v \in \mathcal{U}_{ad}$ implies

$$\overline{\lim_{\varepsilon\to 0}}\mathcal{J}_\varepsilon(u^\varepsilon) \le \widehat{\mathcal{J}}(u^0),$$

and consequently,

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(u^\varepsilon) = \widehat{\mathcal{J}}(u^0). \tag{4.12}$$

We can check that

$$\int_0^T\int_\Omega |y^\varepsilon - z_d|^2 dxdt \to \int_0^T\int_\Omega |y^0 - z_d|^2 dxdt.$$

Using (4.6), we get

$$\int_0^T\int_\Omega \Big(u^\varepsilon - u^0\Big)^2 dxdt = \int_0^T\int_\Omega \Big[(u^\varepsilon)^2 - 2u^\varepsilon u^0 + (u^0)^2\Big]dxdt \to 0, \quad \varepsilon\to 0.$$

Therefore the proof of Proposition 4.1 is complete.

THEOREM 4.1. *Suppose that $\Omega \subset R^n$ is a bounded polygonal convex domain or a bounded smooth domain with a Lipschitz continuous boundary $\partial\Omega$, $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $\Gamma_0\cap\Gamma_1 = \emptyset$, Γ_0 and Γ_1 are, respectively, the Dirichlet and Neumann boundaries, and $\Omega_0 \subset\subset \Omega'' \subset\subset \Omega$. Let $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ be the weak solution of the control system* (1.4)–(1.6) *without any constraints and assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1,2,3$. Let $\{y_s^\varepsilon, \psi_s^\varepsilon, u_s^\varepsilon\}$, $\{Y_s^\varepsilon, \Psi_s^\varepsilon, U_s^\varepsilon\}$, $s = 1,2$, be defined as in* (3.6)–(3.8). *Under assumptions* (A_1)–(A_4), *if $f, z_d \in L^2(0,T;L^2(\Omega)) \cap H^{s,1}(\Omega''\times(0,T))$, $g_0 \in L^2(0,T;H^{1/2}(\Gamma_0))$, $g_1 \in L^2(0,T;L^2(\Gamma_1))$, $\phi_0 \in H^1(\Omega)\cap H^{s+1}(\Omega'')$, $s = 1,2$, $\gamma \equiv C > 0$, then we obtain*

$$\begin{aligned} &\|y^\varepsilon(u^\varepsilon) - Y_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\psi^\varepsilon(u^\varepsilon) - \Psi_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\\ &\quad + \|u^\varepsilon - U_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_s(T)\delta(\varepsilon), \end{aligned} \tag{4.13}$$

where $s = 1, 2$, $C_s(T)$ *is a constant independent of* ε *but dependent on* T, *and* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Under the assumptions of Theorem 4.1, using Theorem 10.1 of [18, p. 351], we can obtain the required interior regularity of the state function $y^0(u^0)$ and the adjoint function $\psi^0(u^0)$ of the homogenized optimal control problem (2.3)–(2.5) without any constraints. Therefore there are multiscale asymptotic solutions y_s^ε and ψ_s^ε defined as in (2.1) and (2.2) in a subdomain $\Omega'' \subset\subset \Omega$. For simplicity, set $y^\varepsilon = y^\varepsilon(u^\varepsilon)$, $\psi^\varepsilon = \psi^\varepsilon(u^\varepsilon)$. Using Proposition 4.1 and repeating the process of Theorem 2.3 of [5, p. 283], we can prove

$$\|y^\varepsilon - y_s^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \le C_s(T)\delta(\varepsilon), \quad \|\psi^\varepsilon - \psi_s^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \le C_s(T)\delta(\varepsilon), \tag{4.14}$$

where $\Omega_0 \subset\subset \Omega'' \subset\subset \Omega$, and $C_s(T)$ is a constant independent of ε but dependent on T. $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Subtracting (3.1) from (1.4) leads to

$$\begin{cases} \dfrac{\partial}{\partial t}(y^\varepsilon - y_s^{\varepsilon,b}) + \mathcal{A}_\varepsilon(y^\varepsilon - y_s^{\varepsilon,b}) + \gamma^{-1}(\psi^\varepsilon - \psi_s^{\varepsilon,b}) = 0, \\ \qquad (x,t) \in \Omega_1 \times (0,T), \\ y^\varepsilon - y_s^{\varepsilon,b} = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ y^\varepsilon - y_s^{\varepsilon,b} = y^\varepsilon - y_s^\varepsilon, \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(y^\varepsilon - y_s^{\varepsilon,b}) = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ (y^\varepsilon - y_s^{\varepsilon,b})|_{t=0} = 0, \quad x \in \Omega_1, \end{cases} \tag{4.15}$$

where $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$ is as shown in Figure 2. Similarly, subtracting (3.2) from (1.5) yields

$$\begin{cases} -\dfrac{\partial}{\partial t}(\psi^\varepsilon - \psi_s^{\varepsilon,b}) + \mathcal{A}_\varepsilon(\psi^\varepsilon - \psi_s^{\varepsilon,b}) - (y^\varepsilon - y_s^{\varepsilon,b}) = 0, \\ \qquad (x,t) \in \Omega_1 \times (0,T), \\ \psi^\varepsilon - \psi_s^\varepsilon = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ \psi^\varepsilon - \psi_s^{\varepsilon,b} = \psi^\varepsilon - \psi_s^\varepsilon, \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(\psi^\varepsilon - \psi_s^{\varepsilon,b}) = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ (\psi^\varepsilon - \psi_s^{\varepsilon,b})|_{t=T} = 0, \quad x \in \Omega_1. \end{cases} \tag{4.16}$$

We define $\kappa^\varepsilon(x,t)$ and $z^\varepsilon(x,t)$, respectively, as follows:

$$\begin{cases} \dfrac{\partial\kappa^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon\kappa^\varepsilon = 0, \quad (x,t) \in \Omega_1 \times (0,T), \\ \kappa^\varepsilon = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ \kappa^\varepsilon = y^\varepsilon - y_s^\varepsilon, \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(\kappa^\varepsilon) = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ \kappa^\varepsilon|_{t=0} = 0, \quad x \in \Omega_1, \end{cases} \tag{4.17}$$

and

$$\begin{cases} -\dfrac{\partial z^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon z^\varepsilon = 0, \quad (x,t) \in \Omega_1 \times (0,T), \\ z^\varepsilon = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \\ z^\varepsilon = \psi^\varepsilon - \psi_s^\varepsilon, \quad (x,t) \in \Gamma^* \times (0,T), \\ \sigma_\varepsilon(z^\varepsilon) = 0, \quad (x,t) \in \Gamma_1 \times (0,T), \\ z^\varepsilon|_{t=T} = 0, \quad x \in \Omega_1. \end{cases} \tag{4.18}$$

By using an a priori estimate for parabolic equations, the trace theorem, and (4.14), we can show that

$$\begin{aligned} \|\kappa^\varepsilon\|_{L^2(0,T;H^1(\Omega_1))} &\le C\|y^\varepsilon - y_s^\varepsilon\|_{L^2(0,T;H^{1/2}(\Gamma^*))} \\ &\le C\|y^\varepsilon - y_s^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \le C_2(T)\delta(\varepsilon). \end{aligned} \tag{4.19}$$

