SIAM J. Optim.
Vol. 21, No. 3, pp. 730–760

STABLE AND TOTAL FENCHEL DUALITY FOR DC OPTIMIZATION PROBLEMS IN LOCALLY CONVEX SPACES*

D. H. FANG[†], C. LI[‡], and X. Q. YANG[§]

Abstract. We consider the DC (difference of two convex functions) optimization problem $(P)\quad \inf_{x\in X}\{(f_1(x)-f_2(x))+(g_1(Ax)-g_2(Ax))\}$, where f_1, f_2, g_1, and g_2 are proper convex functions defined on locally convex Hausdorff topological vector spaces X and Y, and A is a linear continuous operator from X to Y. Adopting different tactics, two types of the Fenchel dual problems of (P) are given. By using the properties of the epigraph of the conjugate functions, some sufficient and necessary conditions for the weak duality of (P) are provided. Sufficient and/or necessary conditions for the strong Fenchel duality, the stable Fenchel duality, and the stable total duality are derived.

Key words. strong Fenchel duality, total Fenchel duality, difference of two convex functions programming

AMS subject classifications. 90C26, 90C46

DOI. 10.1137/100789749

1. Introduction.

Let X and Y be real locally convex Hausdorff topological vector spaces, whose respective dual spaces, X^* and Y^*, are endowed with the weak*-topologies $w^*(X^*,X)$ and $w^*(Y^*,Y)$, respectively. Let $f\colon X\to\bar{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$ and $g\colon Y\to\bar{\mathbb{R}}$ be proper functions, and let $A\colon X\to Y$ be a linear continuous operator satisfying $A(\operatorname{dom} f)\cap\operatorname{dom} g\neq\emptyset$. Consider the primal problem

$$(\mathcal{P})\quad \inf_{x\in X}\{f(x)+g(Ax)\}$$

and its associated Fenchel dual problem

$$(\mathcal{D})\quad \sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\},$$

where f^* and g^* are the Fenchel conjugates of f and g, respectively, and $A^*\colon Y^*\to X^*$ stands for the adjoint operator.

It is well-known that the optimal values of these problems, $v(\mathcal{P})$ and $v(\mathcal{D})$, respectively, satisfy the so-called weak duality (i.e., $v(\mathcal{P})\geq v(\mathcal{D})$), but a duality gap may occur (i.e., we may have $v(\mathcal{P})>v(\mathcal{D})$). A challenge in convex analysis has been to give

*Received by the editors March 22, 2010; accepted for publication (in revised form) May 10, 2011; published electronically August 23, 2011.

http://www.siam.org/journals/siopt/21-3/78974.html

[†]College of Mathematics and Computer Science, Jishou University, Jishou 416000, People's Republic of China, and Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China (dh_fang@jsu.edu.cn). This author was supported by the Scientific Research Fund of Hunan Provincial Education Department grant 08A057 as a visiting research student by the Research Committee of the Hong Kong Polytechnic University (1-ZV96) while this work was partly completed.

[‡]Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China, and Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia (cli@zju.edu.cn). This author's research was supported in part by National Natural Science Foundation of China grant 10731060 and by Zhejiang Provincial Natural Science Foundation of China grant Y6110006.

[§]Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong (mayangxq@polyu.edu.hk). This author's research was supported in part by the Research Grants Council of Hong Kong (PolyU 5334/08E).

sufficient conditions which guarantee the strong duality, i.e., the situation when there is no duality gap and the dual problem has at least an optimal solution. In the case when f and g are proper convex functions, several interiority-type conditions were given in order to preclude the existence of such a duality gap in different settings (see, for instance, [2], [5], [20], and [35, Theorem 2.8.3]). Taking inspiration from Burachik and Jeyakumar [9], [10], some authors approached the strong duality by using some properties of the epigraphs of the functions f^* and g^* (see, for instance, [7], [11]). In particular, Li. et al. considered in [25] the case when f and g are not necessarily convex, and they gave some sufficient and/or necessary conditions for the strong duality between $(\mathcal{P})$ and $(\mathcal{D})$.

Another related and interesting problem is the total duality, which corresponds to the situation in which $v(\mathcal{P}) = v(\mathcal{D})$ and both problems $(\mathcal{P})$ and $(\mathcal{D})$ have optimal solutions. This problem was considered in [2], [25] for the case in which f and g are proper convex functions. But to the best of our knowledge, the total duality has not been considered in the case where the involved functions are not convex.

Recently, the DC (difference of two convex functions) programming problem has received much attention (cf. [1], [6], [8], [13], [14], [15], [16], [21], [28], [34], and the references therein). The reason is, as mentioned in [13], that DC programming problems are very important from both viewpoints of optimization theory and applications: on the one hand, such problems being heavily nonconvex can be considered as a special class of nondifferentiable programming (in particular, quasi-differentiable programming [12]) and thus advanced techniques of variational analysis and generalized differentiation developed, e.g., in [12], [29], [32], can be applied and, on the other hand, the special convex structure of both plus function and minus function offers the possibility to use powerful tools of convex analysis in the study of DC programming.

Inspired by the works mentioned above, we continue to study the optimization problem $(\mathcal{P})$ but with $f := f_1 - f_2$ and $g := g_1 - g_2$ being two DC functions, that is, the primal problem defined by

$$(P) \quad \inf_{x\in X}\{f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax)\}, \tag{1.1}$$

where $f_1, f_2 : X \to \bar{\mathbb{R}}$, $g_1, g_2 : Y \to \bar{\mathbb{R}}$ are proper convex functions. Our interest here is the investigation of strong dualities.

In the case when f_2 and g_2 are lower semicontinuous (lsc in brief), the standard convexification technique can be applied. In fact, in this case, the problem (P) can be reformulated equivalently as the following one:

$$\inf_{\substack{u^*\in\operatorname{dom} f_2^*\\ v^*\in\operatorname{dom} g_2^*}} \inf_{x\in X}\{f_1(x) + f_2^*(u^*) - \langle u^*, x\rangle + g_1(Ax) + g_2^*(v^*) - \langle Ax, v^*\rangle\}.$$

Note that, for each $u^* \in \operatorname{dom} f_2^*$ and $v^* \in \operatorname{dom} g_2^*$, the subproblem

$$(P_{(u^*,v^*)}) \quad \inf_{x\in X}\{f_1(x) + f_2^*(u^*) - \langle u^*, x\rangle + g_1(Ax) + g_2^*(v^*) - \langle Ax, v^*\rangle\}$$

is a convex optimization problem, and its Fenchel dual problem is

$$(D_{(u^*,v^*)}) \quad \sup_{y^*\in Y^*}\{-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\}. \tag{1.2}$$

Thus, this reformulation motivates us to define the following (convexification) dual problem of (P):

$$(1.3)\qquad (D^C)\quad \inf_{\substack{u^*\in \operatorname{dom} f_2^*\\ v^*\in \operatorname{dom} g_2^*}} \sup_{y^*\in Y^*} \{-f_1^*(u^*-A^*y^*)+f_2^*(u^*)-g_1^*(y^*+v^*)+g_2^*(v^*)\}.$$

Here and throughout the whole paper, following [35, page 39], we adapt the convention that $(+\infty)+(-\infty)=(+\infty)-(+\infty)=+\infty$ and $0\cdot\infty=0$. Then, for any two proper convex functions $h_1, h_2 : X\to\bar{\mathbb{R}}$, we have that

$$(1.4)\qquad h_1(x)-h_2(x)\begin{cases} \in\mathbb{R}, & x\in \operatorname{dom} h_1\cap \operatorname{dom} h_2,\\ =-\infty, & x\in \operatorname{dom} h_1\setminus \operatorname{dom} h_2,\\ =+\infty, & x\notin \operatorname{dom} h_1.\end{cases}$$

Hence,

$$(1.5)\qquad h_1-h_2 \text{ is proper} \Leftrightarrow \operatorname{dom} h_1\subseteq \operatorname{dom} h_2.$$

Another approach is to consider the corresponding dual problem $(\mathcal{D})$ of (1.1) (with f_1-f_2 and g_1-g_2 in place of f and g, respectively), which is clearly independent of the special convex structure of DC functions. Note that, in the case when f_2 and g_2 are lsc, the conjugates of DC functions f_1-f_2 and g_1-g_2 can be expressed as

$$(1.6)\qquad (f_1-f_2)^*(-A^*y^*)=\sup_{u^*\in \operatorname{dom} f_2^*}\{f_1^*(u^*-A^*y^*)-f_2^*(u^*)\}$$

and

$$(1.7)\qquad (g_1-g_2)^*(y^*)=\sup_{v^*\in \operatorname{dom} g_2^*}\{g_1^*(v^*+y^*)-g_2^*(v^*)\},$$

respectively, (cf. Lemma 2.3 in section 2). We are using a formulation of (D) for $f:=f_1-f_2$ and $g:=g_1-g_2$ which, in view of the two equalities above, reduces to the following (Fenchel) dual problem of (P):

$$(1.8)\qquad (D^F)\quad \sup_{y^*\in Y^*} \inf_{\substack{u^*\in \operatorname{dom} f_2^*\\ v^*\in \operatorname{dom} g_2^*}} \{-f_1^*(u^*-A^*y^*)+f_2^*(u^*)-g_1^*(y^*+v^*)+g_2^*(v^*)\}.$$

Note that, without assuming the lower semicontinuity of f_2 and g_2, (1.6) and (1.7) do not necessarily hold. Thus, $(\mathcal{D})$ and (D^F) are, in general, not equivalent.

Let $v(P)$, $v(D^F)$, and $v(D^C)$ denote the optimal values of problems (P), (D^F), and (D^C), respectively. Obviously, $v(D^F)\le v(D^C)$. However, the weak dualities between (P) and (D^F) and between (P) and (D^C) do not necessarily hold, in general, as to be shown in Example 3.1 in section 3. Our main aim in the present paper is to use the epigraph technique to provide some new regularity conditions, which characterize the weak dualities, the strong dualities, the stable strong dualities, as well as the total dualities between (P) and (D^C) and between (P) and (D^F). The epigraph technique has been used extensively and has shown great power in convex programming; see, for example, [2], [3], [4], [9], [10], [13], [14], [15], [16], [18], [22], [23], [24], [25], [26], [27]. In general, we assume only that f_1, f_2 and g_1, g_2 are convex functions (not necessarily lsc) and that, to avoid the triviality in our study for (1.1),

$$\Omega := A(\operatorname{dom}(f_1-f_2))\cap \operatorname{dom}(g_1-g_2)\neq\varnothing.$$

Most of the results obtained in the present paper seem new and are proper extensions of the results in [2] and [25] in the special case when $f_2=g_2=0$. As we noted earlier, the equivalence between $(\mathcal{D})$ and (D^F) does not necessarily hold. Also, we consider the strong duality and the total duality between (P) and (D^C), and provide some conditions ensuring the equivalence of the strong dualities between (P) and (D^C) and between (P) and (D^F). In particular, both our dual problems and the regularity conditions introduced here are defined in terms of conjugates of the convex functions f_1, f_2, g_1, and g_2 rather than of the DC functions $f_1 - f_2$ and $g_1 - g_2$, which are different from the consideration in [25] for the general (not necessarily convex) case.

The paper is organized as follows. The next section contains some necessary notation and preliminary results. In section 3, some new constraint qualifications are introduced to study the weak dualities and the strong dualities. The stable strong dualities and the stable total dualities are considered in sections 4 and 5, respectively.

2. Notation and preliminaries. The notation used in the present paper is standard (cf. [19], [35]). In particular, we assume throughout the whole paper that X and Y are real locally convex Hausdorff topological vector spaces, and let X^* denote the dual space, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$ we shall denote the value of the functional $x^* \in X^*$ at $x \in X$; i.e., $\langle x^*, x\rangle = x^*(x)$. Let Z be a set in X. The closure of Z is denoted by $\operatorname{cl} Z$. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The indicator function δ_Z and the support function σ_Z of the nonempty set Z are, respectively, defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\sigma_Z(x^*) := \sup_{x\in Z}\langle x^*, x\rangle \quad \text{for each } x^* \in X^*.$$

Let $f\colon X \to \bar{\mathbb{R}}$ be a proper function. The effective domain, the conjugate function, and the epigraph of f are denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively; they are defined by

$$\operatorname{dom} f := \{x \in X\colon f(x) < +\infty\},$$

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x)\colon x \in X\} \quad \text{for each } x^* \in X^*,$$

and

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R}\colon f(x) \le r\}.$$

It is well-known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The lsc hull of f, denoted by $\operatorname{cl} f$, is defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

Then (cf. [35, Theorem 2.3.1]),

$$f^* = (\operatorname{cl} f)^*. \tag{2.1}$$

By [35, Theorem 2.3.4], if $\operatorname{cl} f$ is proper and convex, then the following equality holds:

$$f^{**} = \operatorname{cl} f. \tag{2.2}$$

Let $x \in X$. The subdifferential of f at x is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y) \quad \text{for each } y \in X\} \tag{2.3}$$

if $x \in \operatorname{dom} f$, and $\partial f(x) := \varnothing$ otherwise. By definition, the Young–Fenchel inequality below holds:

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle \quad \text{for each pair } (x, x^*) \in X \times X^*. \tag{2.4}$$

Moreover, by [35, Theorem 2.4.2(iii)],

$$f(x) + f^*(x^*) = \langle x^*, x\rangle \quad \text{if and only if } x^* \in \partial f(x) \tag{2.5}$$

(the equality in (2.5) is usually referred to as Young's equality). If g, h are proper, then

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g + h)^*, \tag{2.6}$$

$$g \leq h \Rightarrow g^* \geq h^* \Leftrightarrow \operatorname{epi} g^* \subseteq \operatorname{epi} h^*, \tag{2.7}$$

and

$$\partial g(a) + \partial h(a) \subseteq \partial(g + h)(a) \quad \text{for each } a \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.8}$$

Furthermore, if g, h are convex and $\operatorname{cl} g$, $\operatorname{cl} h$ are proper, then

$$\operatorname{cl} g \leq \operatorname{cl} h \Leftrightarrow \operatorname{epi} g^* \subseteq \operatorname{epi} h^*. \tag{2.9}$$

The following lemma is known in [17] and [35] (cf. [17, Lemma 2.1] for (2.10) and (2.11), and [35, Theorem 2.8.7] for (2.12)).

