Analytical blowup solutions to the 2-dimensional isothermal Euler-Poisson equations of gaseous stars II

Manwai Yuen[a)]
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

(Received 3 September 2010; accepted 30 June 2011; published online 22 July 2011)

In this article, we construct analytical blowup solutions with non-radial symmetry for the 2-dimensional Euler-Poisson equations. Based on the previous solutions with radial symmetry for the 2-dimensional isothermal Euler-Poisson equations, some special blowup solutions with non-radial symmetry are constructed by the separation method. © *2011 American Institute of Physics*. [doi:10.1063/1.3614504]

I. INTRODUCTION

The evolution of self-gravitating Newtonian fluids (gaseous stars) can be formulated by the isentropic Euler-Poisson equations of the following form:

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla P = -\rho\nabla\Phi \\ \Delta\Phi(t,\vec{x}) = \alpha(N)\rho, \end{cases} \tag{1}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N: $\alpha(1) = 2$; $\alpha(2) = 2\pi$ and For $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)}, \tag{2}$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is the Gamma function. And as usual, $\rho = \rho(t,\vec{x})$ and $\vec{u} = \vec{u}(t,\vec{x}) = (u_1, u_2, \ldots, u_N) \in \mathbf{R}^N$ are the density and the velocity, respectively.

In the above system, the self-gravitational potential field $\Phi = \Phi(t,\vec{x})$ is determined by the density ρ through the Poisson equation.

The equation $(1)_3$ is the Poisson equation through which the gravitational potential is determined by the density distribution itself. Thus, we call the system (1) the Euler-Poisson equations. The equations can be viewed as a prefect gas model in galactic dynamics and cosmology.[2,13] The function $P = P(\rho)$ is the pressure. The γ-law can be applied on the pressure $P(\rho)$, i.e.,

$$P(\rho) = K\rho^\gamma := \frac{\rho^\gamma}{\gamma}, \tag{3}$$

which is the common hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where c_P, c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (3). In particular, the fluid is called isothermal if $\gamma = 1$. It can be used for constructing models with non-degenerate isothermal cores, which have a role in connection with the so-called Schonberg-Chandrasekhar limit.[9]

[a)]Electronic mail: nevetsyuen@hotmail.com.

The system can be rewritten as

$$\begin{cases} \rho_t + \nabla\cdot\vec{u}\rho + \nabla\rho\cdot\vec{u} = 0 \\ \rho\left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k}\right) + \frac{\partial}{\partial x_i}P(\rho) = -\rho\frac{\partial}{\partial x_i}\Phi(\rho), \text{ for } i = 1, 2, ...N \\ \Delta\Phi(t,x) = \alpha(N)\rho. \end{cases} \tag{4}$$

For $N = 3$, the system (4) is a classical (non-relativistic) description of a galaxy, in astrophysics. See Refs. 2, 3, and 9 for details about the system.

For the local existence results about the system were shown in Refs. 1, 6, and 11. In particular, the radially symmetric solutions can be expressed by

$$\rho(t,\vec{x}) = \rho(t,r) \quad \text{and} \quad \vec{u}(t,\vec{x}) = \frac{\vec{x}}{r}V(t,r) := \frac{\vec{x}}{r}V, \tag{5}$$

where the radial diameter $r := \left(\sum_{i=1}^N x_i^2\right)^{1/2}$. Historically in astrophysics, Goldreich and Weber constructed the analytical blowup (collapsing) solutions of the 3-dimensional Euler-Poisson equations for $\gamma = 4/3$ for the non-rotating gas spheres.[7] After that, Makino[12] obtained the rigorously mathematical proof of the existence of such kind of blowup solutions. Besides, Deng *et al*. extended the above blowup solutions in R^N $(N \geq 3)$.[4] Then, Yuen obtained the blowup solutions in R^2 with $\gamma = 1$ by a new transformation.[15] The family of the analytical solutions are rewritten as follows:

for $N \geq 3$ and $\gamma = (2N-2)/N$, in Ref. 4

$$\begin{cases} \rho(t,r) = \begin{cases} \frac{1}{a^N(t)} y(\frac{r}{a(t)})^{N/(N-2)}, & \text{for} \quad r < a(t)Z_\mu; \\ 0, \quad \text{for} & a(t)Z_\mu \leq r. \end{cases}, V(t,r) = \frac{\dot{a}(t)}{a(t)}r \\ \ddot{a}(t) = \frac{-\lambda}{a^{N-1}(t)}, \ a(0) = a_1 \neq 0, \ \dot{a}(0) = a_2 \\ \ddot{y}(z) + \frac{N-1}{z}\dot{y}(z) + \frac{\alpha(N)}{(2N-2)K} y(z)^{N/(N-2)} = \mu, \ y(0) = \alpha > 0, \ \dot{y}(0) = 0, \end{cases} \tag{6}$$

where $\mu = [N(N-2)\lambda]/(2N-2)K$ and the finite Z_μ is the first zero of $y(z)$;

for $N = 2$ and $\gamma = 1$, in Ref. 15

$$\begin{cases} \rho(t,r) = \frac{1}{a^2(t)} e^{y(r/a(t))}, \ V(t,r) = \frac{\dot{a}(t)}{a(t)}r \\ \ddot{a}(t) = \frac{-\lambda}{a(t)}, \ a(0) = a_1 > 0, \ \dot{a}(0) = a_2 \\ \ddot{y}(z) + \frac{1}{z}\dot{y}(z) + \frac{\alpha(2)}{K} e^{y(z)} = \mu, \ y(0) = \alpha, \ \dot{y}(0) = 0, \end{cases} \tag{7}$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small λ and α are constants.

And for other special blowup solutions, the readers may see the details in Ref. 14.

In 2009, Yuen extended the above solutions to the pressureless Navier-Stokes-Poisson equations with density-dependent viscosity in Ref. 16:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \frac{N-1}{r}\rho V = 0 \\ \rho(V_t + VV_r) + \frac{\alpha(N)\rho}{r^{N-1}} \int_0^r \rho(t,s)s^{N-1}ds = [\kappa\rho^\theta]_r \left(\frac{N-1}{r}V + V_r\right) \\ \qquad + (\kappa\rho^\theta)(V_{rr} + \frac{N-1}{r}V_r + \frac{N-1}{r^2}V). \end{cases} \tag{8}$$

On the other hand, if we consider the Euler-Poisson equation (1) with a negative background,

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla P = -\rho\nabla\Phi \\ \Delta\Phi(t,\vec{x}) = \alpha(N)\rho - \Lambda \end{cases} \tag{9}$$

some periodic solutions were found by Yuen.[17]

However, the known solutions are all in radial symmetry. In this paper, we continue to construct the analytical solutions with non-radial symmetry for the Euler-Poisson equations by the separation method. Here, we are able to obtain the similar results to the non-radially symmetric cases for the 2-dimensional Euler-Poisson equations (4) in the following theorem:

Theorem 1: *For the* 2*-dimensional isothermal Euler-Poisson equations (4), there exists a family of solutions,*

$$\begin{cases} \rho(t,x,y) = \frac{C}{a(t)^2} e^{-\frac{\Phi\left(\frac{Ax+By}{a(t)}\right)}{K}}, \ \vec{u}(t,x,y) = \frac{\dot{a}(t)}{a(t)}(x,y) \\ a(t) = a_1 + a_2 t \\ \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \ \Phi(0) = \alpha, \ \dot{\Phi}(0) = \beta \end{cases} \tag{10}$$

where $A,\ B,\ C > 0,\ a_1 \neq 0,\ a_2\ ,\ \frac{2\pi C}{A^2+B^2} = \epsilon^* > 0$ *with that* A *and* B *are not both* 0*;* α *and* β *are constants.*

In particular,

(1) $a_1 > 0$ *and* $a_2 < 0$*, the solutions (10) blow up in the finite time* $T = -a_2/a_1$.
(2) $a_1 > 0$ *and* $a_2 > 0$*, the solutions (10) are global.*

We can arbitrarily choose $A,\ B,\ C,\ \alpha,\ \beta,\ a_1$, and a_2 in the theorem. And, our solutions (10) could provide the tool for testing numerical methods.