Similarly we can prove that

$$\|z^\varepsilon\|_{L^2(0,T;H^1(\Omega_1))} \le C_2(T)\delta(\varepsilon). \tag{4.20}$$

Set $\tilde{\eta}^\varepsilon = y^\varepsilon - y_s^{\varepsilon,b} - \kappa^\varepsilon$, $\tilde{\pi}^\varepsilon = \psi^\varepsilon - \psi_s^{\varepsilon,b} - z^\varepsilon$. Given $\gamma \equiv C > 0$, multiplying by $\gamma\tilde{\eta}^\varepsilon$ and $\tilde{\pi}^\varepsilon$ on both sides of the first equations of (4.15) and (4.16), respectively, adding up, and integrating on $\Omega_1 \times (0, t')$ leads to

$$\begin{aligned}
&\frac{\gamma}{2}\int_0^{t'}\int_{\Omega_1} \frac{\partial(\tilde{\eta}^\varepsilon)^2}{\partial t} dxdt + \gamma\int_0^{t'}\int_{\Omega_1} a_{ij}^\varepsilon(x,t)\frac{\partial\tilde{\eta}^\varepsilon}{\partial x_j}\frac{\partial\tilde{\eta}^\varepsilon}{\partial x_i} dxdt \\
&\quad -\frac{1}{2}\int_0^{t'}\int_{\Omega_1} \frac{\partial(\tilde{\pi}^\varepsilon)^2}{\partial t} dxdt + \int_0^{t'}\int_{\Omega_1} a_{ij}^\varepsilon(x,t)\frac{\partial\tilde{\pi}^\varepsilon}{\partial x_j}\frac{\partial\tilde{\pi}^\varepsilon}{\partial x_i} dxdt \\
&= \int_0^{t'}\int_{\Omega_1} \Big(\kappa^\varepsilon\tilde{\pi}^\varepsilon - z^\varepsilon\tilde{\eta}^\varepsilon\Big) dxdt.
\end{aligned} \tag{4.21}$$

Since $\tilde{\eta}^\varepsilon|_{t=0} = 0$ and $\tilde{\pi}^\varepsilon|_{t=T} = 0$, we get

$$\begin{aligned}
&\gamma\int_0^{T}\int_{\Omega_1} a_{ij}^\varepsilon(x,t)\frac{\partial\tilde{\eta}^\varepsilon}{\partial x_j}\frac{\partial\tilde{\eta}^\varepsilon}{\partial x_i} dxdt + \int_0^{T}\int_{\Omega_1} a_{ij}^\varepsilon(x,t)\frac{\partial\tilde{\pi}^\varepsilon}{\partial x_j}\frac{\partial\tilde{\pi}^\varepsilon}{\partial x_i} dxdt \\
&\le \int_0^{T}\int_{\Omega_1} \Big(|\kappa^\varepsilon\tilde{\pi}^\varepsilon| + |z^\varepsilon\tilde{\eta}^\varepsilon|\Big) dxdt.
\end{aligned} \tag{4.22}$$

Thanks to (A_3), using the Young inequality, and combining (4.19), (4.20), and (4.22) it leads to

$$\|\tilde{\eta}^\varepsilon\|_{L^2(0,T;H^1(\Omega_1))} + \|\tilde{\pi}^\varepsilon\|_{L^2(0,T;H^1(\Omega_1))} \le C_s(T)\delta(\varepsilon). \tag{4.23}$$

It follows from (4.14), (4.19), (4.20), and (4.23) that

$$\|y^\varepsilon - Y_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\psi^\varepsilon - \Psi_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_s(T)\delta(\varepsilon). \tag{4.24}$$

Furthermore, combining (1.7), (3.8), and (4.24) yields

$$\|u^\varepsilon - U_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_s(T)\delta(\varepsilon). \tag{4.25}$$

Therefore the proof of Theorem 4.1 is complete.

THEOREM 4.2. *Suppose that $\partial\Omega \in C^4$ is a pure Dirichlet boundary. Let $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ be the weak solution of the control system* (1.4)–(1.6) *without any constraints and assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1, 2, 3$. Let y_2^ε, ψ_2^ε, u_2^ε be the second order multiscale asymptotic solutions defined in* (2.1), (2.2), *and* (3.8), *respectively. Under assumptions* (A_1)–(A_4), *if $f, z_d \in H^{2,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{7}{2},\frac{7}{4}}(\partial\Omega \times (0,T))$, $\phi_0 \in H^3(\Omega)$, the compatibility relations* (2.7) *are satisfied (see also* [21, p. 32]*), and $\gamma \equiv C > 0$, then it holds that*

$$\begin{aligned}
&\|y^\varepsilon(u^\varepsilon) - y_2^\varepsilon\|_{L^2(0,T;V)} + \|\psi^\varepsilon(u^\varepsilon) - \psi_2^\varepsilon\|_{L^2(0,T;V)} \\
&\quad + \|u^\varepsilon - u_2^\varepsilon\|_{L^2(0,T;V)} \le C_2(T)\varepsilon^{1/2},
\end{aligned} \tag{4.26}$$

where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and $C_2(T)$ is a constant independent of ε but dependent on T.

Proof. For simplicity, set $y^\varepsilon(u^\varepsilon) = y^\varepsilon$, $\psi^\varepsilon(u^\varepsilon) = \psi^\varepsilon$. As mentioned in section 2.2, under the assumptions of Theorem 4.2, we can show that $y^0(u^0), \psi^0(u^0), u^0 \in H^{4,2}(\Omega\times(0,T))$. Set $\hat{\eta}^\varepsilon = y^\varepsilon - y_2^\varepsilon$, $\hat{\pi}^\varepsilon = \psi^\varepsilon - \psi_2^\varepsilon$. If $(x,t)\in\Omega\times(0,T)$, from (2.1)–(2.2), using the definitions of cell functions $N_{\alpha_1}(\xi,\tau)$, $N_{\alpha_1\alpha_2}(\xi,\tau)$ (see [1]), where $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-k}t$, $k = 1,2,3$, then we obtain the following equations which hold in the sense of distributions:

$$(4.27)\qquad \begin{cases} \dfrac{\partial \hat{\eta}^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon \hat{\eta}^\varepsilon + \gamma^{-1}\hat{\pi}^\varepsilon = \varepsilon F_0(x,t,\varepsilon), \\ -\dfrac{\partial \hat{\pi}^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon \hat{\pi}^\varepsilon - \hat{\eta}^\varepsilon = \varepsilon F_1(x,t,\varepsilon), \end{cases}$$

where $F_0(x,t,\varepsilon)$, $F_1(x,t,\varepsilon)$ are the sums of functions such as N_{α_1}, $N_{\alpha_1\alpha_2}$, $\frac{\partial N_{\alpha_1}}{\partial \xi_j}$, $\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}$, $\frac{\partial N_{\alpha_1}}{\partial \tau}$, $\frac{\partial N_{\alpha_1\alpha_2}}{\partial \tau}$, $\frac{\partial^m y^0(u^0)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_m}}$, $\frac{\partial^{l+1} y^0(u^0)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}\partial t}$, $\frac{\partial^m \psi^0(u^0)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_m}}$, $\frac{\partial^{l+1} \psi^0(u^0)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}\partial t}$; $0\le m\le 4$; $0\le l\le 2$; $\alpha_1,\dots,\alpha_l,\dots,\alpha_m = 1,2,\dots,n$.