LEMMA 2.1. *Let $g, h : X \to \bar{\mathbb{R}}$ be proper convex functions satisfying* $\operatorname{dom} g \cap \operatorname{dom} h \neq \varnothing$.

(i) *If g, h are lsc, then*

$$\operatorname{epi}(g + h)^* = \operatorname{cl}(\operatorname{epi} g^* + \operatorname{epi} h^*). \tag{2.10}$$

(ii) *If either g or h is continuous at some point of* $\operatorname{dom} g \cap \operatorname{dom} h$, *then*

$$\operatorname{epi}(g + h)^* = \operatorname{epi} g^* + \operatorname{epi} h^*, \tag{2.11}$$

and

$$\partial(g + h)(x) = \partial g(x) + \partial h(x) \quad \textit{for each} \quad x \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.12}$$

The following lemma is a direct consequence of the definitions of a conjugate function and an epigraph. In particular, statements (i) and (ii) were used in [35, Theorem 2.13(i)] and [26, equation (2.5)], respectively.

LEMMA 2.2. *Let I be an index set, and let $\{f_i : i \in I\}$ be a family of functions. Then the following statements hold:*

(i) $\operatorname{epi}(\sup_{i\in I} f_i) = \cap_{i\in I} \operatorname{epi} f_i$.

(ii) $(\inf_{i\in I} f_i)^* = \sup_{i\in I} f_i^*$; *consequently,* $\operatorname{epi}(\inf_{i\in I} f_i)^* = \cap_{i\in I} \operatorname{epi} f_i^*$.

LEMMA 2.3. *Let $f, g\colon X \to \bar{\mathbb{R}}$ be proper functions. Suppose that g is lsc and convex. Then for each $p \in X^*$,*

$$(f-g)^*(p) = \sup_{u^*\in \operatorname{dom} g^*} \{f^*(p+u^*) - g^*(u^*)\}. \tag{2.13}$$

Proof. Let $p \in X^*$. By (2.2),

$$g(\cdot) = g^{**}(\cdot) = \sup_{x^*\in \operatorname{dom} g^*} \{\langle x^*, \cdot\rangle - g^*(x^*)\}. \tag{2.14}$$

Then, by definition and (2.14), we have that

$$\begin{aligned}
(f-g)^*(p) &= \sup_{x\in X}\{\langle p, x\rangle - (f-g)(x)\}\\
&= \sup_{x\in X}\sup_{x^*\in \operatorname{dom} g^*}\{(\langle x^*, x\rangle - g^*(x^*)) - (f(x) - \langle p, x\rangle)\}\\
&= \sup_{x^*\in \operatorname{dom} g^*}\sup_{x\in X}\{\langle p+x^*, x\rangle - f(x) - g^*(x^*)\}\\
&= \sup_{x^*\in \operatorname{dom} g^*}\{\sup_{x\in X}\{\langle p+x^*, x\rangle - f(x)\} - g^*(x^*)\}\\
&= \sup_{x^*\in \operatorname{dom} g^*}\{f^*(p+x^*) - g^*(x^*)\}.
\end{aligned}$$

Hence, (2.13) holds, and the proof is complete. ∎

We end this section with a remark that an element $p \in X^*$ can be naturally regarded as a function on X in such a way that

$$p(x) := \langle p, x\rangle \quad \text{for each } x \in X. \tag{2.15}$$

Thus, the following facts are clear for any $a \in \mathbb{R}$ and any function $h\colon X \to \bar{\mathbb{R}}$:

$$(h+p+a)^*(x^*) = h^*(x^*-p) - a \quad \text{for each } x^* \in X^*, \tag{2.16}$$

$$\operatorname{epi}(h+p+a)^* = \operatorname{epi} h^* + (p, -a). \tag{2.17}$$

3. The further regularity condition (*FRC*) and strong dualities. Let X and Y be real locally convex Hausdorff topological vector spaces, and let $A\colon X \to Y$ be a linear continuous operator. Let $f_1, f_2\colon X \to \bar{\mathbb{R}}$ and $g_1, g_2\colon Y \to \bar{\mathbb{R}}$ be proper convex functions such that $f_1 - f_2, g_1 - g_2$ are proper functions and such that

$$\boldsymbol{\Omega} = A(\operatorname{dom}(f_1 - f_2)) \cap \operatorname{dom}(g_1 - g_2) \neq \emptyset.$$

Then, by (1.5), we have that

$$\emptyset \neq \operatorname{dom} f_1 \subseteq \operatorname{dom} f_2 \quad \text{and} \quad \emptyset \neq \operatorname{dom} g_1 \subseteq \operatorname{dom} g_2. \tag{3.1}$$

For simplicity, we denote

$$H^* := \operatorname{dom} f_2^* \times \operatorname{dom} g_2^*.$$

To make the dual problems considered here well-defined, we further assume that $\mathrm{cl}\, f_2$ and $\mathrm{cl}\, g_2$ are proper. Then $H^* \neq \varnothing$. Consider the DC optimization problem (1.1), that is,

$$(P) \quad \inf_{x\in X}\{f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax)\}, \tag{3.2}$$

and the dual problems defined, respectively, by (1.3) and (1.8), that is,

$$(D^C) \quad \inf_{(u^*,v^*)\in H^*} \sup_{y^*\in Y^*} \{-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\} \tag{3.3}$$

and

$$(D^F) \quad \sup_{y^*\in Y^*} \inf_{(u^*,v^*)\in H^*} \{-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\}. \tag{3.4}$$

This section is devoted to the study of the weak dualities and the strong dualities between (P) and (D^F) and between (P) and (D^C). Recall that $v(P)$, $v(D^C)$, $v(D^F)$, and $v(D_{(u^*,v^*)})$ denote the optimal values of (P), (D^C), (D^F), and $(D_{(u^*,v^*)})$, respectively, where $(D_{(u^*,v^*)})$ for $(u^*, v^*) \in H^*$ is the dual problem defined by (1.2), that is,

$$(D_{(u^*,v^*)}) \quad \sup_{y^*\in Y^*} \{-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\}. \tag{3.5}$$

Then

$$v(D^F) \leq v(D^C). \tag{3.6}$$

Definition 3.1. *We say that*

(a) *the weak F-duality holds (between (P) and (D^F)) if $v(D^F) \leq v(P)$;*
(b) *the weak C-duality holds (between (P) and (D^C)) if $v(D^C) \leq v(P)$;*
(c) *the strong F-duality holds (between (P) and (D^F)) if $v(P) = v(D^F)$ and the problem (D^F) has an optimal solution;*
(d) *the strong C-duality holds (between (P) and (D^C)) if $v(P) = v(D^C)$ and for each $(u^*, v^*) \in H^*$ satisfying $v(D_{(u^*,v^*)}) = v(D^C)$, the dual problem $(D_{(u^*,v^*)})$ has an optimal solution.*

Remark 3.1. By definition, it is easy to see that the strong C-duality holds if and only if $v(P) = v(D^C)$ and for each $(u^*, v^*) \in H^*$ there is $y^* \in Y^*$ satisfying

$$-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) \geq v(D^C). \tag{3.7}$$

Moreover, in the special case when $f_2 = g_2 = 0$, the strong F-duality and the strong C-duality coincide with the strong duality for convex optimization problems.

Clearly, if f_2 and g_2 are lsc, then by (3.6), we have that

$$v(D^F) \leq v(D^C) \leq v(P); \tag{3.8}$$

that is, the weak F-duality and the weak C-duality hold. The following example shows that, in general, the weak F-duality and the weak C-duality do not necessarily hold.

Example 3.1. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$ by $f_1 := \delta_{\{0\}}$, $g_1 := \delta_{[0,+\infty)}$, $g_2 := 0$, and

$$f_2(x) := \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ +\infty, & x > 0. \end{cases}$$

Then f_1, f_2, g_1, g_2 are proper convex functions and $v(P) = -1$. Clearly, $f_1^* = 0$, $f_2^* = \delta_{[0,+\infty)}$, $g_1^* = \delta_{(-\infty,0]}$, and $g_2^* = \delta_{\{0\}}$. Hence, $H^* = [0, +\infty) \times \{0\}$. Let $y^* \in \mathbb{R}$. Then for each $(u^*, v^*) \in H^*$,

$$-f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) = -g_1^*(y^*).$$

Hence, $v(D^F) = 0$. This implies that $v(P) < v(D^F)$. Consequently, the weak F-duality and the weak C-duality do not hold.

Remark 3.2. Assume that the weak C-duality holds. Then the weak F-duality holds by (3.6), and the following implication holds by definitions:

$$\text{the strong } F\text{-duality} \Rightarrow \text{the strong } C\text{-duality}. \tag{3.9}$$

The following example shows that the converse does not hold even in the case when f_2 and g_2 are lsc.

Example 3.2. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$, respectively, by $f_1 := \delta_{[0,+\infty)}$, $f_2(x) := x^2$ for each $x \in \mathbb{R}$, $g_1 := \delta_{(-\infty,1]}$, and

$$g_2(x) := \begin{cases} x^2, & x \leq 0, \\ 0, & 0 \leq x \leq 1, \\ +\infty, & x > 1. \end{cases}$$

Then f_1, f_2, g_1, g_2 are proper convex lsc functions. Moreover, for each $x \in \mathbb{R}$,

$$f_1(x) - f_2(x) + g_1(x) - g_2(x) = \begin{cases} -x^2, & 0 \leq x \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence, $v(P) = -1$. Note that $f_1^* = \delta_{(-\infty,0]}$, $f_2^*(x^*) = \frac{x^{*2}}{4}$ for each $x^* \in \mathbb{R}$,

$$g_1^*(x^*) = \begin{cases} x^*, & x^* \geq 0, \\ +\infty, & x < 0, \end{cases} \quad \text{and} \quad g_2^*(x^*) = \begin{cases} \frac{x^{*2}}{4}, & x^* \leq 0, \\ x^*, & x > 0. \end{cases}$$

Then $\operatorname{dom} f_2^* = \operatorname{dom} g_2^* = \mathbb{R}$ and $H^* = \mathbb{R} \times \mathbb{R}$. Below we show that the strong C-duality holds but not the strong F-duality. To do this, consider the function $h : \mathbb{R}^3 \to \bar{\mathbb{R}}$ defined by

$$h(x_1, x_2, x_3) := \begin{cases} \frac{x_1^2}{4} - (x_2 + x_3) + \frac{x_2^2}{4}, & x_3 \geq \max\{x_1, -x_2\}, \quad x_2 \leq 0, \\ \frac{x_1^2}{4} - x_3, & x_3 \geq \max\{x_1, -x_2\}, \quad x_2 > 0, \\ -\infty & \text{otherwise} \end{cases} \tag{3.10}$$

for any $(x_1, x_2, x_3) \in \mathbb{R}^3$. Note that, for each $x_3 \in \mathbb{R}$, one has $h(x_3 + 1, -x_3 - 1, x_3) = -\infty$. Hence,

$$\sup_{x_3 \in \mathbb{R}} \inf_{(x_1,x_2) \in \mathbb{R}^2} h(x_1, x_2, x_3) = -\infty. \tag{3.11}$$

Obviously, for each $(x_1, x_2) \in \mathbb{R}^2$, the function $h(x_1, x_2, \cdot)$ attains the maximum at $x_3 = \max\{x_1, -x_2\}$. Therefore, for each $(x_1, x_2) \in \mathbb{R}^2$,

$$\sup_{x_3\in\mathbb{R}} h(x_1,x_2,x_3) = \begin{cases} \min\left\{\frac{x_1^2}{4}+\frac{x_2^2}{4}-x_1-x_2, \frac{x_1^2}{4}+\frac{x_2^2}{4}\right\}, & x_2 \le 0, \\ \min\left\{\frac{x_1^2}{4}-x_1, \frac{x_1^2}{4}+x_2\right\}, & x_2 > 0. \end{cases}$$

Then it follows that

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{x_3\in\mathbb{R}} h(x_1,x_2,x_3) = -1 \tag{3.12}$$

and

$$\{(x_1,x_2)\in\mathbb{R}^2 : \sup_{x_3\in\mathbb{R}} h(x_1,x_2,x_3) = -1\} = \{(2,0)\}. \tag{3.13}$$

By definitions, we can check that, for any $(u^*, v^*, y^*) \in \mathbb{R}^3$,

$$-f_1^*(u^*-y^*) + f_2^*(u^*) - g_1^*(v^*+y^*) + g_2^*(v^*) = h(u^*, v^*, y^*). \tag{3.14}$$

Therefore,

$$v(D^C) = \inf_{(u^*,v^*)\in\mathbb{R}^2} \sup_{y^*\in\mathbb{R}} h(u^*,v^*,y^*) = -1$$

by (3.12) and $h(2,0,y^*)$ attains its maximum at $y^* = 2$. Thus, the strong C-duality holds by definition because any pair $(u^*, v^*) \in \mathbb{R}^2$ satisfying $v(D_{(u^*,v^*)}) = v(D^C)$ is (2,0) by (3.13). However, by (3.11) and (3.14),

$$v(D^F) = \sup_{y^*\in\mathbb{R}} \inf_{(u^*,v^*)\in\mathbb{R}^2} h(u^*,v^*,y^*) = -\infty,$$

and so the strong F-duality does not hold.

