Journal of Mathematical Physics

New method for blowup of the Euler-Poisson system

Cite as: J. Math. Phys. **57**, 083501 (2016); https://doi.org/10.1063/1.4960472
Submitted: 29 January 2016 . Accepted: 25 July 2016 . Published Online: 05 August 2016

Man Kam Kwong, and Manwai Yuen

JOURNAL OF MATHEMATICAL PHYSICS **57**, 083501 (2016)

New method for blowup of the Euler-Poisson system

Man Kam Kwong[1,a)] and Manwai Yuen[2,b)]
[1]*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*
[2]*Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong*

(Received 29 January 2016; accepted 25 July 2016; published online 5 August 2016)

In this paper, we provide a new method for establishing the blowup of C^2 solutions for the pressureless Euler-Poisson system with attractive forces for R^N $(N \geq 2)$ with $\rho(0,x_0) > 0$ and $\Omega_{0ij}(x_0) = \frac{1}{2}\left[\partial_i u^j(0,x_0) - \partial_j u^i(0,x_0)\right] = 0$ at some point $x_0 \in R^N$. By applying the generalized Hubble transformation $\operatorname{div} u(t,x_0(t)) = \frac{N\dot{a}(t)}{a(t)}$ to a reduced Riccati differential inequality derived from the system, we simplify the inequality into the Emden equation $\ddot{a}(t) = -\frac{\lambda}{a(t)^{N-1}}$, $a(0) = 1$, $\dot{a}(0) = \frac{\operatorname{div} u(0,x_0)}{N}$. Known results on its blowup set allow us to easily obtain the blowup conditions of the Euler-Poisson system. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4960472]

I. INTRODUCTION

The compressible Euler-Poisson system in R^N refers to the equations,

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] = \delta \rho \nabla \Phi, \\ \Delta \Phi(t,x) = \rho, \end{cases} \tag{1}$$

where $\rho = \rho(t,x) \geq 0$ and $u = u(t,x) \in \mathbf{R}^N$ are the density and the velocity, respectively, of the fluid under study. If $\delta = -1$, the system has been used to model self-gravitating fluids such as gaseous stars in cosmology.[1–3] In particular, details of the connection between the Euler-Poisson system and Einstein's field system are given in Longair.[4] If $\delta = 0$, equation set (1) constitutes the compressible Euler system, which is a classical model in fluid mechanics; see, for example, Ref. 5. If $\delta = 1$, the system can be viewed as a semiconductor model; see, for example, Ref. 6.

For further analysis of the Euler-Poisson system, interested readers are referred to Refs. 7–20. In addition, explicit blowup or global (periodical) solutions to the Euler-Poisson system can be found in Refs. 21–25.

In 2008, Chae and Tadmor[17] established the finite-time blowup for the pressureless Euler-Poisson system (1) with attractive forces ($\delta = -1$) under the initial condition,

$$S := \{ x_0 \in R^N \,|\, \rho_0(x_0) > 0, \quad \Omega_0(x_0) = 0, \quad \operatorname{div} u(0,x_0) < 0\} \neq \phi, \tag{2}$$

where $u = (u^1, u^2, \ldots, u^N)$ and $\Omega_0(x_0)$ is the vorticity matrix defined by

$$\Omega_{0ij}(x_0) = \frac{1}{2}\left[\partial_i u^j(0,x_0) - \partial_j u^i(0,x_0)\right]. \tag{3}$$

Applying spectral dynamics analysis, they derived the Riccati differential inequality,

$$\frac{D \operatorname{div} u(t,x_0(t))}{Dt} \leq -\frac{1}{N}[\operatorname{div} u(t,x_0(t))]^2 \tag{4}$$

a)E-mail address: mankwong@polyu.edu.hk.
b)Author to whom correspondence should be addressed. Electronic mail: nevetsyuen@hotmail.com.

along the characteristic line $\frac{dx_0(t)}{dt} = u(t, x_0(t))$. The corresponding solution of inequality (4) blows up at or before $T = -N/\text{div}\, u(0, x_0(0))$ with an initial condition requiring that $\text{div}\, u(0, x_0(0))$ takes a non-vacuum form. An improved blowup condition for the Euler-Poisson system (1) was obtained by Cheng and Tadmor[18] in 2009.

In this paper, relative to the methods in Refs. 13, 17, and 18, we provide a new and shorter proof for the blowup of C^2 solutions for the pressureless Euler-Poisson system (1) for attractive forces ($\delta = -1$) in R^N and repulsive forces ($\delta = 1$) in R.