For example, for the case $k=1$, we have

$$\begin{aligned} F_0(x,t,\varepsilon) \;=\;& \frac{\partial N_{\alpha_1\alpha_2}(\xi,\tau)}{\partial\tau}\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + N_{\alpha_1}(\xi,\tau)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial t} \\ & - a_{ij}(\xi,\tau)\frac{\partial N_{\alpha_1\alpha_2}(\xi,\tau)}{\partial \xi_j}\frac{\partial^3 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} \\ & - \frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j} - a_{ij}(\xi,\tau)N_{\alpha_1}(\xi,\tau)\frac{\partial^3 y^0(u^0)}{\partial x_{\alpha_1}\partial x_i\partial x_j} \\ & - \gamma^{-1}N_{\alpha_1}(\xi,\tau)\frac{\partial \psi^0(u^0)}{\partial x_{\alpha_1}} + \varepsilon N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^3 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial t} \\ & - \varepsilon a_{ij}(\xi,\tau)N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^4 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_1}\partial x_i\partial x_j} \\ & - \varepsilon\gamma^{-1}N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 \psi^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}. \end{aligned}$$

Using Lemma 2.1 and the regularity of $y^0(u^0)$ and $\psi^0(u^0)$, we can prove that

$$\|F_\sigma\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \sigma = 0,1,$$

where C is a constant independent of ε.

For $(x,t)\in\partial\Omega\times(0,T)$, we have

$$(4.28)\qquad y^\varepsilon - y_2^\varepsilon = \varepsilon G_1^\varepsilon, \quad \psi^\varepsilon - \psi_2^\varepsilon = \varepsilon G_2^\varepsilon,$$

where

$$\begin{aligned} G_1^\varepsilon &= -N_{\alpha_1}(\xi,\tau)\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}} - \varepsilon N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, \\ G_2^\varepsilon &= -N_{\alpha_1}(\xi,\tau)\frac{\partial \psi^0(u^0)}{\partial x_{\alpha_1}} - \varepsilon N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 \psi^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}. \end{aligned}$$

For the initial conditions or the terminal conditions, we obtain

$$(4.29)\qquad \hat{\eta}^\varepsilon|_{t=0} = \varepsilon\Psi_\varepsilon^1, \quad \hat{\pi}^\varepsilon|_{t=T} = \varepsilon\Psi_\varepsilon^2,$$

where

$$\begin{aligned} \Psi_\varepsilon^1 &= -N_{\alpha_1}(\xi,\tau)\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}|_{t=0} - \varepsilon N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}|_{t=0}, \\ \Psi_\varepsilon^2 &= -N_{\alpha_1}(\xi,\tau)\frac{\partial \psi^0(u^0)}{\partial x_{\alpha_1}}|_{t=T} - \varepsilon N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 \psi^0(u^0)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}|_{t=T}. \end{aligned}$$

For any fixed $t \in (0,T)$, using Lemma 2.1 and repeating the proof of (1.22) of [24, p. 126], we can show that

$$\|G_1^\varepsilon\|_{H^{1/2}(\partial\Omega)} \le C\varepsilon^{-1/2}\|y^0(u^0)\|_{H^3(\Omega)}, \quad \|G_2^\varepsilon\|_{H^{1/2}(\partial\Omega)} \le C\varepsilon^{-1/2}\|\psi^0(u^0)\|_{H^3(\Omega)}. \tag{4.30}$$

Using Lemma 2.1 and the regularity of $y^0(u^0), \psi^0(u^0)$, it is not difficult to show that

$$\|\Psi_\varepsilon^1\|_{L^2(\Omega)} \le C, \quad \|\Psi_\varepsilon^2\|_{L^2(\Omega)} \le C, \tag{4.31}$$

where C is a constant independent of ε.

Similarly to (4.17) and (4.18), we define $\zeta^\varepsilon(x,t)$ and $\mu^\varepsilon(x,t)$, respectively, as follows:

$$\begin{cases} \dfrac{\partial\zeta^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon\zeta^\varepsilon = \varepsilon F_0(x,t,\varepsilon), \quad (x,t) \in \Omega \times (0,T), \\ \zeta^\varepsilon = \varepsilon G_1^\varepsilon, \quad (x,t) \in \partial\Omega \times (0,T), \\ \zeta^\varepsilon|_{t=0} = \varepsilon\Psi_\varepsilon^1, \quad x \in \Omega, \end{cases} \tag{4.32}$$

$$\begin{cases} -\dfrac{\partial\mu^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon\mu^\varepsilon = \varepsilon F_1(x,t,\varepsilon), \quad (x,t) \in \Omega \times (0,T), \\ \mu^\varepsilon = \varepsilon G_2^\varepsilon, \quad (x,t) \in \partial\Omega \times (0,T), \\ \mu^\varepsilon|_{t=T} = \varepsilon\Psi_\varepsilon^2, \quad x \in \Omega. \end{cases} \tag{4.33}$$

Applying an a priori estimate for parabolic equations and combining (4.28)–(4.31) gives

$$\begin{aligned} \|\zeta^\varepsilon\|_{L^2(0,T;V)} &\le C\{\varepsilon\|F_0\|_{L^2(0,T;L^2(\Omega))} + \varepsilon\|G_1^\varepsilon\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \\ &\quad + \varepsilon\|\Psi_\varepsilon^1\|_{L^2(\Omega)}\} \le C(T)\varepsilon^{1/2}, \\ \|\mu^\varepsilon\|_{L^2(0,T;V)} &\le C\{\varepsilon\|F_1\|_{L^2(0,T;L^2(\Omega))} + \varepsilon\|G_2^\varepsilon\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \\ &\quad + \varepsilon\|\Psi_\varepsilon^2\|_{L^2(\Omega)}\} \le C(T)\varepsilon^{1/2}. \end{aligned} \tag{4.34}$$

Subtracting (4.27) from (4.32) and (4.33) leads to

$$\begin{cases} \dfrac{\partial(\hat\eta^\varepsilon - \zeta^\varepsilon)}{\partial t} + \mathcal{A}_\varepsilon(\hat\eta^\varepsilon - \zeta^\varepsilon) + \gamma^{-1}\hat\pi^\varepsilon = 0, \quad (x,t) \in \Omega \times (0,T), \\ -\dfrac{\partial(\hat\pi^\varepsilon - \mu^\varepsilon)}{\partial t} + \mathcal{A}_\varepsilon(\hat\pi^\varepsilon - \mu^\varepsilon) - \hat\eta^\varepsilon = 0, \quad (x,t) \in \Omega \times (0,T), \\ \hat\eta^\varepsilon - \zeta^\varepsilon = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \hat\pi^\varepsilon - \mu^\varepsilon = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ (\hat\eta^\varepsilon - \zeta^\varepsilon)|_{t=0} = 0, \quad (\hat\pi^\varepsilon - \mu^\varepsilon)|_{t=T} = 0. \end{cases} \tag{4.35}$$

