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DEVELOPING FINITE ELEMENT METHODS FOR MAXWELL'S EQUATIONS IN A COLE–COLE DISPERSIVE MEDIUM*

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Abstract. In this paper, we consider the time-dependent Maxwell's equations when Cole–Cole dispersive medium is involved. The Cole–Cole model contains a fractional time derivative term, which couples with the standard Maxwell's equations in free space and creates some challenges in developing and analyzing time-domain finite element methods for solving this model as mentioned in our earlier work [J. Li, *J. Sci. Comput.*, 47 (2001), pp. 1–26]. By adopting some techniques developed for the fractional diffusion equations [V.J. Ervin, N. Heuer, and J.P. Roop, *SIAM J. Numer. Anal.*, 45 (2007), pp. 572–591], [Y. Lin and C. Xu, *J. Comput. Phys.*, 225 (2007), pp. 1533–1552], [F. Liu, P. Zhuang, V. Anh, I. Turner, and K. Burrage, *Appl. Math. Comput.*, 191 (2007), pp. 12–20], we propose two fully discrete mixed finite element schemes for the Cole–Cole model. Numerical stability and optimal error estimates are proved for both schemes. The proposed algorithms are implemented and detailed numerical results are provided to justify our theoretical analysis.

Key words. Maxwell's equations, dispersive medium, Cole–Cole model, finite element method

AMS subject classifications. 65N30, 35L15, 78-08

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1. Introduction. In electromagnetics, if a medium's permittivity or permeability depends on the wave frequency, then this medium is called dispersive medium. Biological tissue, ionosphere, water, soil, plasma, radar absorbing material, and optical fiber are some examples of dispersive media. Therefore study of wave propagation in dispersive media is a very important subject.

In the early 1990's, engineers started the investigation of numerical simulation of wave propagation in dispersive media. The early numerical techniques were limited to the finite-difference time-domain (FDTD) methods, which have a major disadvantage for complex geometry problems. In 2001, Jiao and Jin [12] introduced the time-domain finite element method (TDFEM) for solving Maxwell's equations when dispersive media are involved. Their method is based on a second-order vector wave equation obtained from the Maxwell's equations. In 2003, Lu, Zhang, and Cai [20] developed a time-domain discontinuous Galerkin (DG) method for solving dispersive media models written in first-order Maxwell's equations. Since 2006, various TDFEMs [1, 11, 13, 14, 15, 28] have been developed and analyzed for three popular dispersive media models: the cold plasma model, the Debye model, and the Lorentz model. However, all of the above mentioned TDFEMs developed so far cannot be easily extended to

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the so-called Cole–Cole dispersive medium model [24, 27] as we mentioned in our previous work [13]. The Cole–Cole model contains a fractional derivative term, which is quite different from the standard dispersive media models such as plasma, Debye, and Lorentz models.

In this paper, by combining many techniques we developed for the standard dispersive media models [13] with those developed for the fractional diffusion equations [8, 18, 19], we propose two fully-discrete schemes for solving the Cole–Cole model: one implicit (the Crank–Nicolson type) and one explicit (the leap-frog type). Detailed stability analysis and error estimates are carried out. The proposed algorithms are implemented and numerical results supporting our analysis are provided. Though there exist many excellent work on finite element analysis and implementation for Maxwell's equations in free space (e.g., papers [2, 3, 5, 6, 10, 16, 29], books [7, 9, 22], and references cited therein), to the best of our knowledge, no finite element schemes have been investigated and analyzed for the Cole–Cole model.

We like to remark that there is some numerical work done in the FDTD framework for a Cole–Cole dispersive medium (e.g., [4, 25, 26, 27] and references cited therein). Generally speaking, the FDTD methods can be classified into two big categories: one way is to approximate the induced polarization via a time convolution of the electric field [4]; another way is to introduce some numerical scheme to approximate the auxiliary differential equation (ADE) obtained for the induced polarization and the electric field [25, 26, 27]. Compared to the convolution approach, the application of ADE based approach is quite easy and straightforward. Our scheme here is ADE based, and is different from [25, 26, 27], which employs several Debye terms to approximate the Cole–Cole model. Hence the resulting methods are quite time consuming, and precise stability and convergence estimate are yet unavailable.