The following example shows that the implication (3.9) does not hold if the weak C-duality assumption is dropped.

Example 3.3. Let $X = Y = \mathbb{R}^2$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R}^2 \to \bar{\mathbb{R}}$ by $f_2(x_1,x_2) := x_1^2$ for each $(x_1,x_2)\in\mathbb{R}^2$,

$$f_1(x_1,x_2) := \begin{cases} 0, & x_1 \ge 0, \\ +\infty, & x_1 < 0, \end{cases} \qquad g_1(x_1,x_2) := \begin{cases} 0, & x_1 \le 1, \\ +\infty, & x_1 > 1, \end{cases}$$

and

$$g_2(x_1,x_2) := \begin{cases} x_1^2, & x_1 \le 0, \\ 0, & 0 \le x_1 < 1, \\ |x_2|, & x_1 = 1, \\ +\infty, & x_1 > 1. \end{cases}$$

Then f_1, f_2, g_1, g_2 are proper convex functions and, for each $(x_1,x_2)\in\mathbb{R}^2$,

$$f_1(x_1,x_2) - f_2(x_1,x_2) + g_1(x_1,x_2) - g_2(x_1,x_2) = \begin{cases} -x_1^2, & 0 \le x_1 < 1, \\ -x_1^2 - |x_2|, & x_1 = 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence, $v(P) = -\infty$. Note that, for each $(x_1^*, x_2^*) \in \mathbb{R}^2$,

$$f_1^*(x_1^*, x_2^*) = \begin{cases} 0, & x_1^* \leq 0, x_2^* = 0, \\ +\infty & \text{otherwise}, \end{cases} \qquad f_2^*(x_1^*, x_2^*) = \begin{cases} \frac{x_1^{*2}}{4}, & x_2^* = 0, \\ +\infty & \text{otherwise}, \end{cases}$$

$$g_1^*(x_1^*, x_2^*) = \begin{cases} x_1^*, & x_1^* \geq 0,\ x_2^* = 0, \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$g_2^*(x_1^*, x_2^*) = \begin{cases} \frac{x_1^{*2}}{4}, & x^* \leq 0, x_2^* = 0, \\ x_1^*, & x_1^* > 0, x_2^* = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Then $\operatorname{dom} f_2^* = \operatorname{dom} g_2^* = \mathbb{R} \times \{0\}$ and $H^* = (\mathbb{R} \times \{0\})^2$, and, for each $(u^*, v^*) \in H^*$ with $u^* = (u_1^*, 0)$, $v^* = (v_1^*, 0)$ and each $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$, we have that

$$-f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(v^* + y^*) + g_2^*(v^*) = \begin{cases} h(u_1^*, v_1^*, y_1^*), & y_2^* = 0, \\ -\infty, & y_2^* \neq 0, \end{cases} \tag{3.15}$$

where $h : \mathbb{R}^3 \to \bar{\mathbb{R}}$ is defined by (3.10). Therefore, by (3.11) and the definition of $v(D^F)$,

$$v(D^F) = \sup_{y_1^* \in \mathbb{R}} \inf_{(u_1^*, v_1^*) \in \mathbb{R}^2} h(u_1^*, v_1^*, y_1^*) = -\infty = v(P),$$

and $y^* = (0, 0)$ is an optimal solution of problem (D^F). Thus, by definition, the strong F-duality holds. However, by (3.12) and (3.15),

$$v(D^C) = \inf_{(u_1^*, v_1^*) \in \mathbb{R}^2} \sup_{y_1^* \in \mathbb{R}} h(u_1^*, v_1^*, y_1^*) = -1 \neq v(P),$$

and so the strong C-duality does not hold.

In order to characterize the weak dualities and the strong dualities between the primal problem and the dual problems, we need to introduce some new regularity conditions. To this aim, we shall consider the identity operator $\mathrm{Id}_{\mathbb{R}}$ on $\mathbb{R}$, and the image set $(A^* \times \mathrm{Id}_{\mathbb{R}})(Z)$ of a set $Z \subseteq Y^* \times \mathbb{R}$ through the map $A^* \times \mathrm{Id}_{\mathbb{R}} : Y^* \times \mathbb{R} \to X^* \times \mathbb{R}$ defined by

$$(x^*, r) \in (A^* \times \mathrm{Id}_{\mathbb{R}})(Z) \Leftrightarrow \begin{cases} \exists y^* \in Y^* \text{ such that } (y^*, r) \in Z \\ \text{and} \quad A^* y^* = x^* \end{cases}$$

(where the map $A^* \times \mathrm{Id}_{\mathbb{R}}$ was for the first time introduced in [7]). Moreover, we will make use of the following characteristic sets K_L, K_C, and K_F defined, respectively, by

$$K_L := \bigcap_{u^* \in \operatorname{dom} f_2^*} (\operatorname{epi} f_1^* - (u^*, f_2^*(u^*))) + \bigcap_{v^* \in \operatorname{dom} g_2^*} ((A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (A^* v^*, g_2^*(v^*))),$$

$$K_C := \bigcap_{(u^*, v^*) \in H^*} (\operatorname{epi} f_1^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (u^*, f_2^*(u^*)) - (A^* v^*, g_2^*(v^*))),$$

and

$$K_F := \bigcap_{(u^*,v^*)\in H^*} (\text{epi}(f_1 + g_1 \circ A)^* - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*))).$$

Clearly, we have the following inclusions:

$$K_L \subseteq K_C \subseteq K_F. \tag{3.16}$$

The functions $\bar{f}$ and $\bar{g}$, which play a bridging role for our study, are defined, respectively, by

$$\bar{f} := f_1 - \text{cl}\, f_2 \quad \text{and} \quad \bar{g} := g_1 - \text{cl}\, g_2. \tag{3.17}$$

By (3.1), $\bar{f}$ and $\bar{g}$ are proper. The relationships between K_F, K_L, and $\text{epi}(\bar{f} + \bar{g} \circ A)^*$ are described in the following lemma.

Lemma 3.2. *We have the following formulas:*

$$K_F = \text{epi}(\bar{f} + \bar{g} \circ A)^* \tag{3.18}$$

and

$$K_L = \text{epi}\, \bar{f}^* + (A^* \times \text{Id}_{\mathbb{R}})(\text{epi}\, \bar{g}^*). \tag{3.19}$$

Proof. In fact, since $\text{cl}\, f_2$ and $\text{cl}\, g_2$ are proper lsc convex functions, it follows from (2.2) that

$$\text{cl}\, f_2 = f_2^{**} \quad \text{and} \quad \text{cl}\, g_2 = g_2^{**}. \tag{3.20}$$

Hence, using Lemma 2.2(ii), one gets that

$$\begin{aligned}(\bar{f} + \bar{g} \circ A)^* &= (\inf\nolimits_{(u^*,v^*)\in H^*} (f_1 + g_1 \circ A - u^* - A^*v^* + f_2^*(u^*) + g_2^*(v^*)))^* \\ &= \sup_{(u^*,v^*)\in H^*} (f_1 + g_1 \circ A - u^* - A^*v^* + f_2^*(u^*) + g_2^*(v^*))^*.\end{aligned}$$

This, together with Lemma 2.2(i) and Lemma 2.1(ii), implies that

$$\begin{aligned}\text{epi}(\bar{f} + \bar{g} \circ A)^* &= \bigcap_{(u^*,v^*)\in H^*} \text{epi}(f_1 + g_1 \circ A - u^* - A^*v^* + f_2^*(u^*) + g_2^*(v^*))^* \\ &= \bigcap_{(u^*,v^*)\in H^*} (\text{epi}(f_1 + g_1 \circ A)^* - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*))),\end{aligned}$$

where the last equality holds because of (2.17). Hence, (3.18) is seen to hold.

To show (3.19), note by definition that if $g_1 = g_2 = 0$, then

$$K_F = \bigcap_{u^*\in \text{dom}\, f_2^*} (\text{epi} f_1^* - (u^*, f_2^*(u^*))).$$

It follows from (3.18) (applied to $\{f_1, f_2, 0, 0\}$ in place of $\{f_1, f_2, g_1, g_2\}$) that

$$\text{epi}\, \bar{f}^* = \bigcap_{u^*\in \text{dom}\, f_2^*} (\text{epi}\, f_1^* - (u^*, f_2^*(u^*))). \tag{3.21}$$

Similarly, we have

$$\operatorname{epi}\bar{g}^* = \bigcap_{v^*\operatorname{dom}\in g_2^*}(\operatorname{epi} g_1^* - (v^*, g_2^*(v^*))); \tag{3.22}$$

hence,

$$\begin{aligned}(A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi}\bar{g}^*) &= \bigcap_{v^*\in\operatorname{dom} g_2^*}(A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^* - (v^*, g_2^*(v^*)))\\ &= \bigcap_{v^*\in\operatorname{dom} g_2^*}((A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (A^*v^*, g_2^*(v^*))).\end{aligned}$$

This together with (3.21) implies that (3.19) holds. ∎

In particular, in the case when f_2 and g_2 are lsc, the following assertion holds:

$$K_C \subseteq \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = K_F. \tag{3.23}$$

The following example shows that "$\subseteq$" and "$=$" in (3.23) do not hold in general.

Example 3.4. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$ by $g_1 = g_2 := 0$, $f_1 := \delta_{(-\infty,0]}$, and

$$f_2(x) := \begin{cases} 0, & x < 0,\\ 1, & x = 0,\\ +\infty, & x > 0.\end{cases}$$

Then f_1, f_2, g_1, and g_2 are proper convex functions. By definition (noting that $g_1 = g_2 = 0$), we have

$$K_C = K_F. \tag{3.24}$$

Clearly, $(f_1 - \operatorname{cl} f_2)^* = \delta_{[0,+\infty)}$. It follows that $\operatorname{epi}(f_1 - \operatorname{cl} f_2)^* = [0, +\infty) \times [0, +\infty)$. This together with (3.18) implies that

$$K_F = \operatorname{epi}(f_1 - \operatorname{cl} f_2)^* = [0, +\infty) \times [0, +\infty).$$

Moreover, it is easy to see that, for each $x^* \in \mathbb{R}$,

$$(f_1 - f_2)^*(x^*) = \begin{cases} 1, & x^* \geq 0,\\ +\infty, & x^* < 0.\end{cases} \tag{3.25}$$

Hence,

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \operatorname{epi}(f_1 - f_2)^* = [0, +\infty) \times [1, +\infty).$$

Therefore, $K_F \neq \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ and $K_C \not\subseteq \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ by (3.24).

Example 3.5 below shows that, in general, $\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \subseteq K_C$ does not necessarily hold.

Example 3.5. Let $X = Y = \mathbb{R}^2$, and let A be the identity. Let

$$A := \{(x - 1, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and

$$B := \{(1-x, -y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Define $f_1, f_2, g_1, g_2 : \mathbb{R}^2 \to \bar{\mathbb{R}}$ by $f_2 = g_2 := 0$, $f_1 := \delta_A$, and $g_1 := \delta_B$. Then $f_1 + g_1 = \delta_{\{0\}}$ and $\operatorname{epi}(f_1 + g_1)^* = \mathbb{R}^2 \times [0, +\infty)$. Clearly, for each $x^* \in \mathbb{R}^2$, we have

$$f_1^*(x^*) = \|x^*\| - x_1^* \quad \text{and} \quad g_1^*(x^*) = \|x^*\| + x_1^*.$$

Hence,

$$\operatorname{epi} f_1^* + \operatorname{epi} g_1^* = \{(r_1, r_2, \alpha) : r_2 = 0, \alpha \geq 0\} \cup \{(r_1, r_2, \alpha) : r_2 \neq 0, \alpha > 0\}. \tag{3.26}$$

This together with the definition of K_C implies that

$$K_C = \operatorname{epi} f_1^* + \operatorname{epi} g_1^* = \{(r_1, r_2, \alpha) : r_2 = 0, \alpha \geq 0\} \cup \{(r_1, r_2, \alpha) : r_2 \neq 0, \alpha > 0\}$$

(as $f_2 = g_2 = 0$ and A is the identity mapping). Therefore,

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \operatorname{epi}(f_1 + g_1)^* \nsubseteq K_C.$$

Considering the possible inclusions among $\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ and K_F, K_C, we introduce the following definition.