Given $\gamma \equiv C > 0$, multiplying by $\gamma(\hat\eta^\varepsilon - \zeta^\varepsilon)$ and $(\hat\pi^\varepsilon - \mu^\varepsilon)$ on both sides of $(4.35)_1$ and $(4.35)_2$, respectively, adding up, and integrating on $\Omega \times (0,t')$ gives

$$\begin{aligned} &\frac{\gamma}{2}\int_0^{t'}\int_\Omega \frac{\partial(\hat\eta^\varepsilon - \zeta^\varepsilon)^2}{\partial t}dxdt + \gamma\int_0^{t'}\int_\Omega a_{ij}^\varepsilon(x,t)\frac{\partial(\hat\eta^\varepsilon - \zeta^\varepsilon)}{\partial x_j}\frac{\partial(\hat\eta^\varepsilon - \zeta^\varepsilon)}{\partial x_i}dxdt \\ &-\frac{1}{2}\int_0^{t'}\int_\Omega \frac{\partial(\hat\pi^\varepsilon - \mu^\varepsilon)^2}{\partial t}dxdt + \int_0^{t'}\int_\Omega a_{ij}^\varepsilon(x,t)\frac{\partial(\hat\pi^\varepsilon - \mu^\varepsilon)}{\partial x_j}\frac{\partial(\hat\pi^\varepsilon - \mu^\varepsilon)}{\partial x_i}dxdt \\ &= \int_0^{t'}\int_\Omega \Big(\zeta^\varepsilon\hat\pi^\varepsilon - \mu^\varepsilon\hat\eta^\varepsilon\Big)dxdt. \end{aligned} \tag{4.36}$$

Since $(\hat{\eta}^\varepsilon - \zeta^\varepsilon)|_{t=0} = 0$, $(\hat{\pi}^\varepsilon - \mu^\varepsilon)|_{t=T} = 0$, and (A_3) holds, using the Young inequality and (4.34), we obtain

$$\|\hat{\eta}^\varepsilon - \zeta^\varepsilon\|_{L^2(0,T;V)} + \|\hat{\pi}^\varepsilon - \mu^\varepsilon\|_{L^2(0,T;V)} \leq C_2(T)\varepsilon^{1/2}. \tag{4.37}$$

It follows from (4.37) and (4.34) that

$$\|y^\varepsilon - y_s^\varepsilon\|_{L^2(0,T;V)} + \|\psi^\varepsilon - \psi_s^\varepsilon\|_{L^2(0,T;V)} \leq C_s(T)\varepsilon^{1/2}. \tag{4.38}$$

Furthermore, combining (1.7), (3.8), and (4.38) gives

$$\|u^\varepsilon - u_s^\varepsilon\|_{L^2(0,T;V)} \leq C_s(T)\varepsilon^{1/2}. \tag{4.39}$$

Therefore we complete the proof of Theorem 4.2.

COROLLARY 4.1. *Let* $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ *be the weak solution of the control system* (1.4)–(1.6) *without any constraints and assume that* $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1,2,3$. *Let* Y_2^ε, Ψ_2^ε, U_2^ε *be defined as in* (3.6), (3.7), *and* (3.8), *respectively. Under the assumptions of Theorem* 4.2, *we have*

$$\begin{aligned}\|y^\varepsilon(u^\varepsilon) - Y_2^\varepsilon\|_{L^2(0,T;V)} + \|\psi^\varepsilon(u^\varepsilon) - \Psi_2^\varepsilon\|_{L^2(0,T;V)} \\ + \|u^\varepsilon - U_2^\varepsilon\|_{L^2(0,T;V)} \leq C_2(T)\varepsilon^{1/2},\end{aligned} \tag{4.40}$$

where $C_2(T)$ *is a constant independent of* ε *but dependent of* T.

Proof. Repeating the process of (4.15)–(4.23) and using the trace theorem and Theorem 4.2, we can complete the proof of Corollary 4.1.

COROLLARY 4.2. *Let* $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ *be the weak solution of the control system* (1.4)–(1.6) *without any constraints and assume that* $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, t)$. $\Omega \subset R^n$ *is a bounded Lipschitz smooth domain with boundary* $\partial\Omega \in C^3$, $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ *with* $\Gamma_0 \cap \Gamma_1 = \emptyset$, *and* Γ_0 *and* Γ_1 *denoting, respectively, the Dirichlet and Neumann boundaries with meas*$(\Gamma_1) > 0$. *If* $f, z_d \in H^{1,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{5}{2},\frac{5}{4}}(\Gamma_0 \times (0,T))$, $g_1 \in H^{\frac{3}{2},\frac{5}{4}}(\Gamma_1 \times (0,T))$, $\phi_0 \in H^2(\Omega)$, $\gamma \equiv C > 0$, *and the compatibility relations* (2.7) *are satisfied, then it holds that*

$$\begin{aligned}\|y^\varepsilon(u^\varepsilon) - y_1^\varepsilon\|_{L^2(0,T;V)} + \|\psi^\varepsilon(u^\varepsilon) - \psi_1^\varepsilon\|_{L^2(0,T;V)} \\ + \|u^\varepsilon - u_1^\varepsilon\|_{L^2(0,T;V)} \leq C_1(T)\varepsilon,\end{aligned} \tag{4.41}$$

where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, *and* $C_1(T)$ *is a constant independent of* ε *but dependent on* T.

Proof. For simplicity, set $y^\varepsilon(u^\varepsilon) = y^\varepsilon$ and $\psi^\varepsilon(u^\varepsilon) = \psi^\varepsilon$. As stated in section 2.2, under the assumptions of Corollary 4.2, we can show that $y^0(u^0), p^0(u^0), u^0 \in H^{3,1}(\Omega \times (0,T))$.