Remark 2: *The solutions have real applications in engineering. We may interpret the solutions (10) as a line source or sink in terms of cylindrical coordinates, when we take the z direction to lie along the characteristic line of the source or sink. For the physical significance of such kind of solutions, the interested readers may refer to pp. 409-410 of Ref. 8 for details.*

II. SEPARABLE BLOWUP SOLUTIONS

Before presenting the proof of Theorem 1, we prepare some lemmas first.

Lemma 3: For the continuity equation $(4)_1$ *in* R^2*, there exist solutions,*

$$\rho(t,x,y) = \frac{f\left(\frac{Ax+By}{a(t)}\right)}{a^2(t)},\ \vec{u}(t,x,y) = \frac{\dot{a}(t)}{a(t)}(x,y), \tag{11}$$

where the scalar function $f(s) \geq 0 \in C^1$ *and* $a(t) \neq 0 \in C^1$.

Proof: We plug the solutions (10) into the continuity equation $(4)_1$,

$$\rho_t + \nabla\cdot\vec{u}\rho + \nabla\rho\cdot\vec{u} \tag{12}$$

$$= \frac{\partial}{\partial t}\left[\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\right] + \nabla\cdot\frac{\dot{a}(t)}{a(t)}(x,y)\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2} + \nabla\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\cdot\frac{\dot{a}(t)}{a(t)}(x,y) \tag{13}$$

$$= \frac{-2\dot{a}(t)}{a(t)^3}f\left(\frac{Ax+By}{a(t)}\right) + \frac{1}{a(t)^2}\frac{\partial}{\partial t}f\left(\frac{Ax+By}{a(t)}\right) + \frac{\dot{a}(t)}{a(t)}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2} \tag{14}$$

$$+\frac{\dot{a}(t)}{a(t)}\left[\frac{\partial}{\partial x}\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\cdot x + \frac{\partial}{\partial y}\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\cdot y\right] \tag{15}$$

$$= \frac{-2\dot{a}(t)}{a(t)^3}f\left(\frac{Ax+By}{a(t)}\right) - \frac{1}{a(t)^2}\dot{f}\left(\frac{Ax+By}{a(t)}\right)\frac{(Ax+By)\dot{a}(t)}{a(t)^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{f\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2} \tag{16}$$

$$+\frac{\dot{a}(t)}{a(t)}\left[\frac{\dot{f}\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\frac{Ax}{a(t)} + \frac{\dot{f}\left(\frac{Ax+By}{a(t)}\right)}{a(t)^2}\frac{By}{a(t)}\right] \tag{17}$$

$$= 0. \tag{18}$$

The proof is completed. ■

The technique for proving the global existence of the ordinary differential equation $(10)_3$ is the standard energy method in classical mechanics.[5,10]

Lemma 4: The equation,

$$\begin{cases} \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0 \\ \Phi(0) = \alpha,\ \dot{\Phi}(0) = \beta \end{cases} \tag{19}$$

where $\epsilon^* > 0$, α *and* β *are constants, has a solution in* $\Phi(s) \in C^2(-\infty, +\infty)$ *and* $\lim_{s\to\pm\infty}\Phi(s) = +\infty$.

Proof: The proof is similar to Lemma 3 in Ref. 17.

For the equation (19), multiply $\dot{a}(t)$ and then integrate it, as follows:

$$(\ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}})\dot{\Phi}(s) = 0, \tag{20}$$

$$\int_0^s \dot{\Phi}(\eta)d\dot{\Phi}(\eta) - \epsilon^*\int_0^s e^{-\frac{\Phi(\eta)}{K}}d\Phi(\eta) = 0, \tag{21}$$

$$\frac{1}{2}\dot{\Phi}(s)^2 + \epsilon^* e^{-\frac{\Phi(s)}{K}} = \theta, \tag{22}$$

with the constant $\theta = \frac{1}{2}\dot{\Phi}(0)^2 + \epsilon^* e^{-\frac{\Phi(0)}{K}} > 0$.