The rest of this paper is organized as follows. In next section, we introduce the Cole–Cole model and carry out the stability analysis. Then in section 3, we develop two fully discrete mixed finite element schemes: one implicit (the Crank–Nicolson scheme) and one explicit (the leap-frog scheme). Numerical stabilities are proved for both schemes. Section 4 is devoted to the error analysis of both schemes. Optimal convergence rates in both time and space are proved under proper regularity assumptions. Detailed numerical results consistent with the theoretical analysis are presented in section 5. Finally, we conclude the paper in section 6.

In this paper, C (sometimes with subindex) denotes a generic constant, which is independent of the finite element mesh size h and time step size τ. Let $(H^\alpha(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_\alpha$ and seminorm $|\cdot|_\alpha$. In particular, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. We also use some common notation

$$\begin{aligned} H^\alpha(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in (H^\alpha(\Omega))^3;\ \ \nabla\times\boldsymbol{v} \in (H^\alpha(\Omega))^3\},\\ H_0(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in H(\mathrm{curl};\Omega);\ \ \mathbf{n}\times\boldsymbol{v} = 0 \ \text{ on } \ \partial\Omega\}, \end{aligned}$$

where $\alpha \geq 0$ is a real number, and Ω is a bounded and convex Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. When $\alpha = 0$, we simply denote $H^0(\mathrm{curl};\Omega) = H(\mathrm{curl};\Omega)$. Furthermore, $H(\mathrm{curl};\Omega)$ and $H^\alpha(\mathrm{curl};\Omega)$ are equipped with the norm

$$\begin{aligned} \|\boldsymbol{v}\|_{0,\mathrm{curl}} &= \left(\|\boldsymbol{v}\|_0^2 + \|\mathrm{curl}\ \boldsymbol{v}\|_0^2\right)^{1/2},\\ \|\boldsymbol{v}\|_{\alpha,\mathrm{curl}} &= \left(\|\boldsymbol{v}\|_\alpha^2 + \|\mathrm{curl}\ \boldsymbol{v}\|_\alpha^2\right)^{1/2}. \end{aligned}$$

Finally, we denote $C^m(0,T;X)$ the space of m times continuously differentiable functions from $[0,T]$ into the Hilbert space X.

2. The Cole–Cole dispersive medium model. In a Cole–Cole dispersive medium, the relative permittivity is expressed as

$$\epsilon_r(\omega) = \epsilon_\infty + (\epsilon_s - \epsilon_\infty)/(1 + (j\omega\tau_0)^\alpha), \quad 0 < \alpha < 1, \tag{2.1}$$

where $\epsilon_\infty, \epsilon_s, \tau_0$ are, respectively, the infinite-frequency permittivity, the static permittivity, and the relaxation time. Furthermore, $j = \sqrt{-1}$ denotes the imaginary unit, and ω denotes a general frequency. Note that the Cole–Cole model requires that $\epsilon_s > \epsilon_\infty$.

In the frequency domain, the induced polarization field $\hat{\boldsymbol{P}}$, and the electric field $\hat{\boldsymbol{E}}$ are related by the expression

$$\hat{\boldsymbol{P}} = \epsilon_0(\epsilon_r - \epsilon_\infty)\hat{\boldsymbol{E}} = \epsilon_0(\epsilon_s - \epsilon_\infty)/(1 + (j\omega\tau_0)^\alpha)\hat{\boldsymbol{E}}, \tag{2.2}$$

where ϵ_0 is the permittivity in the free space. Assuming a time-harmonic variation of $\exp(j\omega t)$ (i.e., $\boldsymbol{E}(\boldsymbol{x}, t) = Re(\exp(j\omega t)\hat{\boldsymbol{E}}(\boldsymbol{x}))$), we can transform (2.2) into time-domain as follows:

$$\tau_0^\alpha \frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha} + \boldsymbol{P}(t) = \epsilon_0(\epsilon_s - \epsilon_\infty)\boldsymbol{E}(t), \tag{2.3}$$

where $\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}$ represents the Letnikov fractional derivative given by

$$\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \boldsymbol{P}(s) ds = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t s^{-\alpha} \boldsymbol{P}(t-s) ds. \tag{2.4}$$

Here $\alpha \in (0, 1)$ is the differentiation order, and $\Gamma(\cdot)$ is the gamma function.