For $(x,t) \in \Gamma_0 \times (0,T)$, we get

$$y^\varepsilon - y_1^\varepsilon = \varepsilon J_\varepsilon^1, \quad \psi^\varepsilon - \psi_1^\varepsilon = \varepsilon J_\varepsilon^2, \tag{4.42}$$

where

$$J_\varepsilon^1 = -N_{\alpha_1}(\xi,\tau)\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}, \quad J_\varepsilon^2 = -N_{\alpha_1}(\xi,\tau)\frac{\partial \psi^0(u^0)}{\partial x_{\alpha_1}}.$$

Similarly to (4.31), for any fixed $t \in (0,T)$, we can show that

$$\begin{aligned}\|J_\varepsilon^1\|_{H^{1/2}(\Gamma_0)} \leq C\varepsilon^{-1/2}\|y^0(u^0)\|_{H^2(\Omega)}, \\ \|J_\varepsilon^2\|_{H^{1/2}(\Gamma_0)} \leq C\varepsilon^{-1/2}\|\psi^0(u^0)\|_{H^2(\Omega)}.\end{aligned} \tag{4.43}$$

We observe that

$$\sigma_\varepsilon(y_1^\varepsilon) \equiv \nu_i a_{ij}\left(\frac{x}{\varepsilon}, t\right)\frac{\partial y_1^\varepsilon(x,t)}{\partial x_j} = \left[\nu_i a_{ij}(\xi,t)\frac{\partial N_{\alpha_1}}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi,t)\right]\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}} + \varepsilon\nu_i a_{ij}(\xi,t)N_{\alpha_1}(\xi,t)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_j}. \tag{4.44}$$

For $(x,t)\in\Gamma_1\times(0,T)$, we have

$$\begin{aligned}\sigma_\varepsilon(y^\varepsilon - y_1^\varepsilon) &= g(x,t) - \left[\nu_i a_{ij}(\xi,t)\frac{\partial N_{\alpha_1}}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi,t)\right]\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}\\ &+ \varepsilon\nu_i a_{ij}(\xi,t)N_{\alpha_1}(\xi,t)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_j} = g(x,t) - \nu_i\hat{a}_{i\alpha_1}\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}\\ &-\nu_i\Big(a_{ij}(\xi,t)\frac{\partial N_{\alpha_1}(\xi,t)}{\partial \xi_j} + a_{i\alpha_1}(\xi,t) - \hat{a}_{i\alpha_1}\Big)\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}\\ &-\varepsilon\nu_i a_{ij}(\xi,t)N_{\alpha_1}(\xi,t)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_j} = -\nu_i\beta^{i\alpha_1}(\xi,t)\frac{\partial y^0(u^0)}{\partial x_{\alpha_1}}\\ &-\varepsilon\nu_i a_{ij}(\xi,t)N_{\alpha_1}(\xi,t)\frac{\partial^2 y^0(u^0)}{\partial x_{\alpha_1}\partial x_j},\end{aligned} \tag{4.45}$$

where $\beta^{i\alpha_1}(\xi,t) = a_{i\alpha_1}(\xi,t) + a_{ij}(\xi,t)\frac{\partial N_{\alpha_1}(\xi,t)}{\partial\xi_j} - \hat{a}_{i\alpha_1}$.

We can verify that $\beta^{i\alpha_1}(\xi,t)$ satisfies all conditions of Lemma 2.2 of [24, Chap. II] by using Lemma 2.1 and the definitions of $N_{\alpha_1}(\xi,t)$, $\alpha_1 = 1,2,\ldots,n$. For any fixed $t\in(0,T)$, it follows from Lemma 2.2 of [24, Chap. II] that, for any $v\in L^2(0,T;H^1(\Omega))$,

$$\left|\int_{\Gamma_1}\sigma_\varepsilon(y^\varepsilon - y_1^\varepsilon)v(x,t)d\Gamma_x\right| \le C\varepsilon^{1/2}\left(\int_\Omega|\nabla_x v(x,t)|^2dx\right)^{1/2}. \tag{4.46}$$

Similarly, we get

$$\left|\int_{\Gamma_1}\sigma_\varepsilon(\psi^\varepsilon - \psi_1^\varepsilon)v(x,t)d\Gamma_x\right| \le C\varepsilon^{1/2}\left(\int_\Omega|\nabla_x v(x,t)|^2dx\right)^{1/2}. \tag{4.47}$$

Therefore, we can complete the proof of Corollary 4.2 by following the reasoning of the proof of Theorem 4.2.

Remark 4.1. In particular, if $\partial\Omega$ is a pure Dirichlet boundary, Corollary 4.2 is also valid.

Remark 4.2. If we replace $y_1^\varepsilon, \psi_1^\varepsilon, u_1^\varepsilon$ with $Y_1^\varepsilon, \Psi_1^\varepsilon, U_1^\varepsilon$ in Corollary 4.2, we can also obtain similar estimates.

5. The convergence results in the case of constraints. In order to obtain the convergence results for the multiscale asymptotic expansions (2.1) and (2.2) for the optimal control problem (1.4)–(1.6) in the case with constraints, the higher order regularity of the state function $y^0(u^0)$ and the adjoint function $\psi^0(u^0)$ of the homogenized optimal control problem associated with (1.4)–(1.6) is required. As discussed in section 2.2, two specific situations with constraints are considered, i.e., (2.8) and (2.9). To this end, the convergence theorems in this section are confined to the two specific cases (2.8) and (2.9).

To begin, we consider the initial-boundary value problem given by

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} + \mathcal{A}_\varepsilon w^\varepsilon = f^\varepsilon(x,t), \quad (x,t) \in \Omega \times (0,T), \\ w^\varepsilon(x,t) = g_0(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \\ w^\varepsilon(x,t)|_{t=0} = \phi_0(x), \end{cases} \tag{5.1}$$

where the operators $\mathcal{A}_\varepsilon$ is given as in section 1. $f^\varepsilon(x,t)$, $g_0(x,t)$, $\phi_0(x)$ are known functions.

Similarly to (1.4), we define the multiscale asymptotic expansion of the solution of problem (5.1) as follows:

$$w_s^\varepsilon(x,t) = \begin{cases} w^0(x,t) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial w^0(x,t)}{\partial x_{\alpha_1}}, \quad s=1, \\ w^0(x,t) + \varepsilon N_{\alpha_1}(\xi,\tau)\dfrac{\partial w^0(x,t)}{\partial x_{\alpha_1}} \\ \qquad + \varepsilon^2 N_{\alpha_1\alpha_2}(\xi,\tau)\dfrac{\partial^2 w^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, \quad s=2, \end{cases} \tag{5.2}$$

where cell functions $N_{\alpha_1}(\xi,\tau)$, $N_{\alpha_1\alpha_2}(\xi,\tau)$, $\alpha_1,\alpha_2 = 1,2,\dots,n$ are the same as given in (2.1) and $w^0(x,t)$ is the unique solution of the homogenized initial-boundary value problem given by

$$\begin{cases} \dfrac{\partial w^0}{\partial t} + \mathcal{A}_0 w^0 = f^0(x,t), \quad (x,t) \in \Omega \times (0,T), \\ w^0(x,t) = g_0(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \\ w^0(x,t)|_{t=0} = \phi_0(x), \quad x \in \Omega, \end{cases} \tag{5.3}$$

where the operator $\mathcal{A}_0$ is the same as given in (2.3), and $f^\varepsilon \to f^0$ in $L^2(0,T;L^2(\Omega))$ strongly as $\varepsilon \to 0$.