Theorem 1. *For the pressureless Euler-Poisson system (1) with* $\rho(0, x_0) > 0$ *and* $\Omega_{0ij}(x_0) = 0$ *(see (6)) at some point* x_0*,*

- *(I) with attractive forces* ($\delta = -1$)*, and one of the two following conditions satisfied, i.e.,*
  - *(Ia)* $N = 1$ *or* 2*, or*
  - *(Ib)* $N \geq 3$*, satisfying*

$$\text{div}\, u(0, x_0) < \sqrt{\frac{2N\rho(0, x_0)}{(N-2)}}, \tag{5}$$

or

- *(II) with repulsive forces* ($\delta = 1$)*, and* $N = 1$ *and satisfying*

$$\text{div}\, u(0, x_0) \leq -\sqrt{2\rho(0, x_0)}, \tag{6}$$

the C^2 *solutions blow up in finite time* T.

II. BLOWUP FOR THE EMDEN EQUATION WITH $N \geq 2$

We now apply the generalized Hubble transformation for $N \geq 2$,

$$\text{div}\, u(t, x_0(t)) = \frac{N\dot{a}(t)}{a(t)}, \tag{7}$$

to shorten the proofs in Refs. 13, 17, and 18. We remark that when $N = 1$, transformation (7) is the classical Hubble transformation in Astrophysics. We have applied this transformation previously in studying the exact solutions to the compressible Euler system ($\delta = 0$) in Ref. 26.

Proof. Because the mass equation $(1)_1$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot u = 0 \tag{8}$$

with convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla) \tag{9}$$

can be integrated to give

$$\rho(t, x_0(t)) = \rho(0, x_0)e^{-\int_0^t \text{div}\, u(s, x_0(s))ds} \geq 0 \tag{10}$$

for $\rho_0(0, x_0) \geq 0$, the density function $\rho(t, x_0(t))$ retains its non-negative nature in the classical solution.

For the momentum equations $(1)_2$ with $\delta = -1$ and the solutions with a non-vacuum form (that is, $\rho(t, x_0) > 0$ at some point x_0), we have

$$u_t + u\nabla \cdot u = -\nabla\Phi. \tag{11}$$

We use the divergence of this equation to obtain

$$\nabla \cdot (u_t + u\nabla \cdot u) = -\Delta\Phi. \tag{12}$$

If the initial condition

$$\Omega_{0ij}(x_0) = \frac{1}{2}\left[\partial_i u^j(0, x_0) - \partial_j u^i(0, x_0)\right] = 0 \tag{13}$$

is fulfilled, the spectral dynamics technique in Refs. 17 and 18 (more specifically, refer to Equation (2.6) in Ref. 17, or (4.1) in Ref. 18) yields

$$\frac{D}{Dt}\operatorname{div} u(t, x_0(t)) + \frac{1}{N}[\operatorname{div} u(t, x_0(t))]^2 \leq -\rho(t, x_0(t)). \tag{14}$$

Alternatively, we can obtain inequality (14) as follows. Take the divergence of equations $(1)_2$ and write the time derivative of $\operatorname{div} u$ as a function of

$$\frac{D}{Dt}(\operatorname{div} u) + \sigma^2 - \Omega^2 = -\rho, \tag{15}$$

where σ is the symmetric shear rate tensor and Ω the antisymmetric vorticity tensor. Since the initial condition stipulates that $\Omega(x_0)$ is 0 at $t = 0$, it remains 0 in the fluid evolution, by one of the Helmholtz theorems (see Section 1.2 in Chorin and Marsden's book[27]). Finally, we apply the inequality

$$Tr(\sigma)^2 \geq \frac{1}{N}(Tr(\sigma))^2 = \frac{(\operatorname{div} u)^2}{N} \tag{16}$$

to obtain inequality (14).

We notice that the advancement of this short paper starts here. We know from Equation (10) that

$$\frac{D\operatorname{div} u(t, x_0(t))}{Dt} + \frac{1}{N}[\operatorname{div} u(t, x_0(t))]^2 \leq -\rho(0, x_0)e^{-\int_0^t \operatorname{div} u(s, x_0(s))ds}. \tag{17}$$

By applying the generalized Hubble transformation,

$$\operatorname{div} u(t, x_0(t)) = \frac{N\dot{a}(t)}{a(t)}, \tag{18}$$

it becomes

$$\frac{D}{Dt}\frac{N\dot{a}(t)}{a(t)} + \frac{1}{N}\left[\frac{N\dot{a}(t)}{a(t)}\right]^2 \leq -\rho(0, x_0)e^{-\int_0^t \frac{N\dot{a}(s)}{a(s)}ds}, \tag{19}$$

$$\frac{-N\dot{a}(t)^2}{a(t)^2} + \frac{N\ddot{a}(t)}{a(t)} + \frac{N\dot{a}(t)^2}{a(t)^2} \leq -\rho(0, x_0)e^{-N\ln a(t) + N\ln a(0)}, \tag{20}$$

$$\frac{N\ddot{a}(t)}{a(t)} \leq -\rho(0, x_0)a(0)^N e^{\ln\frac{1}{a(t)^N}}, \tag{21}$$

$$\ddot{a}(t) \leq -\frac{\rho(0, x_0)a(0)^N}{Na(t)^{N-1}}, \text{ for } a(t) > 0. \tag{22}$$