Definition 3.3. *The family $(f_1, f_2, g_1, g_2; A)$ is said to satisfy*

(a) *the further regularity condition ((FRC)) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{0\} \times \mathbb{R}); \tag{3.27}$$

(b) *the semi-(FRC) ($(SFRC)$) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) \supseteq K_C \cap (\{0\} \times \mathbb{R}); \tag{3.28}$$

(c) *the lower semicontinuity closure at 0 ($(LSC)_0$) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) = K_F \cap (\{0\} \times \mathbb{R}). \tag{3.29}$$

Remark 3.3.

(a) By (3.16), the $(LSC)_0$ implies the $(SFRC)$, while the converse implication is not true which will be shown in Example 3.6.

(b) By (3.18), if f_2 and g_2 are lsc, then the family $(f_1, f_2, g_1, g_2; A)$ satisfies the $(LSC)_0$ and therefore the $(SFRC)$.

(c) Note that

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \subseteq \operatorname{epi}(f_1 - \operatorname{cl} f_2 + g_1 \circ A - (\operatorname{cl} g_2) \circ A)^* \tag{3.30}$$

holds automatically. It follows from (3.18) that (3.29) can be equivalently replaced by

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) \supseteq K_F \cap (\{0\} \times \mathbb{R}). \tag{3.31}$$

(d) Recall from [25, Definition 4.2] that the triple $(\bar{f}, \bar{g}; A)$ satisfies the $(FRC)_A$ if

$$(\bar{f} + \bar{g} \circ A)^*(0) \geq (\bar{f}^* \,\square\, A^* \bar{g}^*)(0), \tag{3.32}$$

and there exists $x^* \in X^*$ such that $(\bar{f}^* \,\Box\, A^*\bar{g}^*)(0) = \bar{f}^*(-x^*) + (A^*\bar{g}^*)(x^*)$ and the infimum in the definition of $(A^*\bar{g}^*)(x^*)$ is attained, where $\bar{f}^* \,\Box\, A^*\bar{g}^*$ denotes the infimal convolution of $\bar{f}^*$ and $A^*\bar{g}^*$ (see, for example, [33] or [35, page 43] for the definition). By [25, Proposition 4.3], the $(FRC)_A$ for the triple $(\bar{f}, \bar{g}; A)$ is equivalent to

$$\operatorname{epi}(\bar{f} + \bar{g} \circ A)^* \cap (\{0\} \times \mathbb{R}) = (\operatorname{epi} \bar{f}^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} \bar{g}^*)) \cap (\{0\} \times \mathbb{R}). \tag{3.33}$$

This together with (3.18), (3.19), and (3.16) implies that

$$K_F \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{0\} \times \mathbb{R}) = K_L \cap (\{0\} \times \mathbb{R}). \tag{3.34}$$

Therefore, if the $(LSC)_0$ holds, then the $(FRC)_A$ for the triple $(\bar{f}, \bar{g}; A)$ implies the (FRC) for the family $(f_1, f_2, g_1, g_2; A)$. It should be noted that, for the triple $(\bar{f}, \bar{g}; A)$, the $(FRC)_A$ is different from the (FRC_A), which was for the first time introduced in [7, section 4] (cf. [25]).

Let (P^{cl}) denote the optimization problem $(\mathcal{P})$ with f and g defined by (3.17):

$$(P^{\mathrm{cl}}) \quad \inf_{x \in X} \{f_1(x) - \operatorname{cl} f_2(x) + g_1(Ax) - (\operatorname{cl} g_2)(Ax)\},$$

and let $v(P^{\mathrm{cl}})$ denote the optimal value of the problem (P^{cl}). We need the following lemma.

LEMMA 3.4. *Let $r \in \mathbb{R}$. Then the following assertions hold:*

(i) $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ *if and only if* $v(P) \geq -r$.

(ii) $(0, r) \in K_F$ *if and only if* $v(P^{\mathrm{cl}}) \geq -r$.

(iii) $(0, r) \in K_C$ *if and only if* $v(D^C) \geq -r$ *and for each* $(u^*, v^*) \in H^*$ *there is* $y^* \in Y^*$ *satisfying*

$$-f_1^*(u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) \geq -r. \tag{3.35}$$

Proof.

(i) By the definition of the conjugate function, one has

$$v(P) = -(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*(0). \tag{3.36}$$

Hence, the result is clear.

(ii) By (3.18), we have $(0, r) \in K_F \Leftrightarrow (0, r) \in \operatorname{epi}(f_1 - \operatorname{cl} f_2 + g_1 \circ A - (\operatorname{cl} g_2) \circ A)^*$. Thus, (i) is applied to obtain the conclusion.

(iii) Let $(0, r) \in K_C$, and let $(u^*, v^*) \in H^*$. Then

$$(0, r) \in \operatorname{epi} f_1^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*)).$$

Thus, there exist $(x_1^*, r_1) \in \operatorname{epi} f_1^*$ and $(y_1^*, r_2) \in \operatorname{epi} g_1^*$ such that

$$x_1^* + A^*y_1^* - u^* - A^*v^* = 0 \tag{3.37}$$

and

$$r_1 + r_2 - f_2^*(u^*) - g_2^*(v^*) = r. \tag{3.38}$$

Since

$$f_1^*(x_1^*) \le r_1 \quad \text{and} \quad g_1^*(y_1^*) \le r_2, \tag{3.39}$$

it follows from (3.37) and (3.38) that

$$\begin{aligned} &-f_1^*(u^* - A^*(y_1^* - v^*)) + f_2^*(u^*) - g_1^*(y_1^*) + g_2^*(v^*) \\ &\quad = -f_1^*(x_1^*) + f_2^*(u^*) - g_1^*(y_1^*) + g_2^*(v^*) \\ &\quad \ge -r_1 + f_2^*(u^*) - r_2 + g_2^*(v^*) \\ &\quad = -r. \end{aligned}$$

Recall that $v(D^C)$ is the optimal value of the problem (D^C) (cf. (3.3)). It follows that $v(D^C) \ge -r$ and $y^* := y_1^* - v^*$ satisfy (3.35).

Conversely, suppose that $v(D^C) \ge -r$ and for each $(u^*, v^*) \in H^*$ there is $y^* \in Y^*$ satisfying (3.35). Let $(u^*, v^*) \in H^*$, and let $y_0^* \in Y^*$ satisfy (3.35); hence,

$$g_1^*(y_0^* + v^*) \le r - f_1^*(u^* - A^*y_0^*) + f_2^*(u^*) + g_2^*(v^*).$$

This means that

$$(A^*(y_0^* + v^*),\ r - f_1^*(u^* - A^*y_0^*) + f_2^*(u^*) + g_2^*(v^*)) \in (A^* \times \mathrm{Id}_{\mathbb{R}})(\mathrm{epi}\, g_1^*). \tag{3.40}$$

Since

$$0 = (u^* - A^*y_0^*) + A^*(y_0^* + v^*) - u^* - A^*v^*$$

and

$$r = f_1^*(u^* - A^*y_0^*) + (r - f_1^*(u^* - A^*y_0^*) + f_2^*(u^*) + g_2^*(v^*)) - f_2^*(u^*) - g_2^*(v^*),$$

it follows from (3.40) that

$$(0, r) \in \mathrm{epi}\, f_1^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\mathrm{epi}\, g_1^*) - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*)).$$

Noting that $(u^*, v^*) \in H^*$ is arbitrary, we have that

$$(0, r) \in \bigcap_{(u^*, v^*) \in H^*} (\mathrm{epi}\, f_1^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\mathrm{epi}\, g_1^*) - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*))) = K_C.$$

Thus, we complete the proof. ∎

The following proposition establishes the connection among the problems (P) and (P^{cl}) and the regularity condition $(LSC)_0$

PROPOSITION 3.5. *The family $(f_1, f_2, g_1, g_2; A)$ satisfies the $(LSC)_0$ if and only if $v(P) = v(P^{\mathrm{cl}})$.*

Proof. Suppose that the $(LSC)_0$ holds. Then (3.29) holds. To show $v(P) = v(P^{\mathrm{cl}})$, it suffices to show that $v(P) \ge v(P^{\mathrm{cl}})$ since $v(P) \le v(P^{\mathrm{cl}})$ holds automatically. To do this, suppose, on the contrary, that $v(P) < v(P^{\mathrm{cl}})$. Then there exists $r \in \mathbb{R}$ such that $v(P) < -r < v(P^{\mathrm{cl}})$. Thus, by Lemma 3.4(ii), $(0, r) \in K_F$, and so $(0, r) \in \mathrm{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ by (3.29). It follows from Lemma 3.4(i) that $v(P) \ge -r$. This contradicts $v(P) < -r$ and completes the proof of the inequality $v(P) \ge v(P^{\mathrm{cl}})$.

Conversely, suppose that $v(P) = v(P^{\rm cl})$. By Remark 3.3(c), it suffices to show that (3.31) holds. To do this, let $(0, r) \in K_F$. Then, by Lemma 3.4(ii), $v(P^{\rm cl}) \geq -r$ and so $v(P) \geq -r$. Hence, by Lemma 3.4(i), $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$. Therefore, (3.31) is proved. ∎

Our first theorem of this section shows that the $(SFRC)$ is a sufficient and necessary condition for the weak C-duality to hold.

THEOREM 3.6. *The inequalities*

$$v(D^F) \leq v(D^C) \leq v(P) \tag{3.41}$$

hold if and only if the family $(f_1, f_2, g_1, g_2; A)$ *satisfies the* $(SFRC)$.

Proof. Suppose that (3.41) holds. Let $(0, r) \in K_C$. Then, by Lemma 3.4(iii), we have $v(D^C) \geq -r$. By (3.41), one has $v(P) \geq -r$, which implies that $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$, thanks to Lemma 3.4(i). Hence, (3.28) holds; that is, the $(SFRC)$ holds.

Conversely, suppose that the family $(f_1, f_2, g_1, g_2; A)$ satisfies the $(SFRC)$. To show (3.41), it suffices to show $v(D^C) \leq v(P)$. To do this, suppose, on the contrary, that $v(P) < v(D^C)$. Then there exists $r \in \mathbb{R}$ such that $v(P) < -r < v(D^C)$. Thus, from (3.3) we have that for each $(u^*, v^*) \in H^*$ there exists $y^* \in Y^*$ satisfying (3.35). Hence, $(0, r) \in K_C$ by Lemma 3.4(iii), and $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ by the $(SFRC)$. This together with Lemma 3.4(i) implies that $v(P) \geq -r$, which contradicts $v(P) < -r$. Consequently, we have $v(D^C) \leq v(P)$ and complete the proof. ∎

The following theorem provides a characterization for the strong C-duality to hold in terms of the (FRC)

THEOREM 3.7. *The family* $(f_1, f_2, g_1, g_2; A)$ *satisfies the* (FRC) *if and only if the strong* C*-duality holds.*

Proof. Suppose that the family $(f_1, f_2, g_1, g_2; A)$ satisfies the (FRC). Then the f amily $(f_1, f_2, g_1, g_2; A)$ satisfies the $(SFRC)$, and so $v(D^C) \leq v(P)$ by Theorem 3.6. Thus, to prove the strong C-duality, by Remark 3.1, it suffices to show that $v(D^C) \geq v(P)$ and that for each $(u^*, v^*) \in H^*$ there exists $y^* \in Y^*$ satisfying (3.7). Note that the conclusion holds trivially if $v(P) = -\infty$. Below we consider only the case when $-r := v(P) \in \mathbb{R}$. By Lemma 3.4(i), $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ and hence $(0, r) \in K_C$, thanks to the assumed (FRC). Thus, by Lemma 3.4(iii), we have that $v(D^C) \geq -r$ and for each $(u^*, v^*) \in H^*$ there exists $y^* \in Y^*$ satisfying (3.35). Hence, the strong C-duality holds.