Lemma 5.1 (see Corollary 2.2 of [1]). *Suppose that $\Omega \subset R^n$ is a bounded smooth domain with a pure Dirichlet boundary $\partial\Omega \in C^3$. Let $w^\varepsilon(x,t)$ be the unique solution of the initial-boundary value problem* (5.1)*, and assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon},t)$. Let $w_1^\varepsilon(x,t)$ be the first order multiscale asymptotic solution as given in* (5.2) *associated with $w^\varepsilon(x,t)$. If $f^\varepsilon \in L^2(0,T;L^2(\Omega))$, $f^0 \in H^{1,1}(\Omega\times(0,T))$, $g_0 \in H^{\frac{5}{2},\frac{5}{4}}(\partial\Omega\times(0,T))$, $\phi_0 \in H^2(\Omega)$, and the compatibility relations* (2.7) *are satisfied, then we have*

$$\sup_{0\le t\le T}\int_\Omega (w^\varepsilon(x,t) - w_1^\varepsilon(x,t))^2 dx + \int_0^T \|w^\varepsilon - w_1^\varepsilon\|_{H^1(\Omega)}^2 dt \le C\{\varepsilon + \|f^\varepsilon - f^0\|_{L^2(0,T;L^2(\Omega))}\}, \tag{5.4}$$

where C is a constant independent of ε, and $f^\varepsilon \to f^0$ in $L^2(0,T;L^2(\Omega))$ strongly as $\varepsilon \to 0$.

Using Lemma 5.1, the following proposition is obvious.

Proposition 5.1. *Assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon},t)$. Let $w^\varepsilon(x,t)$ and $w^0(x,t)$ be the solutions of the initial-boundary value problem* (5.1) *and the corresponding homogenized problem* (5.3)*, respectively. Under the assumptions of Lemma* 5.1*, it holds that*

$$\sup_{0\le t\le T}\int_\Omega (w^\varepsilon(x,t) - w^0(x,t))^2 dx + \int_0^T \|w^\varepsilon - w^0\|_{L^2(\Omega)}^2 dt \le C\Big\{\varepsilon + \|f^\varepsilon - f^0\|_{L^2(0,T;L^2(\Omega))}\Big\}. \tag{5.5}$$

LEMMA 5.2 (see Corollary 2.1 of [1]). *Suppose that $\Omega \subset R^n$ is a bounded smooth domain with a pure Dirichlet boundary $\partial\Omega \in C^4$. Let $w^\varepsilon(x,t)$ be the unique solution of the initial-boundary value problem* (5.1)*, and assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1,2,3$. Let $w_2^\varepsilon(x,t)$ be the second order multiscale asymptotic solution as given in* (5.2) *associated with $w^\varepsilon(x,t)$. If $f^\varepsilon \in L^2(0,T;L^2(\Omega))$, $f^0 \in H^{2,1}(\Omega\times(0,T))$, $g_0 \in H^{\frac{7}{2},\frac{7}{4}}(\partial\Omega\times(0,T))$, $\phi_0 \in H^3(\Omega)$, and the compatibility relations* (2.7) *are satisfied, then it holds*

$$\sup_{0\le t\le T}\int_\Omega (w^\varepsilon(x,t) - w_2^\varepsilon(x,t))^2dx + \int_0^T \|w^\varepsilon - w_2^\varepsilon\|_{H^1(\Omega)}^2 dt \le C_2\{\varepsilon + \|f^\varepsilon - f^0\|_{L^2(0,T;L^2(\Omega))}\}, \tag{5.6}$$

where C_2 is a constant independent of ε, and $f^\varepsilon \to f^0$ in $L^2(0,T;L^2(\Omega))$ strongly as $\varepsilon \to 0$.

Applying Lemma 5.2, we have the following proposition.

PROPOSITION 5.2. *Assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1,2,3$. Let $w^\varepsilon(x,t)$ and $w^0(x,t)$ be the solutions of the initial-boundary value problem* (5.1) *and the corresponding homogenized problem* (5.3)*, respectively. Under the assumptions of Lemma* 5.2*, we obtain*

$$\sup_{0\le t\le T}\int_\Omega (w^\varepsilon(x,t) - w^0(x,t))^2dx + \int_0^T \|w^\varepsilon - w^0\|_{L^2(\Omega)}^2 dt \le C_2\{\varepsilon + \|f^\varepsilon - f^0\|_{L^2(0,T;L^2(\Omega))}\}. \tag{5.7}$$

Before giving the main theorems of this section, let us introduce the notation and some lemmas. We recall the cost function $\mathcal{J}_\varepsilon(v)$ of the optimal control problem (1.1). Let

$$\mathcal{J}_\varepsilon(v) = g_\varepsilon(v) + j(v),$$

where

$$g_\varepsilon(v) = \int_0^T\int_\Omega |y^\varepsilon(v) - z_d|^2dxdt, \quad j(v) = \int_0^T\int_{\Omega_U} \gamma v^2 dxdt.$$

Let $\Omega_U \subset \Omega$ be a control domain, $\mathcal{U} = L^2(0,T;L^2(\Omega_U))$ be endowed with the scalar product $(\phi,\psi)_\mathcal{U} = \int_0^T\int_{\Omega_U}\phi\psi dxdt$, and $\mathcal{U}_{ad} \subset \mathcal{U}$ be the set of admissible controls. We denote by $\mathcal{J}_\varepsilon'$, g_ε', j' the Gâteaux derivatives of $\mathcal{J}_\varepsilon$, g_ε, and j, respectively.

LEMMA 5.3. *One can prove that*

$$(j'(u) - j'(v), u - v)_\mathcal{U} \ge \gamma_0\|u - v\|_\mathcal{U}^2, \quad u, v \in \mathcal{U}_{ad}. \tag{5.8}$$

Proof. Using the Hölder inequality, we have

$$|(j'(u) - j'(v))\cdot w| \le \int_0^T\int_\Omega |\gamma(u-v)w|dx \le M\|u-v\|_\mathcal{U}\|w\|_\mathcal{U} \tag{5.9}$$

and

$$\|(j'(u) - j'(v))\|_{\mathcal{U}\to\mathcal{U}} \le M\|u - v\|_\mathcal{U}. \tag{5.10}$$

Hence $j'(v)$ is Lipschitz continuous. On the other hand,

$$(j'(u) - j'(v), u - v)_{\mathcal{U}} = j'(u) \cdot (u - v) - j'(v) \cdot (u - v) \\ = \int_0^T \int_\Omega \gamma (u - v)^2 dxdt \geq \gamma_0 \|u - v\|_{\mathcal{U}}^2. \tag{5.11}$$