We define the kinetic energy as:

$$F_{kin}(t) := \frac{\dot{\Phi}(t)^2}{2} \tag{23}$$

and the potential energy as:

$$F_{pot}(t) = \epsilon^* e^{-\frac{\Phi(t)}{K}}. \tag{24}$$

And the total energy is conserved:

$$\frac{d}{dt}(F_{kin}(t) + F_{pot}(t)) = 0 \tag{25}$$

$$F_{kin}(t) + F_{pot}(t) = \theta. \tag{26}$$

By the classical energy method for conservative systems (in Sec. 4.3 of Ref. 10), the solutions have a trajectory for the potential function (24). We may calculate the required time for traveling the whole orbit:

$$T = \int_0^T \frac{d\Phi(\eta)}{\sqrt{2(\theta - \epsilon^* e^{-\frac{\Phi(\eta)}{K}})}} = \int_\alpha^{+\infty} \frac{d\eta}{\sqrt{2(\theta - \epsilon^* e^{-\frac{\eta}{K}})}} \geq \int_\alpha^{+\infty} \frac{d\eta}{\sqrt{2\theta}} = +\infty. \tag{27}$$

Therefore, the solution $\Phi(s)$ exists globally for $s \in [0, +\infty)$ and $\lim_{s\to+\infty} \Phi(s) = +\infty$.

For the interval $s \in (-\infty, 0]$, we let $t = -s$ to have the equation (19)

$$\begin{cases} \ddot{\Phi}(t) - \epsilon^* e^{-\frac{\Phi(t)}{K}} = 0 \\ \Phi(0) = \alpha,\ \dot{\Phi}(0) = -\beta. \end{cases} \tag{28}$$

For $t \geq 0$, it is similar to have $\lim_{t\to+\infty} \Phi(t) = +\infty$.

That is $\lim_{s\to\pm\infty} \Phi(s) = +\infty$ to show that the lemma is true.

The proof is completed. ■

On the other hand, the following lemma handles the Poisson equation $(4)_3$ for our solutions (10):

Lemma 5: The solutions,

$$\rho = \frac{C}{a(t)^2} e^{-\frac{\Phi\left(\frac{Ax+By}{a(t)}\right)}{K}} \tag{29}$$

with the second-order ordinary differential equation:

$$\begin{cases} \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0 \\ \Phi(0) = \alpha,\ \dot{\Phi}(0) = \beta \end{cases} \tag{30}$$

where $s := (Ax + By)/a(t)$ *and* C, $\frac{2\pi C}{A^2+B^2} = \epsilon^* > 0$, α *and* β *are constants, fit into the Poisson equation* $(4)_3$ *in* R^2.

Proof: We check that our potential function $\Phi(t,x,y)$ satisfies the Poisson equation $(4)_3$:

$$\Delta\Phi(t,x,y)-2\pi\rho \tag{31}$$

$$=\frac{\partial}{\partial x}\left[\dot{\Phi}(\frac{Ax+By}{a(t)})\frac{A}{a(t)}\right]+\frac{\partial}{\partial y}\left[\dot{\Phi}(\frac{Ax+By}{a(t)})\frac{B}{a(t)}\right]-\frac{2\pi C}{a(t)^2}e^{-\frac{\Phi(\frac{Ax+By}{a(t)})}{K}} \tag{32}$$

$$=\frac{A^2+B^2}{a(t)^2}\left(\ddot{\Phi}(s)-\frac{2\pi C}{A^2+B^2}e^{-\frac{\Phi(s)}{K}}\right), \tag{33}$$

where A and B are not both 0.
Then, we choose $s:=(Ax+By)/a(t)$ and the ordinary differential equation:

$$\begin{cases}\ddot{\Phi}(s)-\epsilon^{*}e^{-\frac{\Phi(s)}{K}}=0\\ \Phi(0)=\alpha,\ \dot{\Phi}(0)=\beta\end{cases} \tag{34}$$

with $\frac{2\pi C}{A^2+B^2}=\epsilon^*$, α and β are constants in Lemma 4. Therefore, our solutions (29) satisfy the Poisson equation $(4)_3$.

The proof is completed. ■

With the above Lemmas, readers are easy to check that the solutions fit into the Euler-Poisson equations (4). For the main technique of the proof, we use the pressure term $\nabla K\rho$ to balance the potential term $-\rho\nabla\Phi$ for the momentum equations. Therefore, we omit the details here.