Conversely, suppose that the strong C-duality holds. Let $(u^*, v^*) \in H^*$. Then, by Remark 3.1, $v(P) = v(D^C)$ and there exists $y^* \in Y^*$ satisfying (3.7). Thus, by Theorem 3.6, (3.28) holds, and so we need to verify only that the set on the left-hand side of (3.27) is contained in the set on the right-hand side. To do this, let $(0, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$. Then, by Lemma 3.4(i), $v(P) \geq -r$. Hence, $v(D^C) = v(P) \geq -r$ and $y^* \in Y^*$ satisfies (3.35). This together with Lemma 3.4(iii) implies that $(0, r) \in K_C$ as $(u^*, v^*) \in H^*$ is arbitary. Hence, $\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) \subseteq K_C \cap (\{0\} \times \mathbb{R})$, and this completes the proof of the implication (ii) $\Rightarrow$ (i). ∎

Theorem 3.8 below describes the relationship between the strong F-duality and the strong C-duality. Consider the condition

$$K_C \cap (\{0\} \times \mathbb{R}) = K_L \cap (\{0\} \times \mathbb{R}), \tag{3.42}$$

which, by (3.16), is clearly equivalent to the following one:

$$K_C \cap (\{0\} \times \mathbb{R}) \subseteq K_L \cap (\{0\} \times \mathbb{R}). \tag{3.43}$$

THEOREM 3.8. *Suppose that the family $(f_1, f_2, g_1, g_2; A)$ satisfies the $(LSC)_0$. Then the following assertions are equivalent.*

(i) *The strong F-duality holds.*

(ii) *The strong C-duality and* (3.42) *hold.*

Proof. Consider the problem $(P^{\rm cl})$ and its Fenchel dual problem

$$(D^{\rm cl}) \quad \sup_{y^* \in Y^*} \{-\bar{f}^*(-A^* y^*) - \bar{g}^*(y^*)\},$$

where $\bar{f} = f_1 - \mathrm{cl}\, f_2$ and $\bar{g} = g_1 - \mathrm{cl}\, g_2$ are defined by (3.17). Note by Lemma 2.3 (applied to $\{f_1, \mathrm{cl}\, f_2\}$ and $\{g_1, \mathrm{cl}\, g_2\}$ in place of $\{f, g\}$, respectively) that

$$-\bar{f}^*(-A^* y^*) - \bar{g}^*(y^*) = \inf_{(u^*, v^*) \in H^*} \{-f_1^*(u^* - A^* y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\}.$$

Then, the dual problem $(D^{\rm cl})$ coincides with (D^F). Moreover, we have $v(P) = v(P^{\rm cl})$ by Proposition 3.5 (noting the $(LSC)_0$ holds as assumed). Hence, we have the chain of equivalences

$$\begin{aligned}
\text{(i)} &\Leftrightarrow \text{the strong Fenchel duality between } (P^{\rm cl}) \text{ and } (D^{\rm cl})\\
&\Leftrightarrow \text{the } (FRC)_A \text{ for the family } (\bar{f}, \bar{g}; A)\\
&\Leftrightarrow (3.33)\\
&\Leftrightarrow K_F \cap (\{0\} \times \mathbb{R}) = K_L \cap (\{0\} \times \mathbb{R}),
\end{aligned}$$

where the second equivalence and the third one follow from Remark 3.3(d) while the last equivalence holds by (3.18) and (3.19). Moreover, by the assumed $(LSC)_0$, we see that (3.29) holds. Hence,

$$\text{the strong } C\text{-duality} \Leftrightarrow \text{the } (FRC) \Leftrightarrow K_C \cap (\{0\} \times \mathbb{R}) = K_F \cap (\{0\} \times \mathbb{R}),$$

where the first equivalence follows from Theorem 3.7 and the last equivalence holds by (3.29). Therefore, by (3.16),

$$\text{(ii)} \Leftrightarrow K_F \cap (\{0\} \times \mathbb{R}) = K_L \cap (\{0\} \times \mathbb{R}),$$

and the proof is complete. ◻

Remark 3.4. The conclusion of Theorem 3.8 may not be true if the $(LSC)_0$ assumption is dropped; see Example 3.3.

The following example shows that the $(LSC)_0$ is not necessary for (i) and (ii) in Theorem 3.8 to be equivalent.

Example 3.6. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$ by $f_1 := \delta_{\{0\}}$, $g_2 := 0$,

$$f_2(x) := \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ +\infty, & x > 0, \end{cases} \quad \text{and} \quad g_1(x) := \begin{cases} 0, & x > 0, \\ 1, & x = 0, \\ +\infty, & x < 0. \end{cases}$$

Then f_1, f_2, g_1, g_2 are proper convex functions. Since $f_1 - f_2 + g_1 - g_2 = \delta_{\{0\}}$, it follows that $v(P) = 0$. Clearly,

$$f_1^* = 0, \quad f_2^* = \delta_{[0,+\infty)}, \quad g_1^* = \delta_{(-\infty,0]}, \quad \text{and} \quad g_2^* = \delta_{\{0\}}.$$

Hence,

$$\operatorname{dom} f_2^* = [0,+\infty), \quad \operatorname{dom} g_2^* = \{0\}, \quad \text{and} \quad H^* = [0,+\infty) \times \{0\}.$$

Let $y^* \in \mathbb{R}$ and $(u^*, v^*) \in H^*$. Then

$$-f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) = -g_1^*(y^*). \tag{3.44}$$

Hence,

$$v(D^F) = \sup_{y^* \in \mathbb{R}} \inf_{(u^*,v^*) \in H^*} \{-f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\} = 0 = v(P),$$

and $y^* = 0$ is an optimal solution of (D^F). Thus, by definition, the strong F-duality holds; that is, assertion (i) of Theorem 3.8 holds. Below we show that assertion (ii) of Theorem 3.8 holds too. For this purpose, we note by (3.44) that

$$v(D^C) = \inf_{(u^*,v^*) \in H^*} \sup_{y^* \in \mathbb{R}} \{-f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\} = 0 = v(P).$$

Hence, the weak C-duality holds. Thus, by Remark 3.2, the strong C-duality holds. Moreover, since $f_1 - \operatorname{cl} f_2 = f_1$ and $g_1 - \operatorname{cl} g_2 = g_1$, it follows that

$$\operatorname{epi}(f_1 - \operatorname{cl} f_2)^* = \operatorname{epi} f_1^* = \mathbb{R} \times [0,+\infty)$$

and

$$\operatorname{epi}(g_1 - \operatorname{cl} g_2)^* = \operatorname{epi} g_1^* = (-\infty,0] \times [0,+\infty).$$

Hence,

$$K_C = \bigcap_{u^* \in [0,+\infty)} (\operatorname{epi} f_1^* - (u^*, f_2^*(u^*)) + \operatorname{epi} g_1^*) = \mathbb{R} \times [0,+\infty),$$

and, applying (3.19), we get that

$$K_L = \operatorname{epi}(f_1 - \operatorname{cl} f_2)^* + \operatorname{epi}(g_1 - \operatorname{cl} g_2)^* = \mathbb{R} \times [0,+\infty).$$

Thus, $K_C = K_L$ and (3.42) holds. This means that assertion (ii) of Theorem 3.8 holds. However, since, for each $x \in \mathbb{R}$,

$$f_1(x) - (\operatorname{cl} f_2)(x) + g_1(x) - (\operatorname{cl} g_2)(x) = \begin{cases} 1, & x = 0, \\ +\infty, & x \neq 0, \end{cases}$$

it follows that $v(P^{\text{cl}}) = 1 \neq v(P)$, and so the $(LSC)_0$ does not hold by Proposition 3.5.

The remainder of this section provides some equivalent conditions for the $(LSC)_0$ and the strong C-duality. For this purpose, we consider the following condition, which plays an important role in our study:

$$K_F \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{0\} \times \mathbb{R}). \tag{3.45}$$

Clearly, the strong C-duality together with the $(LSC)_0$ implies (3.45), while the converse is not true as shown by the following example.

Example 3.7. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$ by $f_1 = g_1 := \delta_{[1,+\infty)}$,

$$f_2(x) := \begin{cases} 0, & x > 1, \\ 1, & x = 1, \\ +\infty, & x < 1, \end{cases} \quad \text{and} \quad g_2(x) := \begin{cases} -2x, & x > 1, \\ -1, & x = 1, \\ +\infty, & x < 1. \end{cases}$$

Then f_1, f_2, g_1, and g_2 are proper convex functions. By definition, one calculates that, for each $x^* \in \mathbb{R}$,

$$f_1^*(x^*) = f_2^*(x^*) = g_1^*(x^*) = \begin{cases} x^*, & x^* \leq 0, \\ +\infty, & x^* > 0, \end{cases}$$

$$g_2^*(x^*) = \begin{cases} x^* + 2, & x^* \leq -2, \\ +\infty, & x^* > -2, \end{cases}$$

$$(f_1 - f_2 + g_1 - g_2)^*(x^*) = \begin{cases} x^*, & x^* \leq 2, \\ +\infty, & x^* > 2, \end{cases}$$

and

$$(f_1 - \mathrm{cl}\, f_2 + g_1 - \mathrm{cl}\, g_2)^*(x^*) = \begin{cases} x^* - 2, & x^* \leq 2, \\ +\infty, & x^* > 2. \end{cases}$$

Hence, $H^* = (-\infty, 0] \times (-\infty, -2]$,

$$\mathrm{epi} f_1^* = \mathrm{epi} g_1^* = \{(x, y) : x \leq 0, y \geq x\},$$

and

$$\mathrm{epi}(f_1 - f_2 + g_1 - g_2)^* = \{(x, y) \in \mathbb{R}^2 : x \leq 2, y \geq x\}.$$

Thus,

$$K_C = \bigcap_{(u^*, v^*) \in H^*} (\mathrm{epi}\, f_1^* - (u^*, u^*) + \mathrm{epi}\, g_1^* - (v^*, v^* + 2)) = \{(x, y) \in \mathbb{R}^2 : x \leq 2, y \geq x - 2\},$$

and, by (3.18),

$$K_F = \mathrm{epi}(f_1 - \mathrm{cl}\, f_2 + g_1 - \mathrm{cl}\, g_2)^* = \{(x, y) \in \mathbb{R}^2 : x \leq 2, y \geq x - 2\}.$$

Hence,

$$K_F \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{0\} \times \mathbb{R}) = \{0\} \times [-2, +\infty).$$

Noting that

$$\mathrm{epi}(f_1 - f_2 + g_1 - g_2)^* \cap (\{0\} \times \mathbb{R}) = \{0\} \times [0, +\infty),$$

we see that

$$\operatorname{epi}(f_1 - f_2 + g_1 - g_2)^* \cap (\{0\} \times \mathbb{R}) \neq K_F \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{0\} \times \mathbb{R}).$$

This means that neither the $(LSC)_0$ nor the strong C-duality holds.

By definition of the $(LSC)_0$ and Theorem 3.7, we get the following proposition straightforwardly.

PROPOSITION 3.9. *Suppose that one of* (3.45)*, the strong C-duality, or the $(LSC)_0$ holds. Then the other two are equivalent.*

4. The condition (*CC*) and stable strong dualities.

Recall that X and Y are real locally convex Hausdorff topological vector spaces, $A\colon X \to Y$ is a linear continuous operator, and $f_1, f_2\colon X \to \bar{\mathbb{R}}$, $g_1, g_2\colon Y \to \bar{\mathbb{R}}$ are proper convex functions such that $f_1 - f_2$, $g_1 - g_2$ are proper and $\Omega \neq \emptyset$. Given $p \in X^*$, we consider in this section the following DC optimization problem with a linear perturbation:

$$(P_p)\colon \quad \inf_{x\in X}\{f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) - \langle p, x\rangle\}. \tag{4.1}$$

Then the corresponding dual problems are

$$(D_p^F)\colon \quad \sup_{y^*\in Y^*} \inf_{(u^*,v^*)\in H^*} \{-f_1^*(p + u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\} \tag{4.2}$$

and

$$(D_p^C)\colon \quad \inf_{(u^*,v^*)\in H^*} \sup_{y^*\in Y^*} \{-f_1^*(p + u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*)\}. \tag{4.3}$$

In particular, in the case when $p = 0$, problem (P_p) as well as its dual problems (D_p^F) and (D_p^C) are reduced to the problem (P) and its dual problems (D^F) and (D^C), respectively. Clearly, the following inequality holds:

$$v(D_p^F) \leq v(D_p^C) \quad \text{for each } p \in X^*. \tag{4.4}$$

Following [24], [25], we say that the stable weak F-duality (resp., the stable weak C-duality) holds if the weak F-duality (resp., the weak C-duality) between (P_p) and (D_p^F) (resp., (D_p^C)) holds for each $p \in X^*$, and that the stable strong F-duality (resp., the stable strong C-duality) holds if the strong F-duality (resp., the strong C-duality) between (P_p) and (D_p^F) (resp., (D_p^C)) holds for each $p \in X^*$.

This section is devoted to the study of the stable strong dualities. For this purpose, we introduce the following regularity conditions.

DEFINITION 4.1. *The family $(f_1, f_2, g_1, g_2; A)$ is said to satisfy*

(a) *the closure condition ((CC)) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = K_C; \tag{4.5}$$

(b) *the semi-(CC) ((SCC)) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \supseteq K_C; \tag{4.6}$$

(c) *the lower semcontinuity closure ((LSC)) if*

$$\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = K_F. \tag{4.7}$$

Remark 4.1.

(a) By (3.16), the (LSC) implies the (SCC), while the inverse implication is not true. For example, let f_1, f_2, g_1, g_2, and A be defined by Example 3.6. Then $\mathrm{epi}(f_1 - f_2 + g_1 - g_2)^* = K_C = \mathbb{R} \times [0, +\infty)$. Thus, the (SCC) holds, but the (LSC) does not hold as shown in Example 3.6.

(b) By (3.18), if f_2 and g_2 are lsc, then the family $(f_1, f_2, g_1, g_2; A)$ satisfies the (LSC) and therefore the (SCC).