THEOREM 5.1. *Suppose that $\Omega \subset R^n$ is a bounded smooth domain with a pure Dirichlet boundary $\partial\Omega \in C^3$. Let $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ be the unique solution of the optimal control problem (1.4)–(1.6) with constraints, and assume that $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, t)$. Let $y_1^\varepsilon(x,t)$ and $\psi_1^\varepsilon(x,t)$ be the first order multiscale asymptotic solutions as defined in (2.1) and (2.2), respectively. Let $u^0(x,t)$ be the optimal control function of the homogenized problem (2.3)–(2.5). If $f, z_d \in H^{1,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{5}{2},\frac{3}{4}}(\partial\Omega \times (0,T))$, $\phi_0 \in H^2(\Omega)$, $\gamma \geq \gamma_0 > 0, \gamma \in H^{1,1}(\Omega \times (0,T)) \cap L^\infty(\Omega \times (0,T))$, and the compatibility relations (2.7) are satisfied, then we have*

$$\|u^\varepsilon - u^0\|_{\mathcal{U}} \leq C\varepsilon^{1/2}, \tag{5.12}$$

$$\sup_{0 \leq t \leq T} \int_\Omega (y^\varepsilon(u^\varepsilon) - y_1^\varepsilon(x,t))^2 dx + \int_0^T \|y^\varepsilon(u^\varepsilon) - y_1^\varepsilon\|_{H^1(\Omega)}^2 dt \leq C\varepsilon, \tag{5.13}$$

$$\sup_{0 \leq t \leq T} \int_\Omega (\psi^\varepsilon(u^\varepsilon) - \psi_1^\varepsilon(x,t))^2 dx + \int_0^T \|\psi^\varepsilon(u^\varepsilon) - \psi_1^\varepsilon\|_{H^1(\Omega)}^2 dt \leq C\varepsilon, \tag{5.14}$$

where C is a constant independent of ε.

Proof. Note that we set

$$\mathcal{L}_\varepsilon \equiv \frac{\partial}{\partial t} + \mathcal{A}_\varepsilon, \quad \tilde{\mathcal{L}}_\varepsilon \equiv -\frac{\partial}{\partial t} + \mathcal{A}_\varepsilon.$$

We rewrite (1.4), (1.5), and (1.6) as follows:

$$\mathcal{L}_\varepsilon y^\varepsilon(u^\varepsilon) = f + u^\varepsilon, \tag{5.15}$$

$$\tilde{\mathcal{L}}_\varepsilon \psi^\varepsilon(u^\varepsilon) = y^\varepsilon(u^\varepsilon) - z_d, \tag{5.16}$$

$$(\psi^\varepsilon(u^\varepsilon) + \gamma u^\varepsilon, v - u^\varepsilon)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad},\ u^\varepsilon \in \mathcal{U}_{ad}. \tag{5.17}$$

Similarly to Lemma 5.3, we can prove

$$(g_\varepsilon'(y^\varepsilon(u)) - g_\varepsilon'(y^\varepsilon(v)), y^\varepsilon(u) - y^\varepsilon(v))_{\mathcal{U}} \geq 0, \quad u, v \in \mathcal{U}_{ad}. \tag{5.18}$$

From (5.17), we have

$$\mathcal{J}_\varepsilon'(u) \cdot v = (\gamma u + \psi^\varepsilon(u), v)_{\mathcal{U}}. \tag{5.19}$$

Using (5.18) leads to

$$\begin{aligned}(\mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), u - v)_{\mathcal{U}} &= (\gamma(u - v), u - v)_{\mathcal{U}} + (\psi^\varepsilon(u) - \psi^\varepsilon(v), u - v)_{\mathcal{U}} \\ &= (\gamma(u - v), u - v)_{\mathcal{U}} \\ &\quad + (\psi^\varepsilon(u) - \psi^\varepsilon(v), \mathcal{L}_\varepsilon(y^\varepsilon(u)) - \mathcal{L}_\varepsilon(y^\varepsilon(v)))_{\mathcal{U}} \\ &= (\gamma(u - v), u - v)_{\mathcal{U}} \\ &\quad + (g_\varepsilon'(y^\varepsilon(u)) - g_\varepsilon'(y^\varepsilon(v)), y^\varepsilon(u) - y^\varepsilon(v))_{\mathcal{U}} \\ &\geq (\gamma(u - v), u - v)_{\mathcal{U}} \geq \gamma_0 \|u - v\|_{\mathcal{U}}^2.\end{aligned} \tag{5.20}$$

Let us turn to the proof of Theorem 5.1.

Suppose that $u^0 \in \mathcal{U}_{ad}$ is the optimal control function of the homogenized problem (2.3)–(2.5). We first consider the following auxiliary problems:

$$\begin{cases} \mathcal{L}_\varepsilon y^\varepsilon(u^0) = f(x,t) + u^0(x,t), \quad (x,t) \in \Omega \times (0,T), \\ y^\varepsilon(u^0) = g_0(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \\ y^\varepsilon(u^0)|_{t=0} = \phi_0(x), \quad x \in \Omega, \end{cases} \tag{5.21}$$

$$\begin{cases} \tilde{\mathcal{L}}_\varepsilon \psi^\varepsilon(u^0) = y^\varepsilon(u^0) - z_d(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \psi^\varepsilon(u^0) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \psi^\varepsilon(u^0)|_{t=T} = 0, \quad x \in \Omega. \end{cases} \tag{5.22}$$

For a given $u^0 \in \mathcal{U}_{ad}$, existence and uniqueness of the solutions of problems (5.21) and (5.22) can be established.

Let u^ε and $u^0 \in \mathcal{U}_{ad}$ be the optimal control functions of problems (1.4)–(1.6) and (2.3)–(2.5), respectively. From (1.6) and (5.20), we have

$$\begin{aligned} \|u^\varepsilon - u^0\|_{\mathcal{U}}^2 &\le C(\mathcal{J}_\varepsilon'(u^\varepsilon) - \mathcal{J}_\varepsilon'(u^0), u^\varepsilon - u^0)_{\mathcal{U}} \\ &= C(\mathcal{J}_\varepsilon'(u^\varepsilon), u^\varepsilon - u^0)_{\mathcal{U}} - C(\mathcal{J}_\varepsilon'(u^0), u^\varepsilon - u^0)_{\mathcal{U}} \\ &\le C(\mathcal{J}_\varepsilon'(u^0), u^0 - u^\varepsilon)_{\mathcal{U}} = (\gamma u^0 + \psi^\varepsilon(u^0), u^0 - u^\varepsilon)_{\mathcal{U}} \\ &= C(\widehat{\mathcal{J}}'(u^0), u^0 - u^\varepsilon)_{\mathcal{U}} + C(\psi^\varepsilon(u^0) - \psi^0(u^0), u^0 - u^\varepsilon)_{\mathcal{U}} \\ &\le C(\psi^\varepsilon(u^0) - \psi^0(u^0), u^0 - u^\varepsilon)_{\mathcal{U}} \\ &\le C\|\psi^\varepsilon(u^0) - \psi^0(u^0)\|_{\mathcal{U}} \|u^0 - u^\varepsilon\|_{\mathcal{U}} \\ &\le C\|\psi^\varepsilon(u^0) - \psi^0(u^0)\|_{L^2(0,T;L^2(\Omega))} \|u^0 - u^\varepsilon\|_{\mathcal{U}}, \end{aligned} \tag{5.23}$$

and consequently,

$$\|u^\varepsilon - u^0\|_{\mathcal{U}} \le C\|\psi^\varepsilon(u^0) - \psi^0(u^0)\|_{L^2(0,T;L^2(\Omega))}, \tag{5.24}$$

where $\widehat{\mathcal{J}}$ as given in (4.2) denotes the cost function of the homogenized problem (2.3)–(2.5), and $\widehat{\mathcal{J}}'$ is the Gâteaux derivative of functional $\widehat{\mathcal{J}}$.