On the other hand, using (2.1), $\boldsymbol{P}$ can be defined as [24, eq. (2.3)]

$$\boldsymbol{P}(\boldsymbol{x}, t) = \int_0^t \xi_\alpha(t-s)\boldsymbol{E}(\boldsymbol{x}, s) ds, \quad t > 0, \tag{2.5}$$

where $\xi_\alpha(t) = \mathcal{L}^{-1}\{\frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{1 + (j\omega\tau_0)^\alpha}\}$ is the Cole–Cole time-domain susceptibility kernel. Here $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Equation (2.5) implies that the initial value $\boldsymbol{P}(\boldsymbol{x}, 0) = 0$.

Substituting the constitutive relations

$$\boldsymbol{D} = \epsilon_0\epsilon_\infty \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{B} = \mu_0 \boldsymbol{H}$$

into the general Maxwell's equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t},$$

we have

$$\epsilon_0\epsilon_\infty \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{P}}{\partial t}, \tag{2.6}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E}, \tag{2.7}$$

which, along with (2.3), form the governing equations for the Cole–Cole dispersive medium model. In the above, $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, μ_0 is the permeability of free space. To complete the problem, we assume a perfect conducting boundary condition

$$\mathbf{n} \times \boldsymbol{E} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2.8}$$

and the initial conditions

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{P}(\boldsymbol{x}, 0) = \boldsymbol{P}_0(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \Omega, \tag{2.9}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are some given functions.

We recall that a real valued kernel $\beta(t) \in L_1(0, T)$ is called positive-definite [21, eq. (1.2)] (also [17]) if for each $T > 0$, β satisfies

$$\int_0^T \phi(t) \int_0^t \beta(t-s)\phi(s) ds dt \geq 0 \quad \forall\, \phi \in C[0, T]. \tag{2.10}$$

Hence β is positive-definite if and only if

$$\text{Re}\ \hat{\beta}(j\omega) \equiv \int_0^\infty \beta(t) \cos(\omega t) dt \geq 0 \quad \forall\, \omega > 0, \tag{2.11}$$

where $\hat{\beta}$ denotes the Laplace transform of β. Note that

$$\int_0^\infty t^{-\alpha} \cos(\omega t) dt = \frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}} \cos \frac{(1-\alpha)\pi}{2},$$

which is positive for any $\alpha \in (0, 1)$ and $\omega > 0$. Hence the kernel $\beta(t) = t^{-\alpha}$ is positive-definite.

For the Cole–Cole model, we have the following stability.

LEMMA 2.1. *Assume that* $\boldsymbol{E}(t), \boldsymbol{H}(t), \boldsymbol{P}(t)$ *are the solutions of* (2.6)–(2.7) *and* (2.3) *satisfying the boundary condition* (2.8) *and the initial condition* (2.9), *then we have*

$$\begin{aligned} &\epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty \|\boldsymbol{E}(t)\|_0^2 + \mu_0 \|\boldsymbol{H}(t)\|_0^2) + \|\boldsymbol{P}(t)\|_0^2 \\ &\quad \leq \epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty \|\boldsymbol{E}(0)\|_0^2 + \mu_0 \|\boldsymbol{H}(0)\|_0^2) + \|\boldsymbol{P}(0)\|_0^2 \quad \forall\, t \in [0, T]. \end{aligned} \tag{2.12}$$

Proof. Multiplying (2.6) by $\boldsymbol{E}$, integrating over Ω, and using boundary condition (2.8), we have

$$\epsilon_0\epsilon_\infty \left(\frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{E}\right) - (\boldsymbol{H}, \nabla \times \boldsymbol{E}) + \left(\frac{\partial \boldsymbol{P}}{\partial t}, \boldsymbol{E}\right) = 0. \tag{2.13}$$