(c) By (3.30) and (3.18), one sees that (4.7) can be equivalently replaced by

$$\mathrm{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \supseteq K_F. \tag{4.8}$$

(d) Recall from [25, Definition 3.1] that the triple $(\bar{f}, \bar{g}; A)$ satisfies the $(CC)_A$ if

$$\mathrm{epi}(\bar{f} + \bar{g} \circ A)^* = \mathrm{epi}\bar{f}^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\mathrm{epi}\bar{g}^*). \tag{4.9}$$

Let $\bar{f}$ and $\bar{g}$ be defined by (3.17), and suppose that the (LSC) holds. Then (4.7) holds, and we have from (3.18) that

$$\mathrm{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \mathrm{epi}(\bar{f} + \bar{g} \circ A)^*.$$

This together with (3.16) and (3.19) shows that the $(CC)_A$ for the triple $(\bar{f}, \bar{g}; A)$ implies the (CC) for the family $(f_1, f_2, g_1, g_2; A)$ if the (LSC) holds.

(e) By (2.7) and (3.18), if

$$\mathrm{cl}(f_1 - f_2 + g_1 \circ A - g_2 \circ A) = \mathrm{cl}(f_1 - \mathrm{cl}\, f_2 + g_1 \circ A - (\mathrm{cl}\, g_2) \circ A), \tag{4.10}$$

then the (LSC) holds. The converse is not true, in general, as will be shown by Example 4.1 below.

Example 4.1. Let $X = Y = \mathbb{R}$, and let A be the identity. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \bar{\mathbb{R}}$ by $f_1 := \delta_{(-\infty,2)}$, $g_1 := \delta_{[1,+\infty)}$,

$$f_2(x) := \begin{cases} -\frac{1}{x-2}, & x < 2, \\ +\infty, & x \geq 2, \end{cases} \quad \text{and} \quad g_2(x) := \begin{cases} -2x, & x > 1, \\ -1, & x = 1, \\ +\infty, & x < 1. \end{cases}$$

Then f_1, f_2, g_1, g_2 are proper convex functions and

$$f_2 = \mathrm{cl}\, f_2 \quad \text{and} \quad (\mathrm{cl}\, g_2)(x) = \begin{cases} -2x, & x \geq 1, \\ +\infty, & x < 1. \end{cases}$$

Hence,

$$(f_1 - f_2 + g_1 - g_2)(x) = \begin{cases} 2x + \frac{1}{x-2}, & 1 < x < 2, \\ 0, & x = 1, \\ +\infty, & x \geq 2 \quad \text{or} \quad x < 1, \end{cases}$$

and

$$(f_1 - \mathrm{cl}\, f_2 + g_1 - \mathrm{cl}\, g_2)(x) = \begin{cases} 2x + \frac{1}{x-2}, & 1 \leq x < 2, \\ +\infty, & x \geq 2 \quad \text{or} \quad x < 1. \end{cases}$$

It follows that

$$\operatorname{cl}(f_1 - f_2 + g_1 - g_2)(x) = \begin{cases} -\infty, & x = 2, \\ 2x + \frac{1}{x-2}, & 1 < x < 2, \\ 0, & x = 1, \\ +\infty, & x > 2 \quad \text{or} \quad x < 1, \end{cases}$$

and

$$\operatorname{cl}(f_1 - \operatorname{cl} f_2 + g_1 - \operatorname{cl} g_2)(x) = \begin{cases} -\infty, & x = 2, \\ 2x + \frac{1}{x-2}, & 1 \leq x < 2, \\ +\infty, & x > 2 \quad \text{or} \quad x < 1. \end{cases}$$

This implies that $\operatorname{epi}(f_1 - f_2 + g_1 - g_2)^* = \operatorname{epi}(f_1 - \operatorname{cl} f_2 + g_1 - \operatorname{cl} g_2)^* = \emptyset$. Hence, it follows from (3.18) that (4.7) holds. However, $\operatorname{cl}(f_1 - f_2 + g_1 - g_2) \neq \operatorname{cl}(f_1 - \operatorname{cl} f_2 + g_1 - \operatorname{cl} g_2)$.

The following proposition describes the relationship between the (CC) (resp., the (SCC) and the (LSC)) and the (FRC) (resp., the $(SFRC)$ and the $(LSC)_0$).

PROPOSITION 4.2. *The family $(f_1, f_2, g_1, g_2; A)$ satisfies the (CC) (resp., the (SCC) and the (LSC)) if and only if for each $p \in X^*$, $(f_1 - p, f_2, g_1, g_2; A)$ satisfies the (FRC) (resp., the $(SFRC)$ and the $(LSC)_0$).*

Proof. Let $p \in X^*$, and let $K_F(p)$ and $K_C(p)$ be the sets defined, respectively, by

$$K_F(p) := \bigcap_{(u^*,v^*)\in H^*} (\operatorname{epi}(f_1 - p + g_1 \circ A)^* - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*)))$$

and

$$K_C(p) := \bigcap_{(u^*,v^*)\in H^*} (\operatorname{epi}(f_1 - p)^* + (A^* \times \mathrm{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (u^*, f_2^*(u^*)) - (A^*v^*, g_2^*(v^*))).$$

Then, by (2.17), the following equalities are clear:

$$K_F(p) = K_F + (-p, 0) \quad \text{and} \quad K_C(p) = K_C + (-p, 0).$$

Hence, we have that

$$K_F(p) \cap (\{0\} \times \mathbb{R}) = K_F \cap (\{p\} \times \mathbb{R}) + (-p, 0) \tag{4.11}$$

and

$$K_C(p) \cap (\{0\} \times \mathbb{R}) = K_C \cap (\{p\} \times \mathbb{R}) + (-p, 0). \tag{4.12}$$

Moreover, using (2.17), we conclude that

$$\operatorname{epi}(f_1 - p - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{0\} \times \mathbb{R}) = \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \cap (\{p\} \times \mathbb{R}) + (-p, 0).$$

Thus, the conclusion holds by definitions, and the proof is complete. ∎

By Theorems 3.6 and 3.7 and Proposition 4.2, we get the following theorems straightforwardly.

THEOREM 4.3. *The inequalities*

$$v(D_p^F) \leq v(D_p^C) \leq v(P_p) \quad for\ each\ \ p \in X^* \tag{4.13}$$

hold if and only if the family $(f_1, f_2, g_1, g_2; A)$ *satisfies the* (SCC).

THEOREM 4.4. *The family* $(f_1, f_2, g_1, g_2; A)$ *satisfies the* (CC) *if and only if the stable strong C-duality holds.*

For the following theorem on the relationship between the stable strong F-duality and the stable strong C-duality, we consider the set $K_L(p)$ for any $p \in X^*$, defined by

$$\begin{aligned} K_L(p) \coloneqq & \bigcap_{u^* \in \operatorname{dom} f_2^*} (\operatorname{epi}(f_1 - p)^* - (u^*, f_2^*(u^*))) \\ & + \bigcap_{v^* \in \operatorname{dom} g_2^*} ((A^* \times \operatorname{Id}_{\mathbb{R}})(\operatorname{epi} g_1^*) - (A^* v^*, g_2^*(v^*))). \end{aligned}$$

Then, by (2.17), we have that $K_L(p) = K_L + (-p, 0)$; hence,

$$K_L(p) \cap (\{0\} \times \mathbb{R}) = K_L \cap (\{p\} \times \mathbb{R}) + (-p, 0) \quad \text{for each } p \in X^*.$$

This together with (4.12) in the proof of Proposition 4.2 implies that $K_L = K_C$ holds if and only if the condition (3.42) holds with $f_1 - p$ in place of f_1 for each $p \in X^*$. Thus, by Theorem 3.8 and Proposition 4.2, we get the following theorem directly.

THEOREM 4.5. *Suppose that the family* $(f_1, f_2, g_1, g_2; A)$ *satisfies the* (LSC)*. Then the following statements are equivalent:*

(i) *The stable strong F-duality holds.*

(ii) *The stable strong C-duality and $K_L = K_C$ hold.*

The result below follows directly from the definition of the (LSC) and Theorem 4.4.

PROPOSITION 4.6. *Suppose that one of the conditions $K_F = K_C$, the stable C-duality, or the (LSC) holds. Then the other two are equivalent.*

5. The Moreau–Rockafellar formula and total dualities. Recall that the problem (P_p) and the corresponding dual problem (D_p^C) are defined by (4.1) and (4.3), respectively. For each $p \in X^*$, we use $S_P(p)$ to denote the optimal solution set of (P_p). In particular, we write S_P for $S_P(0)$, the optimal solution set of the problem (P). This section is devoted to the study of characterizing the total dualities. Unlike the convex case, the cases for DC optimization problems are more complicated. We begin with the following definition.

DEFINITION 5.1. *Let X_0 be a subset of X. Between the problems (P) and (D^C), we say that*

(i) *the X_0-total C-duality holds if the strong C-duality holds provided that $S_P \cap X_0 \neq \emptyset$;*

(ii) *the stable X_0-total C-duality holds if, for each $p \in X^*$, the strong C-duality holds between (P_p) and (D_p^C) provided that $S_P(p) \cap X_0 \neq \emptyset$.*

In particular, in the case when $X_0 = X$, the X_0-total C-duality and the stable X_0-total C-duality are called the total C-duality and the stable total C-duality, respectively.

Let $x \in X$, and set

$$\partial H(x) \coloneqq \partial f_2(x) \times \partial g_2(Ax).$$

Let $\mathbf{\Omega}_0 \coloneqq \operatorname{dom}(\partial H)$, the set of all points $x \in X$ such that $\partial H(x) \neq \emptyset$.

LEMMA 5.2. *Suppose that $S_P \cap \Omega_0 \neq \varnothing$. Then the family $(f_1, f_2, g_1, g_2; A)$ satisfies the $(LSC)_0$.*

Proof. Let $x_0 \in S_P \cap \Omega_0$. Then

$$v(P) = f_1(x_0) - f_2(x_0) + g_1(Ax_0) - g_2(Ax_0). \tag{5.1}$$

Since $x_0 \in \Omega_0$, it follows from [35, Theorem 2.4.1] that f_2 is lsc at x_0 and g_2 is lsc at Ax_0. Then for each $x \in X$,

$$\begin{aligned} &f_1(x_0) - (\mathrm{cl}\, f_2)(x_0) + g_1(Ax_0) - (\mathrm{cl}\, g_2)(Ax_0) \\ &\quad = f_1(x_0) - f_2(x_0) + g_1(Ax_0) - g_2(Ax_0) \\ &\quad \leq f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) \\ &\quad \leq f_1(x) - (\mathrm{cl}\, f_2)(x) + g_1(Ax) - (\mathrm{cl}\, g_2)(Ax). \end{aligned}$$

This implies that $v(P) = v(P^{\mathrm{cl}})$. Hence, by Proposition 3.5, the $(LSC)_0$ holds. ∎

Remark 5.1. If $p \in X^*$ satisfies $S_P(p) \cap \Omega_0 \neq \varnothing$, then $v(D_p^C) \leq v(P_p)$. In fact, by Lemma 5.2, the family $(f_1 - p, f_2, g_1, g_2; A)$ satisfies the $(LSC)_0$ if $S_P(p) \cap \Omega_0 \neq \varnothing$. Hence, applying Remark 3.3 and Theorem 3.6 to $f_1(\cdot) - \langle p, \cdot \rangle$ in place of $f_1(\cdot)$, we have the desired result.

Below we will make use of the subdifferential $\partial h(x)$ for a general proper function (not necessarily convex) $h\colon X \to \bar{\mathbb{R}}$; see (2.3). Clearly, the following equivalence holds:

$$x_0 \text{ is a minimizer of } h \text{ if and only if } 0 \in \partial h\,(x_0). \tag{5.2}$$

The following proposition provides an estimate for the subdifferential of the DC function $f_1 - f_2 + g_1 \circ A - g_2 \circ A$ in terms of the subdifferentials of the convex functions involved. Following [35, page 2], we adapt the convention that $\bigcap_{t \in \varnothing} S_t = X$.

PROPOSITION 5.3. *Let $x_0 \in \Omega$. Then*

$$\partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0) \subseteq \bigcap_{(u^*, v^*) \in \partial H(x_0)} (\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^* v^*). \tag{5.3}$$

If additionally $x_0 \in \Omega_0$ and the (LSC) holds, then

$$\bigcap_{(u^*, v^*) \in H^*} (\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^* v^*) \subseteq \partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0). \tag{5.4}$$

Proof. We first recall from [30] or [29, page 90] that the Fréchet subdifferential of ϕ at a point $x_0 \in \mathrm{dom}\, \phi$ with $|\phi(x_0)| < \infty$ is defined by

$$\hat{\partial}\phi(x_0) := \left\{ x^* \in X^* \colon \liminf_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle x^*, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\},$$

where $\phi\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is an extended real valued function, and the related fact (cf. [30, Theorem 3.1]),

$$\hat{\partial}(\phi_1 - \phi_2)(x_0) \subseteq \bigcap_{u^* \in \partial \phi_2(x_0)} (\partial \phi_1(x_0) - u^*), \tag{5.5}$$

where ϕ_1, ϕ_2 are two proper convex functions. Clearly, $\partial\phi(x_0) \subseteq \hat{\partial}\phi(x_0)$ for each $x_0 \in \mathrm{dom}\, \phi$ with $|\phi(x_0)| < \infty$. Let $p \in \partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0)$. Then

$$p \in \hat{\partial}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0). \tag{5.6}$$

Applying (5.5) to $f_1 + g_1 \circ A$ and $f_2 + g_2 \circ A$ in place of ϕ_1 and ϕ_2, respectively, and noting that

$$\partial f_2(x_0) + A^*\partial g_2(Ax_0) \subseteq \partial(f_2 + g_2 \circ A)(x_0)$$

(cf [7, Theorem 3.1]), we conclude that

$$\begin{aligned}\hat{\partial}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0) &\subseteq \bigcap_{u^* \in \partial(f_2+g_2\circ A)(x_0)} (\partial(f_1 + g_1 \circ A)(x_0) - u^*)\\ &\subseteq \bigcap_{(u^*,v^*)\in\partial H(x_0)} (\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^*v^*).\end{aligned}$$

This together with (5.6) implies that $p \in \bigcap_{(u^*,v^*)\in\partial H(x_0)}(\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^*v^*)$. Hence, (5.3) is seen to hold.