We consider the following initial-boundary value problem:

$$\begin{cases} -\dfrac{\partial \psi^\varepsilon(u^0)}{\partial t} + \mathcal{A}_\varepsilon \psi^\varepsilon(u^0) = y^\varepsilon(u^0) - z_d, \quad (x,t) \in \Omega \times (0,T), \\ \psi^\varepsilon(u^0) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \psi^\varepsilon(u^0)|_{t=T} = 0, \quad x \in \Omega, \end{cases} \tag{5.25}$$

and the corresponding homogenized initial-boundary value problem is written as follows:

$$\begin{cases} -\dfrac{\partial \psi^0(u^0)}{\partial t} + \mathcal{A}_0 \psi^0(u^0) = y^0(u^0) - z_d, \quad (x,t) \in \Omega \times (0,T), \\ \psi^0(u^0) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \psi^0(u^0)|_{t=T} = 0, \quad x \in \Omega, \end{cases} \tag{5.26}$$

where the operator $\mathcal{A}_\varepsilon$ is given as in section 1, and the operator $\mathcal{A}_0$ is the same as given in (2.3).

As discussed in section 2.2, under the assumptions of Theorem 5.1, we can show that $u^0 \in H^{1,1}(\Omega \times (0,T))$, $y^0(u^0), \psi^0(u^0) \in H^{3,1}(\Omega \times (0,T))$.

Set $f^\varepsilon(x,t) = y^\varepsilon(u^0) - z_d(x,t)$ and $f^0(x,t) = y^0(u^0) - z_d(x,t)$ in Proposition 5.1. From (5.22) and (5.26), using Proposition 5.1, we can show that

$$\|\psi^\varepsilon(u^0) - \psi^0(u^0)\|_{L^2(0,T;L^2(\Omega))} \le \{\varepsilon^{1/2} + \|y^\varepsilon(u^0) - y^0(u^0)\|_{L^2(0,T;L^2(\Omega))}\}. \tag{5.27}$$

From (5.21), setting $f^\varepsilon(x,t) \equiv f^0(x,t) = f(x,t) + u^0(x,t)$ and applying Proposition 5.1 again, we get

$$\|y^\varepsilon(u^0) - y^0(u^0)\|_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{1/2}, \tag{5.28}$$

and consequently,

$$\|\psi^\varepsilon(u^0) - \psi^0(u^0)\|_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{1/2}. \tag{5.29}$$

Equation (5.12) follows from (5.24) and (5.29). Recalling (1.4), (2.3), (5.1), and (5.6), applying Lemma 5.1 and (5.24), we can prove (5.13). Similarly, combining (1.5), (2.4), and (5.28) gives (5.14). Therefore the proof of Theorem 5.1 is complete.

THEOREM 5.2. *Suppose that* $\Omega \subset R^n$ *is a bounded smooth domain with a pure Dirichlet boundary* $\partial\Omega \in C^4$. *Let* $\{y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ *be the unique solution of the optimal control problem* (1.4)–(1.6) *with constraints, and assume that* $a^\varepsilon_{ij}(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 1,2,3$. *Let* $y^\varepsilon_2(x,t)$, $\psi^\varepsilon_2(x,t)$ *be the second order multiscale asymptotic solutions as defined in* (2.1) *and* (2.2), *respectively. Let* $u^0(x,t)$ *be the optimal control function of the homogenized problem* (2.3)–(2.5). *If* $f, z_d \in H^{2,1}(\Omega \times (0,T))$, $g_0 \in H^{\frac{7}{2},\frac{7}{4}}(\partial\Omega\times(0,T))$, $\phi_0 \in H^3(\Omega)$, $\gamma \ge \gamma_0 > 0, \gamma \in H^{1,1}(\Omega\times(0,T))\cap L^\infty(\Omega\times(0,T))$, $y^0(u^0) \in H^{4,1}(\Omega \times (0,T))$, *and the compatibility relations* (2.7) *are satisfied, then we have*

$$\|u^\varepsilon - u^0\|_{\mathcal{U}} \le C\varepsilon^{1/2}, \tag{5.30}$$

$$\sup_{0\le t\le T} \int_\Omega (y^\varepsilon(x,t) - y^\varepsilon_2(x,t))^2 dx + \int_0^T \|y^\varepsilon - y^\varepsilon_2\|^2_{H^1(\Omega)} dt \le C\varepsilon, \tag{5.31}$$

$$\sup_{0\le t\le T} \int_\Omega (\psi^\varepsilon(x,t) - \psi^\varepsilon_2(x,t))^2 dx + \int_0^T \|\psi^\varepsilon - \psi^\varepsilon_2\|^2_{H^1(\Omega)} dt \le C\varepsilon, \tag{5.32}$$

where C *are constants independent of* ε.

Proof. Under the assumptions of Theorem 5.2, using Lemma 5.2 and Proposition 5.2, and repeating the process of the proof of Theorem 5.1, we can complete the proof of Theorem 5.2.

Remark 5.1. It should be emphasized that, for the two specific cases (2.8) and (2.9), under the assumptions of Theorem 5.2, we can show that the adjoint function $\psi^0(u^0) \in H^{4,2}(\Omega \times (0,T))$. However, for the state function $y^0(u^0)$, the best regularity one can expect is $y^0(u^0) \in H^{3,1}(\Omega \times (0,T))$ due to $u^0 \in H^{1,1}(\Omega \times (0,T))$. In order to obtain the convergence results presented in Theorem 5.2, the condition $y^0(u^0) \in H^{4,1}(\Omega \times (0,T))$ is imposed.

Remark 5.2. It should be stated that the regularity of the optimal control is a challenging problem in general cases even for the problem with constant coefficients. The derived error estimates in this paper are valid provided that the regularity of $y^0(u^0)$ and $\psi^0(u^0)$ is satisfied; even so, the formal multiscale asymptotic expansion is particularly useful for developing efficient numerical methods.

Remark 5.3. Since the homogenized coefficients $\hat{a}_{ij}$ are constants or sufficiently regular, we can obtain the higher order regularity of $y^0(u^0)$ and $\psi^0(u^0)$ in some specific cases. However, for the solution $\{p^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon), u^\varepsilon\}$ of the original optimal control problem (1.4)–(1.6), the best regularity one can expect is $u^\varepsilon \in L^2(0, T; L^2(\Omega_U))$, $y^\varepsilon(u^\varepsilon), \psi^\varepsilon(u^\varepsilon) \in L^2(0, T; V)$ due to the discontinuous coefficients, i.e., $a_{ij}^\varepsilon \in L^\infty(\Omega \times (0, T))$, where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. Therefore, for the original optimal control problem (1.4)–(1.6), both the numerical computation and the theoretical analysis are extremely difficult due to the low regularity on the optimal control. It implies that the homogenization method and the multiscale asymptotic methods are necessary and essential for the optimal control.

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