Similarly, multiplying (2.7) by $\boldsymbol{H}$ and integrating over Ω yields

$$\mu_0 \left(\frac{\partial \boldsymbol{H}}{\partial t}, \boldsymbol{H}\right) + (\nabla \times \boldsymbol{E}, \boldsymbol{H}) = 0. \tag{2.14}$$

Adding (2.13) and (2.14) together, we obtain

$$\frac{1}{2}\frac{d}{dt}(\epsilon_0\epsilon_\infty \|\boldsymbol{E}\|_0^2 + \mu_0 \|\boldsymbol{H}\|_0^2) + \left(\frac{\partial \boldsymbol{P}}{\partial t}, \boldsymbol{E}\right) = 0. \tag{2.15}$$

Note that

$$\begin{aligned}
&\left(\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}, \frac{\partial \boldsymbol{P}(t)}{\partial t}\right)\\
&\quad = \frac{1}{\Gamma(1-\alpha)}\left(t^{-\alpha}\boldsymbol{P}(0) + \int_0^t s^{-\alpha}\frac{\partial \boldsymbol{P}(t-s)}{\partial t}ds, \frac{\partial \boldsymbol{P}(t)}{\partial t}\right)\\
&\quad = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left(\int_\Omega \frac{\partial \boldsymbol{P}(s)}{\partial s}\cdot\frac{\partial \boldsymbol{P}(t)}{\partial t}d\boldsymbol{x}\right)ds,
\end{aligned} \tag{2.16}$$

where we used the fact that $\boldsymbol{P}(0) = 0$.

Multiplying (2.3) by $\frac{\partial \boldsymbol{P}}{\partial t}$, integrating over Ω, and using (2.16), we have

$$\frac{\tau_0^\alpha}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha}\frac{\partial \boldsymbol{P}(s)}{\partial s}ds, \frac{\partial \boldsymbol{P}(t)}{\partial t}\right) + \left(\boldsymbol{P}, \frac{\partial \boldsymbol{P}}{\partial t}\right) - \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\boldsymbol{E}, \frac{\partial \boldsymbol{P}}{\partial t}\right) = 0. \tag{2.17}$$

Multiplying (2.15) by $\epsilon_0(\epsilon_s - \epsilon_\infty)$ and adding to (2.17) leads to

$$\begin{aligned}
&\frac{1}{2}\epsilon_0(\epsilon_s - \epsilon_\infty)\frac{d}{dt}(\epsilon_0\epsilon_\infty\|\boldsymbol{E}\|_0^2 + \mu_0\|\boldsymbol{H}\|_0^2) + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{P}\|_0^2\\
&\quad = -\frac{\tau_0^\alpha}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha}\frac{\partial \boldsymbol{P}(s)}{\partial s}ds, \frac{\partial \boldsymbol{P}(t)}{\partial t}\right) \le 0,
\end{aligned} \tag{2.18}$$

where we used (2.16) in the last step.

Integrating (2.18) with respect to t from $t = 0$ to t concludes the proof. ∎

Furthermore, we can prove that Gauss's law holds true if the initial fields are divergence free. More specifically, we present the following lemma.

LEMMA 2.2. *Assume that the initial fields are divergence free, i.e.,*

$$\nabla\cdot\boldsymbol{E}_0 = 0, \quad \nabla\cdot\boldsymbol{H}_0 = 0, \quad \nabla\cdot\boldsymbol{P}_0 = 0. \tag{2.19}$$

Then for any $t > 0$, the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, and the polarization field $\boldsymbol{P}$ are divergence free.

Proof. Taking the divergence of (2.7) and using the assumption $\nabla\cdot\boldsymbol{H}_0 = 0$, we easily have $\nabla\cdot\boldsymbol{H}(t) = 0$.