Suppose that $x_0 \in \mathbf{\Omega}_0$ and the (LSC) holds. Let $p \in \bigcap_{(u^*,v^*)\in H^*}(\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^*v^*)$. Then $0 \in \partial(f_1 + g_1 \circ A)(x_0) - p - u^* - A^*v^*$ for each $(u^*, v^*) \in H^*$. Let $x \in X$. Then for each $(u^*, v^*) \in H^*$,

$$f_1(x_0) + g_1(Ax_0) - \langle p + u^* + A^*v^*, x_0\rangle \le f_1(x) + g_1(Ax) - \langle p + u^* + A^*v^*, x\rangle;$$

hence,

$$\begin{aligned}&f_1(x_0) + g_1(Ax_0) - \langle p, x_0\rangle - \{\langle u^*, x_0\rangle - f_2^*(u^*)\} - \{\langle A^*v^*, x_0\rangle - g_2^*(v^*)\}\\ &\quad \le f_1(x) + g_1(Ax) - \langle p, x\rangle - \{\langle u^*, x\rangle - f_2^*(u^*)\} - \{\langle A^*v^*, x\rangle - g_2^*(v^*)\}.\end{aligned}$$

Taking the infimum over H^*, we get that

$$\begin{aligned}&f_1(x_0) - (\mathrm{cl}\, f_2)(x_0) + g_1(Ax_0) - (\mathrm{cl}\, g_2)(Ax_0) - \langle p, x_0\rangle\\ &\quad \le f_1(x) - (\mathrm{cl}\, f_2)(x) + g_1(Ax) - (\mathrm{cl}\, g_2)(Ax) - \langle p, x\rangle.\end{aligned} \tag{5.7}$$

Since $x_0 \in \mathbf{\Omega}_0$, it follows that $(\mathrm{cl}\, f_2)(x_0) = f_2(x_0)$ and $(\mathrm{cl}\, g_2)(Ax_0) = g_2(Ax_0)$. Thus, by (5.7), we have that

$$v(P_p^{\mathrm{cl}}) = f_1(x_0) - f_2(x_0) + g_1(Ax_0) - g_2(Ax_0) - \langle p, x_0\rangle. \tag{5.8}$$

Moreover, by Proposition 3.5 (applied to $f_1(\cdot) - \langle p, \cdot\rangle$ in place of $f_1(\cdot)$), the (LSC) implies $v(P_p^{\mathrm{cl}}) = v(P_p)$. This together with (5.8) implies that $x_0 \in S_P(p)$. Hence, $p \in \partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0)$ by (5.2) and the inclusion (5.4) is proved. ∎

In the following definition, we extend the Moreau–Rockafellar formula (MRF in brief) to DC functions.

Definition 5.4. *Let $x_0 \in X$. The family $(f_1, f_2, g_1, g_2; A)$ is said to satisfy*

(a) *the quasi MRF at x_0 if*

$$\begin{aligned}&\partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0)\\ &\quad \subseteq \bigcap_{(u^*,v^*)\in\partial H(x_0)} (\partial f_1(x_0) - u^* + A^*\partial g_1(Ax_0) - A^*v^*);\end{aligned} \tag{5.9}$$

(b) *the MRF at x_0 if*

$$\partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0) \subseteq \bigcap_{(u^*,v^*)\in H^*} (\partial f_1(x_0) - u^* + A^*\partial g_1(Ax_0) - A^*v^*); \tag{5.10}$$

(c) *the strong MRF at x_0 if $\partial H(x_0) = \varnothing$ or*

$$\bigcap_{(u^*,v^*)\in \partial H(x_0)} (\partial(f_1 + g_1 \circ A)(x_0) - u^* - A^*v^*) \subseteq \bigcap_{(u^*,v^*)\in H^*} (\partial f_1(x_0) - u^* + A^*\partial g_1(Ax_0) - A^*v^*). \tag{5.11}$$

We say that the family $(f_1, f_2, g_1, g_2; A)$ satisfies the quasi MRF (resp., the MRF and the strong MRF) if it satisfies the quasi MRF (resp., the MRF and the strong MRF) at each point $x_0 \in \Omega$.

Remark 5.2.

(a) The following implications hold at $x_0 \in \Omega$:

$$\text{the strong MRF} \Rightarrow \text{the quasi MRF} \quad \text{and} \quad \text{the MRF} \Rightarrow \text{the quasi MRF}.$$

If $x_0 \in \Omega_0$, then, by Proposition 5.3, the following implications hold at x_0:

$$\text{the strong MRF} \Rightarrow \text{the MRF} \Rightarrow \text{the quasi MRF}.$$

(b) In the special case when $f_2 = g_2 = 0$, the quasi MRF, the MRF, and the strong MRF are reduced to the MRF for the triple $(f_1, g_1; A)$ introduced in [31] (see also [25]).

For our main theorems in this section, the following lemma is helpful.

LEMMA 5.5. *Let $x_0 \in X$, and let $p \in X^*$ be such that $x_0 \in S_P(p)$ and*

$$p \in \bigcap_{(u^*,v^*)\in H^*} (\partial f_1(x_0) - u^* + A^*\partial g_1(Ax_0) - A^*v^*). \tag{5.12}$$

Then $v(D_p^C) \geq v(P_p)$ and for each $(u^*, v^*) \in H^*$, there exists $y^* \in Y^*$ satisfying

$$-f_1^*(p + u^* - A^*y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) \geq v(P_p). \tag{5.13}$$

Proof. Take $(u^*, v^*) \in H^*$. Then

$$p \in \partial f_1(x_0) - u^* + A^*\partial g_1(Ax_0) - A^*v^*.$$

Consequently, there exist $y_1^* \in \partial f_1(x_0)$ and $y_2^* \in \partial g_1(Ax_0)$ such that

$$p = y_1^* - u^* + A^*(y_2^* - v^*). \tag{5.14}$$

By the Young equality (2.5),

$$\langle y_1^*, x_0\rangle = f_1^*(y_1^*) + f_1(x_0), \qquad \langle y_2^*, Ax_0\rangle = g_1^*(y_2^*) + g_1(Ax_0), \tag{5.15}$$

and by the Young–Fenchel inequality (2.4),

$$\langle u^*, x_0\rangle \leq f_2^*(u^*) + f_2(x_0), \qquad \langle v^*, Ax_0\rangle \leq g_2^*(v^*) + g_2(Ax_0). \tag{5.16}$$

Thus, combining (5.14)–(5.16), we have

$$
\begin{aligned}
&-f_1^*(p+u^*-A^*(y_2^*-v^*))+f_2^*(u^*)-g_1^*(y_2^*)+g_2^*(v^*)\\
&\quad=-f_1^*(y_1^*)+f_2^*(u^*)-g_1^*(y_2^*)+g_2^*(v^*)\\
&\quad\geq -\langle y_1^*,x_0\rangle+f_1(x_0)+\langle u^*,x_0\rangle-f_2(x_0)-\langle y_2^*,Ax_0\rangle\\
&\qquad+g_1(Ax_0)+\langle v^*,Ax_0\rangle-g_2(Ax_0)\\
&\quad=f_1(x_0)-f_2(x_0)+g_1(Ax_0)-g_2(Ax_0)-\langle y_1^*-u^*+A^*(y_2^*-v^*),x_0\rangle\\
&\quad=f_1(x_0)-f_2(x_0)+g_1(Ax_0)-g_2(Ax_0)-\langle p,x_0\rangle.
\end{aligned}
$$

Since $x_0 \in S_P(p)$, it follows that

$$-f_1^*(p+u^*-A^*(y_2^*-v^*))+f_2^*(u^*)-g_1^*(y_2^*)+g_2^*(v^*)\geq v(P_p).$$

This together with the definition of (D_p^C) implies that $v(D_p^C)\geq v(P_p)$ and (5.13) holds with $y^*=y_2^*-v^*$. The proof is complete. □

THEOREM 5.6. *Suppose that the family $(f_1,f_2,g_1,g_2;A)$ satisfies the strong MRF. Then the stable $\mathbf{\Omega}_0$-total C-duality holds.*

Proof. By Remark 3.1, we need to show only that for each $p\in X^*$ satisfying $S_P(p)\cap\mathbf{\Omega}_0\neq\varnothing$, $v(P_p)=v(D_p)$ holds and for each $(u^*,v^*)\in H^*$, there exists $y^*\in Y^*$ such that (5.13) holds. To do this, suppose that $p\in X^*$ satisfies $S_P(p)\cap\mathbf{\Omega}_0\neq\varnothing$. Take $x_0\in S_P(p)\cap\mathbf{\Omega}_0$. Then $p\in\partial(f_1-f_2+g_1\circ A-g_2\circ A)(x_0)$ by (5.2). This and (5.3) imply that $p\in\bigcap_{(u^*,v^*)\in\partial H(x_0)}(\partial(f_1+g_1\circ A)(x_0)-u^*-A^*v^*)$. Hence (5.12) holds by the assumed strong MRF. Now Lemma 5.5 is applied to get that $v(D_p^C)\geq v(P_p)$ and for each $(u^*,v^*)\in H^*$ there exists $y^*\in Y^*$ satisfying (5.13). Moreover, we have that $v(D_p^C)\leq v(P_p)$ by Remark 5.1. Therefore, $v(D_p^C)=v(P_p)$. Thus, the stable $\mathbf{\Omega}_0$-total C-duality holds and the proof is complete. □

The following example shows that the $\mathbf{\Omega}_0$-total dualities in the conclusion of Theorem 5.6 cannot be replaced by the total dualities.

Example 5.1. Let $X=Y=\mathbb{R}$ and A be the identity. Define $f_1,f_2,g_1,g_2:\mathbb{R}\to\bar{\mathbb{R}}$, respectively, by

$$f_1 := \delta_{[-1,1]},\qquad f_2(x):=\begin{cases}-\sqrt{1-x^2}, & |x|\leq 1,\\ +\infty, & |x|>1,\end{cases}$$

$$g_1(x):=\begin{cases}0, & x>1,\\ 2, & x=1,\\ +\infty, & x<1,\end{cases}\quad\text{and}\quad g_2(x):=\begin{cases}0, & x>1,\\ 1, & x=1,\\ +\infty, & x<1.\end{cases}$$

Then f_1, f_2, g_1, and g_2 are proper convex functions and

$$\partial f_2(x)=\begin{cases}\left\{\frac{x}{\sqrt{1-|x|^2}}\right\}, & |x|<1,\\ \varnothing, & |x|\geq 1,\end{cases}\quad\text{and}\quad \partial g_2(x)=\begin{cases}\{0\}, & x>1,\\ \varnothing, & x\leq 1.\end{cases}$$

Hence, $\mathbf{\Omega}_0=\varnothing$, and so the strong MRF holds. Below we show that the total C-duality does not hold. It is easy to see that

$$v(P)=f_1(1)-f_2(1)+g_1(1)-g_2(1)=1.$$

Then $1\in S_P$. Moreover, for each $x^*\in\mathbb{R}$,

$$f_1^*(x^*) = \begin{cases} x^*, & x^* \geq 0, \\ -x^*, & x^* < 0, \end{cases} \qquad f_2^*(x^*) = \sqrt{1 + x^{*2}},$$

and

$$g_1^*(x^*) = g_2^*(x^*) = \begin{cases} x^*, & x^* \leq 0, \\ +\infty, & x^* > 0. \end{cases}$$

Then $\operatorname{dom} f_2^* = \mathbb{R}$, $\operatorname{dom} g_2^* = (-\infty, 0]$, and $H^* = \mathbb{R} \times (-\infty, 0]$. Let $\Phi\colon H^* \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\Phi(u^*, v^*, y^*) \coloneqq -f_1^*(u^* - y^*) + f_2^*(u^*) - g_1^*(v^* + y^*) + g_2^*(v^*)$$

for each $(u^*, v^*, y^*) \in H^* \times \mathbb{R}$. Then, for each $(u^*, v^*, y^*) \in H^* \times \mathbb{R}$,

$$\Phi(u^*, v^*, y^*) = \begin{cases} u^* + \sqrt{1 + u^{*2}} - 2y^*, & u^* \leq y^* \leq -v^* \\ \sqrt{1 + u^{*2}} - u^*, & y^* \leq -v^*, \quad y^* < u^*, \\ -\infty, & y^* > -v^*; \end{cases}$$

hence,

$$\sup_{y^* \in \mathbb{R}} \Phi(u^*, v^*, y^*) = \sqrt{1 + u^{*2}} - u^*$$

and

$$v(D^C) = \inf_{(u^*, v^*) \in H^*} \sup_{y^* \in \mathbb{R}} \Phi(u^*, v^*, y^*) = 0 < 1 = v(P).$$

This means that the total C-duality does not hold.