At the same time, it is known that the solution of the Emden equation,

$$\ddot{a}(t) = -\frac{\lambda}{a(t)^{N-1}}, a(0) = a_1 > 0, \ \dot{a}(0) = a_2, \tag{23}$$

blows up if $\lambda > 0$ and one of the following conditions is satisfied:
(1) $N = 2$ (see Yuen[24]), or
(2) $N > 2$ (see Deng, Xiang, and Yang[23]) and

$$a_2 < \sqrt{\frac{2\lambda}{(N-2)a_1^{N-2}}}. \tag{24}$$

By comparing the two ordinary differential equations (22) and (23) for $N \geq 3$, we can set

$$\lambda := \frac{\rho(0, x_0)a(0)^N}{N}, \tag{25}$$

with the initial conditions

$$a(0) = 1 \quad \text{and} \quad \dot{a}(0) = \frac{\operatorname{div} u(0, x_0)}{N}. \tag{26}$$

Using the blowup condition (24) for the Emden equation, we see that the solution of the Emden inequality (22) blows up for $N \geq 3$ if

$$\operatorname{div} u(0, x_0) < \sqrt{\frac{2N\rho(0, x_0)}{(N-2)}}. \tag{27}$$

For $N = 1$ and for $\lambda > 0$, there exists a finite time T such that $a(T) = 0$. Thus, function (18) blows up in finite time T.

For attractive force ($\delta = 1$) and $N = 1$, we have the corresponding result,

$$\frac{D}{Dt}\operatorname{div} u(t, x_0(t)) + [\operatorname{div} u(t, x_0(t))]^2 = \rho(t, x_0(t)). \tag{28}$$

Then, we obtain

$$\ddot{a}(t) - \rho(0, x_0) = 0, \quad a(0) = 1, \quad \text{and} \quad \dot{a}(0) = \operatorname{div} u(0, x_0). \tag{29}$$

The exact solution is

$$a(t) = \frac{1}{2}\rho(0, x_0)t^2 + \operatorname{div} u(0, x_0)t + 1. \tag{30}$$

By the quadratic formula, there exists a finite time

$$T_1 = \frac{-\operatorname{div} u(0, x_0) - \sqrt{[\operatorname{div} u(0, x_0)]^2 - 2\rho(0, x_0)}}{\rho(0, x_0)} > 0 \tag{31}$$

with

$$\operatorname{div} u(0, x_0) \leq -\sqrt{2\rho(0, x_0)} \quad \text{and} \quad \rho(0, x_0) > 0 \tag{32}$$

such that $a(T_1) = 0$. Thus, the function $\operatorname{div} u(t, x_0(t))$ blows up in finite time T.

This completes the proof. ■

III. DISCUSSION AND CONCLUSION

In this paper, by comparison with the arguments in Refs. 13, 17, and 18, we provide a novel and briefer method for proving the blowup of C^2 solutions to the pressureless Euler-Poisson system (1) with attractive forces ($\delta = -1$) for R^N ($N \geq 2$) with $\rho(0, x_0) > 0$ and

$$\Omega_{0ij}(x_0) = \frac{1}{2}\left[\partial_i u^j(0, x_0) - \partial_j u^i(0, x_0)\right] = 0 \tag{33}$$

at some point x_0. By applying the generalized Hubble transformation,

$$\operatorname{div} u(t, x_0(t)) = \frac{N\dot{a}(t)}{a(t)}, \tag{34}$$

to the reduced Riccati differential inequality (14) derived for the system, we can simplify it to the Emden equation

$$\ddot{a}(t) = -\frac{\lambda}{a(t)^{N-1}}, \quad a(0) = 1, \quad \dot{a}(0) = \frac{\operatorname{div} u(0, x_0)}{N}. \tag{35}$$

By using known results on the blowup set of the Emden equation, we readily obtain the blowup conditions of the Euler-Poisson system for attractive forces. Of particular note is that the Emden equation appears in the self-similar solutions for the Euler[28,26] and Euler-Poisson[21–24] systems.

Further research is highly recommended to investigate whether blowup phenomena exist for

$$\Omega_{0ij}(x_0) \neq 0 \tag{36}$$

with attractive forces ($\delta = -1$) for R^N and repulsive forces ($\delta = 1$) for $R^N (N \geq 2)$.

ACKNOWLEDGMENTS

The authors thank the comments from the reviewers and editors for improving the quality of this article.