Similarly, by taking the divergence of (2.6) and using the assumption (2.19), we have

$$\nabla\cdot(\epsilon_0\epsilon_\infty\boldsymbol{E} + \boldsymbol{P})(t) = 0. \tag{2.20}$$

By taking the divergence of (2.3) and using (2.20), we obtain

$$\tau_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}(\nabla\cdot\boldsymbol{P}(t)) + \frac{\epsilon_s}{\epsilon_\infty}\nabla\cdot\boldsymbol{P}(t) = 0,$$

multiplying by $\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{P}(t)$ and integrating over Ω leads to

$$\tau_0^\alpha\left(\frac{\partial^\alpha}{\partial t^\alpha}(\nabla\cdot\boldsymbol{P}(t)), \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{P}(t)\right) + \frac{\epsilon_s}{\epsilon_\infty}\left(\nabla\cdot\boldsymbol{P}(t), \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{P}(t)\right) = 0. \tag{2.21}$$

Finally, integrating (2.21) with respect to t from $t = 0$ to t and using the fact that the first term will be nonnegative due to the positive-definite kernel, we can conclude the proof. ∎

3. Two fully discrete schemes. Before deriving a finite element scheme, let us first consider a weak formulation for our model problem governed by equations (2.6)–(2.7) and (2.3). Multiplying them by some test functions, then integrating over Ω and using the boundary condition (2.8), we can obtain the weak formulation for (2.6)–(2.7) and (2.3): Find $\boldsymbol{E} \in C(0,T;H_0(\mathrm{curl};\Omega)) \cap C^1(0,T;(L_2(\Omega))^3), \boldsymbol{H} \in C^1(0,T;(L_2(\Omega))^3) \cap C(0,T;(L_2(\Omega))^3)$ and $\boldsymbol{P} \in C^1(0,T;(L_2(\Omega))^3)$ such that

$$\epsilon_0\epsilon_\infty\left(\frac{\partial \boldsymbol{E}}{\partial t},\phi\right)+\left(\frac{\partial \boldsymbol{P}}{\partial t},\phi\right)-(\boldsymbol{H},\nabla\times\phi)=0 \quad \forall\, \phi\in H_0(\mathrm{curl};\Omega), \tag{3.1}$$

$$\mu_0\left(\frac{\partial \boldsymbol{H}}{\partial t},\psi\right)+(\nabla\times\boldsymbol{E},\psi)=0 \quad \forall\, \psi\in(L_2(\Omega))^3, \tag{3.2}$$

$$\tau_0^\alpha\left(\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha},\tilde{\phi}\right)+(\boldsymbol{P},\tilde{\phi})=\epsilon_0(\epsilon_s-\epsilon_\infty)(\boldsymbol{E},\tilde{\phi}) \quad \forall\, \tilde{\phi}\in(L_2(\Omega))^3. \tag{3.3}$$

For simplicity, we assume that Ω is partitioned by a family of regular tetrahedral meshes T^h with maximum mesh size h. Considering the usual low regularity of Maxwell's equations, we consider only the lowest order Raviart–Thomas–Nédélec's mixed spaces [23]:

$$\mathbf{V}_h=\{\boldsymbol{v}_h\in H(\mathrm{div};\Omega):\ \boldsymbol{v}_h|_K=\boldsymbol{c}_K+d_K\boldsymbol{x} \quad \forall\, K\in T^h\}, \tag{3.4}$$

$$\mathbf{U}_h=\{\boldsymbol{u}_h\in H(\mathrm{curl};\Omega):\ \boldsymbol{u}_h|_K=\boldsymbol{a}_K+\boldsymbol{b}_K\times\boldsymbol{x} \quad \forall\, K\in T^h\}, \tag{3.5}$$

$$\mathbf{U}_h^0=\{\boldsymbol{u}_h\in\mathbf{U}_h:\ \boldsymbol{n}\times\boldsymbol{u}_h=0 \ \text{ on } \partial\Omega\}, \tag{3.6}$$

where $\boldsymbol{a}_K,\boldsymbol{b}_K,\boldsymbol{c}_K$ are constant vectors in R^3, and d_K is a real constant.