The following theorem provides a partial converse of Theorem 5.6.

THEOREM 5.7. *Suppose that the stable $\boldsymbol{\Omega}_0$-total C-duality holds. Then the family $(f_1, f_2, g_1, g_2; A)$ satisfies the quasi MRF.*

Proof. Let $x_0 \in \boldsymbol{\Omega}$. Obviously, if $x_0 \notin \boldsymbol{\Omega}_0$, then $\partial H(x_0) = \varnothing$ and (5.9) holds trivially as

$$\bigcap_{(u^*, v^*) \in \partial H(x_0)} (\partial f_1(x_0) - u^* + A^* \partial g_1(Ax_0) - A^* v^*) = X^*.$$

Below we assume that $x_0 \in \boldsymbol{\Omega}_0$ and let $p \in \partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0)$. Then, by (5.2), we have that $x_0 \in S_P(p)$. Let $(u^*, v^*) \in \partial H(x_0)$. By the assumed stable $\boldsymbol{\Omega}_0$-total C-duality and Remark 3.1, there exists $y^* \in Y^*$ satisfying

$$-f_1^*(p + u^* - A^* y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) \geq v(D_p^C) = v(P_p).$$

Noting that $x_0 \in S_P(p)$, we have

$$\begin{aligned} &-f_1^*(p + u^* - A^* y^*) + f_2^*(u^*) - g_1^*(y^* + v^*) + g_2^*(v^*) \\ &\quad \geq f_1(x_0) - f_2(x_0) + g_1(Ax_0) - g_2(Ax_0) - \langle p, x_0 \rangle. \end{aligned}$$

Then

$$\begin{aligned}&[f_1^*(p+u^*-A^*y^*)+f_1(x_0)-\langle p+u^*-A^*y^*,x_0\rangle]\\ &\quad+[g_1^*(y^*+v^*)+g_1(Ax_0)-\langle y^*+v^*,Ax_0\rangle]\\ &\quad\le[f_2^*(u^*)+f_2(x_0)-\langle u^*,x_0\rangle]+[g_2^*(v^*)+g_2(Ax_0)-\langle v^*,Ax_0\rangle].\end{aligned}\tag{5.17}$$

By the Young–Fenchel inequality (2.4), we have

$$f_1^*(p+u^*-A^*y^*)+f_1(x_0)-\langle p+u^*-A^*y^*,x_0\rangle\ge 0\tag{5.18}$$

and

$$g_1^*(y^*+v^*)+g_1(Ax_0)-\langle y^*+v^*,Ax_0\rangle\ge 0.\tag{5.19}$$

Therefore,

$$\begin{aligned}0&\le[f_1^*(p+u^*-A^*y^*)+f_1(x_0)-\langle p+u^*-A^*y^*,x_0\rangle]\\ &\quad+[g_1^*(y^*+v^*)+g_1(Ax_0)-\langle y^*+v^*,Ax_0\rangle]\\ &\le[f_2^*(u^*)+f_2(x_0)-\langle u^*,x_0\rangle]+[g_2^*(v^*)+g_2(Ax_0)-\langle v^*,Ax_0\rangle]\\ &=0,\end{aligned}\tag{5.20}$$

where the first inequality holds by (5.18) and (5.19), the second inequality holds because of (5.19), while the last equality holds by the Young equality (2.5) (noting that $u^*\in\partial f_2(x_0)$ and $v^*\in\partial g_2(Ax_0)$). Combining (5.18)–(5.20), we get that

$$f_1^*(p+u^*-A^*y^*)+f_1(x_0)-\langle p+u^*-A^*y^*,x_0\rangle=0$$

and

$$g_1^*(y^*+v^*)+g_1(Ax_0)-\langle y^*+v^*,Ax_0\rangle=0.$$

Hence, $p+u^*-A^*y^*\in\partial f(x_0)$ and $y^*+v^*\in\partial g_1(Ax_0)$ thanks to the Young equality (2.5). Consequently,

$$p+u^*+A^*v^*=p+u^*-A^*y^*+A^*(y^*+v^*)\in\partial f(x_0)+A^*\partial g_1(Ax_0);$$

hence, $p\in\partial f(x_0)-u^*+A^*\partial g_1(Ax_0)-A^*v^*$. This means that

$$p\in\bigcap_{(u^*,v^*)\in\partial H(x_0)}(\partial f_1(x_0)-u^*+A^*\partial g_1(Ax_0)-A^*v^*)$$

as $(u^*,v^*)\in\partial H(x_0)$ is arbitrary. Then, (5.9) is proved, and the proof is complete. ∎

Theorems 5.8 and 5.9 below provide sufficient conditions ensuring the total dualities and the stable total dualities.

THEOREM 5.8. *Suppose that the family $(f_1,f_2,g_1,g_2;A)$ satisfies both the $(SFRC)$ and the MRF. Then the total C-duality holds.*

Proof. Suppose that $S_P\neq\varnothing$, and let $x_0\in S_P$. Then $0\in\partial(f_1-f_2+g_1\circ A-f_2\circ A)(x_0)$, and (5.12) holds by the assumed MRF. Thus, Lemma 5.5 is applied to get that $v(P)\le v(D^C)$ and for each $(u^*,v^*)\in H^*$ there exists $y^*\in Y^*$ such that (5.13) holds. Moreover, $v(P)\ge v(D^C)$ by Theorem 3.6 because of the assumed $(SFRC)$. Hence, $v(P)=v(D^C)$ and for each $(u^*,v^*)\in H^*$ there exists $y^*\in Y^*$ such that (3.7) holds; that is, the strong C-duality holds, thanks to Remark 3.1. This proves the total C-duality and completes the proof. ∎

By Theorem 5.8, the following theorem is direct.

THEOREM 5.9. *Suppose that the family* $(f_1, f_2, g_1, g_2; A)$ *satisfies both the* (SCC) *and the MRF. Then the stable total* C*-duality holds.*

In the case when $f_2 = g_2 = 0$, by Theorems 5.6 and 5.7, we have the following corollary, which was given in [25, Theorem 5.2].

COROLLARY 5.10. *The family* $(f_1, g_1; A)$ *satisfies the MRF if and only if for each* $p \in X^*$ *satisfying* $S_P(p) \neq \varnothing$*, the following formula holds:*

$$\min_{x\in X}\{f_1(x) + g_1(Ax) - \langle p, x\rangle\} = \max_{y^*\in Y^*}\{-f_1^*(p - A^*y^*) - g_1^*(y^*)\}.$$

Acknowledgment. The authors are grateful to both anonymous reviewers for valuable suggestions and remarks which helped to improve the quality of the paper.

REFERENCES

[1] L. T. H. AN AND P. D. TAO, *The DC (difference of convex functions) programming and DCA revisited with DC models of real world non-convex optimization problems*, Ann. Oper. Res., 133 (2005), pp. 23–46.

[2] R. I. BOŢ, E. R. CSETNEK, AND G. WANKA, *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optim., 19 (2008), pp. 217–233.

[3] R. I. BOŢ, S. M. GRAD, AND G. WANKA, *On strong and total Lagrange duality for convex optimization problems*, J. Math. Anal. Appl., 337 (2008), pp. 1315–1325.

[4] R. I. BOŢ, S. M. GRAD, AND G. WANKA, *New regularity conditions for strong and total Fenchel–Lagrange duality in infinite dimensional spaces*, Nonlinear Anal., 69 (2008), pp. 323–336.

[5] R. I. BOŢ, S. M. GRAD, AND G. WANKA, *Duality in Vector Optimization*, Springer-Verlag, Berlin, 2009.

[6] R. I. BOŢ, I. B. HODREA, AND G. WANKA, *Some new Farkas-type results for inequality systems with DC functions*, J. Global Optim., 39 (2007), pp. 595–608.

[7] R. I. BOŢ AND G. WANKA, *A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces*, Nonlinear Anal., 64 (2006), pp. 2787–2804.

[8] R. I. BOŢ AND G. WANKA, *Duality for multiobjective optimization problems with convex objective functions and DC constraints*, J. Math. Anal. Appl., 315 (2006), pp. 526–543.

[9] R. S. BURACHIK AND V. JEYAKUMAR, *A dual condition for the convex subdifferential sum formula with applications*, J. Convex Anal., 12 (2005), pp. 279–290.

[10] R. S. BURACHIK AND V. JEYAKUMAR, *A simpler closure condition for the normal cone intersection formula*, Proc. Amer. Math. Soc., 133 (2005), pp. 1741–1748.

[11] R. S. BURACHIK, V. JEYAKUMAR, AND Z. Y. WU, *Necessary and sufficient conditions for stable conjugate duality*, Nonlinear Anal., 64 (2006), pp. 1998–2006.

[12] V. F. DEMYANOV AND A. M. RUBINOV, *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt, 1995.

[13] N. DINH, B. MORDUKHOVICH, AND T. T. A. NGHIA, *Qualification and optimality conditions for convex and DC programs with infinite constraints*, Acta Math. Vietnam., 34 (2009), pp. 125–155.

[14] N. DINH, B. MORDUKHOVICH, AND T. T. A. NGHIA, *Subdifferentials of value functions and optimality conditions for DC and bilevel infinite and semi-infinite programs*, Math. Program., 123 (2010), pp. 101–138.

[15] N. DINH, T. T. A. NGHIA, AND G. VALLET, *A closedness condition and its applications to DC programs with convex constraints*, Optimization, 59 (2010), pp. 541–560.

[16] N. DINH, G. VALLET, AND T. T. A. NGHIA, *Farkas-type results and duality for DC programs with convex constraints*, J. Convex Anal., 15 (2008), pp. 235–262.

[17] D. H. FANG, C. LI, AND K. F. NG, *Constraint qualifications for extended Farkas's lemmas and Lagrangian dualities in convex infinite programming*, SIAM J. Optim., 20 (2009), pp. 1311–1332.

[18] M. A. GOBERNA, V. JEYAKUMAR, AND M. A. LÓPEZ, *Necessary and sufficient conditions for solvability of systems of infinite convex inequalities*, Nonlinear Anal., 68 (2008), pp. 1184–1194.

[19] J. HIRIART-URRUTY AND C. LEMARÉCHAL, *Convex Analysis and Minimization Algorithms* I, Grundlehren Math. Wiss. 305, Springer-Verlag, New York, 1993.

[20] R. B. Holmes, *Geometric Functional Analysis*, Springer-Verlag, Berlin, 1975.
[21] R. Horst and N. V. Thoai, *DC programming: Overview*, J. Optim. Theory Appl., 103 (1999), pp. 1–43.
[22] V. Jeyakumar, *The strong conical hull intersection property for convex programming*, Math. Programming, 106 (2006), pp. 81–92.
[23] V. Jeyakumar, G. M. Lee, and N. Dinh, *New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs*, SIAM J. Optim., 14 (2003), pp. 534–547.
[24] V. Jeyakumar and G. M. Lee, *Complete characterization of stable Farkas' lemma and cone-convex programming duality*, Math. Programming, 114 (2008), pp. 335–347.
[25] C. Li, D. H. Fang, G. López, and M. A. López, *Stable and total Fenchel duality for convex optimization problems in locally convex spaces*, SIAM J. Optim., 20 (2009), pp. 1032–1051.
[26] C. Li, K. F. Ng, and T. K. Pong, *The SECQ, linear regularity, and the strong CHIP for an infinite system of closed convex sets in normed linear spaces*, SIAM J. Optim., 18 (2007), pp. 643–665.
[27] C. Li, K. F. Ng, and T. K. Pong, *Constraint qualifications for convex inequality systems with applications in constrained optimization*, SIAM J. Optim., 19 (2008), pp. 163–187.
[28] J. E. Martinez-Legaz and M. Volle, *Duality in DC programming: The case of several constraints*, J. Math. Anal. Appl., 237 (1999), pp. 657–671.
[29] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation,* I: *Basic Theory*, Springer-Verlag, Berlin, 2006.
[30] B. S. Mordukhovich, N. M. Nam, and N. D. Yen, *Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming*, Optimization, 55 (2006), pp. 685–708.
[31] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Math. 1364, Springer-Verlag, Berlin, 1993.
[32] R. T. Rockafellar and R. J. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.
[33] T. Strömberg, *The operation of infimal convolution*, Dissertationes Math. (Rozprawy Mat.), 352 (1996), pp. 1–58.
[34] J. F. Toland, *Duality in nonconvex optimization*, J. Math. Anal. Appl., 66 (1978), pp. 399–415.
[35] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ, 2002.