The work of the first author in this paper was supported by the Hong Kong Government GRF Grant No. PolyU 5003/12P and the Hong Kong Polytechnic University Grant Nos. G-UC22 and G-UA10. The work of the second author in this paper was partially supported by the research Grant No. RG 11/2015-2016R from the Education University of Hong Kong.

[1] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, IL, 1939).
[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Fracisco, 1973).
[3] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, NJ, 1994).
[4] M. Longair, *The Cosmic Century: A History of Astrophysics and Cosmology* (Cambridge University Press, Cambridge, 2006).
[5] P. L. Lions, *Mathematical Topics in Fluid Mechanics* (Clarendon Press, Oxford, 1998), Vol. 1 and 2.
[6] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum, New York, 1984).
[7] U. Brauer, A. Rendall, and O. Reula, "The cosmic no-hair theorem and the non-linear stability of homogeneous Newtonian cosmological models," Classical Quantum Gravity **11**, 2283–2296 (1994).
[8] S. Engelberg, "Formation of singularities in the Euler and Euler-Poisson equations," Physica D **98**, 67–74 (1996).
[9] T. Makino, "On a local existence theorem for the evolution equation of gaseous stars," Stud. Math. Appl. **18**, 459–479 (1986).
[10] T. Makino and B. Perthame, "Sur les solutions a symmetric spherique de l'equation d'Euler-Poisson pour l'evolution d'etoiles gazeuses (French) [on radially symmetric solutions of the Euler-Poisson equation for the evolution of gaseous stars]," Jpn. J. Appl. Math. **7**, 165–170 (1990).
[11] B. Perthame, "Nonexistence of global solutions to Euler-Poisson equations for repulsive forces," Jpn. J. Appl. Math. **7**, 363–367 (1990).
[12] U. Brauer, "Breakdown of smooth solutions of the three-dimensional Euler-Poisson system," J. Math. Phys. **39**, 1050–1074 (1998).
[13] S. Engelberg, H. L. Liu, and E. Tadmor, "Critical thresholds in Euler-Poisson equations," Indiana Univ. Math. J. **50**, 109–157 (2001).
[14] Y. B. Deng, T. P. Liu, T. Yang, and Z. A. Yao, "Solutions of Euler-Poisson equations for gaseous stars," Arch. Ration. Mech. Anal. **164**, 261–285 (2002).
[15] H. L. Liu and E. Tadmor, "Critical thresholds in 2D restricted Euler-Poisson equations," SIAM J. Appl. Math. **63**, 1889–1910 (2003).
[16] M. W. Yuen, "Stabilities for Euler-Poisson equations in some special dimensions," J. Math. Anal. Appl. **344**, 145–156 (2008).
[17] D. H. Chae and E. Tadmor, "On the finite time blow-up of the Euler-Poisson equations in R^N," Commun. Math. Sci. **6**, 785–789 (2008).
[18] B. Cheng and E. Tadmor, "An improved local blow-up condition for Euler-Poisson equations with attractive forcing," Physica D **238**, 2062–2066 (2009).
[19] M. W. Yuen, "Blowup for the C^1 solutions of the Euler-Poisson equations of gaseous stars in R^N," J. Math. Anal. Appl. **383**, 627–633 (2011).
[20] M. W. Yuen, "Blowup for C^2 Solutions of the N-dimensional Euler–Poisson equations in Newtonian cosmology," J. Math. Anal. Appl. **415**, 972–978 (2014).
[21] P. Goldreich and S. Weber, "Homologously collapsing stellar cores," Astrophys. J. **238**, 991–997 (1980).
[22] T. Makino, "Blowing up solutions of the Euler-Poission equation for the evolution of the gaseous stars," Transp. Theory Stat. Phys. **21**, 615–624 (1992).
[23] Y. B. Deng, J. L. Xiang, and T. Yang, "Blowup phenomena of solutions to Euler-Poisson equations," J. Math. Anal. Appl. **286**, 295–306 (2003).
[24] M. W. Yuen, "Analytical blowup solutions to the 2-dimensional isothermal Euler-Poisson equations of gaseous stars," J. Math. Anal. Appl. **341**, 445–456 (2008).
[25] M. K. Kwong and M. W. Yuen, "Periodic solutions of 2D isothermal Euler-Poisson equations with possible applications to spiral and disk-like galaxies," J. Math. Anal. Appl. **420**, 1854–1863 (2014).
[26] M. W. Yuen, "Perturbational blowup solutions to the compressible 1-dimensional Euler equations," Phys. Lett. A **375**, 3821–3825 (2011).
[27] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, 3rd ed. Texts in Applied Mathematics (Springer-Verlag, New York, 1993), Vol. 4.
[28] T. Makino, "Exact solutions for the compressible Euler equation," J. Osaka Sangyo Univ., Nat. Sci. **95**, 21–35 (1993).