To construct a fully discrete scheme, we divide the time interval $(0,T)$ into M uniform subintervals using points $0=t_0<t_1<\cdots<t_M=T$, where $t_k=k\tau$. Moreover, we denote $\boldsymbol{u}^k=\boldsymbol{u}(\cdot,k\tau)$ and the following finite difference operators:

$$\delta_\tau\boldsymbol{u}^k=\left(\boldsymbol{u}^k-\boldsymbol{u}^{k-1}\right)/\tau,\quad \overline{\boldsymbol{u}}^k=\left(\boldsymbol{u}^k+\boldsymbol{u}^{k-1}\right)/2.$$

3.1. The Crank–Nicolson scheme. Before formulating our finite element scheme, we need to approximate the fractional derivative $\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}$. Recall the definition (2.4) and taking the time derivative into the integral, we have (cf. [18])

$$\begin{aligned}
\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}\Big|_{t=t_k} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{\partial \boldsymbol{P}(s)}{\partial s}\cdot(t_k-s)^{-\alpha}ds\\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{\partial \boldsymbol{P}(s)}{\partial s}\Big|_{t_{j-\frac{1}{2}}}\int_{t_{j-1}}^{t_j}(t_k-s)^{-\alpha}ds\\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{\boldsymbol{P}(t_j)-\boldsymbol{P}(t_{j-1})}{\tau}\int_{t_{j-1}}^{t_j}(t_k-s)^{-\alpha}ds\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{\boldsymbol{P}(t_j)-\boldsymbol{P}(t_{j-1})}{\tau}\cdot\frac{-1}{(1-\alpha)}(t_k-s)^{1-\alpha}\Big|_{s=t_{j-1}}^{t_j}\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{\boldsymbol{P}(t_j)-\boldsymbol{P}(t_{j-1})}{\tau}\cdot\frac{\tau^{1-\alpha}}{1-\alpha}((k-j+1)^{1-\alpha}-(k-j)^{1-\alpha})\\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}(\boldsymbol{P}(t_{k-l})-\boldsymbol{P}(t_{k-l-1}))b_l, \qquad (3.7)
\end{aligned}$$

where in the last step we used the identity $(1-\alpha)\Gamma(1-\alpha) = \Gamma(2-\alpha)$ and the notation

$$b_l = (l+1)^{1-\alpha} - l^{1-\alpha}.$$

It is easy to check that

$$1 = b_0 > b_1 > \cdots > b_l > 0, \quad b_l \to 0 \ \text{ as } l \to \infty.$$

When $k = 1$, (3.7) becomes $\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha}(t_1) \approx \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(\boldsymbol{P}(\tau) - \boldsymbol{P}(0))$.

Now we can formulate a Crank–Nicolson type finite element scheme for (3.1)–(3.3): Given initial approximations $\boldsymbol{E}_h^0, \boldsymbol{H}_h^0, \boldsymbol{P}_h^0$, for all $k \geq 1$ find $\boldsymbol{E}_h^k, \boldsymbol{P}_h^k \in \mathbf{U}_h^0, \boldsymbol{H}_h^k \in \mathbf{V}_h$ such that

$$\epsilon_0\epsilon_\infty(\delta_\tau \boldsymbol{E}_h^k, \phi) + (\delta_\tau \boldsymbol{P}_h^k, \phi) - (\overline{\boldsymbol{H}}_h^k, \nabla \times \phi) = 0 \quad \forall\, \phi \in \mathbf{U}_h^0, \tag{3.8}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^k, \psi) + (\nabla \times \overline{\boldsymbol{E}}_h^k, \psi) = 0 \quad \forall\, \psi \in \mathbf{V}_h, \tag{3.9}$$

$$\frac{\tau_0^\alpha}{2}(\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k + \tilde{\partial}_t^\alpha \boldsymbol{P}_h^{k-1}, \tilde{\phi}) + (\overline{\boldsymbol{P}}_h^k, \tilde{\phi}) = \epsilon_0(\epsilon_s - \epsilon_\infty)(\overline{\boldsymbol{E}}_h^k, \tilde{\phi}) \quad \forall\, \tilde{\phi} \in \mathbf{U}_h, \tag{3.10}$$

where $\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k$ $(k \geq 1)$ is the approximation of $\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha}(t_k)$ given by (3.7), while

$$\tilde{\partial}_t^\alpha \boldsymbol{P}_h^0 = \tau_0^{-\alpha}[-\boldsymbol{P}_h^0 + \epsilon_0(\epsilon_s - \epsilon_\infty)\boldsymbol{E}_h^0],$$

which is obtained from (2.3) by setting $t = 0$.