Remark 6: *For the case of $A=0$ and $B=0$, the corresponding solutions (10) may be deduced to the special solutions in Ref. 14.*

Remark 7: *From Lemma (4), before the blowup time T, we have*

$$\lim_{Ax+By\to\pm\infty}\rho(t,x,y)=\lim_{Ax+By\to\pm\infty}\frac{C}{a(t)^2}e^{-\frac{\Phi(\frac{Ax+By}{a(t)})}{K}}=0. \tag{35}$$

Remark 8: *Our solutions (10) also work for the isothermal Navier-Stokes-Poisson equations in R^2:*

$$\begin{cases}\rho_t+\nabla\cdot(\rho\vec{u})=0\\ (\rho\vec{u})_t+\nabla\cdot(\rho\vec{u}\otimes\vec{u})+\nabla K\rho=-\rho\nabla\Phi+\mu\Delta\vec{u}\\ \Delta\Phi(t,x)=2\pi\rho\end{cases} \tag{36}$$

where $\mu>0$ is a positive constant.

Additionally, the blowup rate about the solutions is immediately followed:

Corollary 9: The blowup rate of the solutions (10) is

$$\lim_{t\to T}\rho(t,0,0)\,(T-t)^2\geq O(1). \tag{37}$$

In conclusion, due to the novel solutions obtained by the separation method, the author conjectures there exists other analytical solution in non-radial symmetry. Further works will be continued for seeking more particular solutions to understand the nature of the Euler-Poisson equations (4).

[1] Bezard, M., "Existence locale de solutions pour les equations d'Euler-Poisson. (French) [Local existence of solutions for Euler-Poisson equations]," Jpn. J. Ind. Appl. Math. **10**, 431–450 (1993).
[2] Binney, J. and Tremaine, S., *Galactic Dynamics* (Princeton University Press, Princeton, NJ, 1994).

[3] Chandrasekhar, S., *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, 1939).
[4] Deng, Y. B., Xiang, J. L., and Yang, T., "Blowup phenomena of solutions to Euler-Poisson equations," J. Math. Anal. Appl. **286**, 295–306 (2003).
[5] Fowles, G. R. and Cassiday, G. L., *Analytical Mechanics*, 7th ed. (Thomson Brooks/Cole, Belmont, CA, 2004).
[6] Gamblin, P., "Solution reguliere a temps petit pour l'equation d'Euler-Poisson. (French) [Small-time regular solution for the Euler-Poisson equation]," Commun. Partial Differ. Equ. **18**, 731–745 (1993).
[7] Goldreich, P. and Weber, S., "Homologously collapsing stellar cores," Astrophys. J. **238**, 991–997 (1980).
[8] John, J. and Keith, T., *Gas Dynamics*, 3rd ed. (Pearson Prentice-Hall, Englewood Cliffs, NJ, 2006).
[9] Kippenhahn, R. and Weigert, A., *Stellar Sturture and Evolution* (Springer-Verlag, Berlin, 1990).
[10] Lakin, W. D. and Sanchez, D. A., *Topics in Ordinary Differential Equations* (Dover Pub. Inc., New York, 1982).
[11] Makino, T., *On a Local Existence Theorem for the Evolution Equation of Gaseous Stars*, Patterns and waves, 459–479, Stud. Math. Appl., **18** (North-Holland, Amsterdam, 1986).
[12] Makino, T., "Blowing up solutions of the Euler-Poission equation for the evolution of the gaseous stars," Transp. Theory Stat. Phys. **21**, 615–624 (1992).
[13] Straumann, N., *General Relativity and Relativistic Atrophysics* (Springer, New York, 1991).
[14] Yuen, M. W., "Blowup solutions for a class of fluid dynamical equations in R^N," J. Math. Anal. Appl. **329**, 1064–1079 (2007).
[15] Yuen, M. W., "Analytical blowup solutions to the 2-dimensional isothermal Euler-Poisson equations of gaseous stars," J. Math. Anal. Appl. **341**, 445–456 (2008).
[16] Yuen, M. W., "Analytical blowup solutions to the pressureless Navier-Stokes-Poisson equations with density-dependent viscosity in R^N," Nonlinearity **22**, 2261–2268 (2009).
[17] Yuen, M. W., "Analytically periodic solutions to the three-dimensional Euler-Poisson equations of gaseous stars with negative cosmological constant," Class. Quantum Grav. **26**, 235011 (2009).