In practical implementation, the scheme (3.8)–(3.10) can be realized as follows: First, from (3.10), we represent $\boldsymbol{P}_h^k$ using $\boldsymbol{E}_h^k$ and past history of $\boldsymbol{P}$; then substitute $\boldsymbol{P}_h^k$ into (3.8), and solve the resulting equation along with (3.9) for both $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$. The solvability of the system for both $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$ can be proved in the same way as in our previous work [15].

THEOREM 3.1. *For the solutions* $\boldsymbol{E}_h^n, \boldsymbol{H}_h^n, \boldsymbol{P}_h^n$ $(n \geq 1)$ *of* (3.8)–(3.10), *we have the discrete stability:*

$$\begin{aligned}
&\epsilon_0(\epsilon_s - \epsilon_\infty)[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^n\|_0^2 + \mu_0\|\boldsymbol{H}_h^n\|_0^2] \\
&\qquad + \|\boldsymbol{P}_h^n\|_0^2 + \frac{1}{\Gamma(2-\alpha)} \cdot \left(\frac{\tau_0}{\tau}\right)^\alpha \sum_{k=1}^n \left\|\boldsymbol{P}_h^k - \boldsymbol{P}_h^{k-1}\right\|_0^2 \\
&\qquad\qquad \leq C[\epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^0\|_0^2 + \mu_0\|\boldsymbol{H}_h^0\|_0^2) + \|\boldsymbol{P}_h^0\|_0^2].
\end{aligned} \tag{3.11}$$

Remark 3.1. By dropping the summation term $\sum_{k=1}^n \|\boldsymbol{P}_h^k - \boldsymbol{P}_h^{k-1}\|_0^2$, the stability of Theorem 3.1 becomes

$$\begin{aligned}
&\epsilon_0(\epsilon_s - \epsilon_\infty)\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^n\|_0^2 + \mu_0\|\boldsymbol{H}_h^n\|_0^2\right] + \|\boldsymbol{P}_h^n\|_0^2 \\
&\qquad \leq C\left[\epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^0\|_0^2 + \mu_0\|\boldsymbol{H}_h^0\|_0^2) + \|\boldsymbol{P}_h^0\|_0^2\right],
\end{aligned}$$

which has the exact form (if $C = 1$) as the stability obtained in Lemma 2.1 for the continuous case.

Proof. Choosing $\phi = \frac{\tau}{2}(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1})$ in (3.8), $\psi = \frac{\tau}{2}(\boldsymbol{H}_h^k + \boldsymbol{H}_h^{k-1})$ in (3.9), then adding the results together, we obtain

$$\begin{aligned}
&\frac{1}{2}\epsilon_0\epsilon_\infty\left(\|\boldsymbol{E}_h^k\|_0^2 - \|\boldsymbol{E}_h^{k-1}\|_0^2\right) + \frac{1}{2}\mu_0\left(\|\boldsymbol{H}_h^k\|_0^2 - \|\boldsymbol{H}_h^{k-1}\|_0^2\right) \\
&\qquad\qquad + \frac{1}{2}\left(\boldsymbol{P}_h^k - \boldsymbol{P}_h^{k-1}, \boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1}\right) = 0.
\end{aligned} \tag{3.12}$$

From (3.7), we have

$$\begin{aligned}
&\frac{\tau_0^\alpha}{2}\left(\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k+\tilde{\partial}_t^\alpha \boldsymbol{P}_h^{k-1}\right)\\
&\quad=\frac{1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left[\sum_{l=0}^{k-1}\left(\boldsymbol{P}_h^{k-l}-\boldsymbol{P}_h^{k-1-l}\right)b_l+\sum_{l=0}^{k-2}\left(\boldsymbol{P}_h^{k-1-l}-\boldsymbol{P}_h^{k-2-l}\right)b_l\right]\\
&\quad=\frac{1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left[\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}+\sum_{l=0}^{k-2}\left(\boldsymbol{P}_h^{k-1-l}-\boldsymbol{P}_h^{k-2-l}\right)(b_l+b_{l+1})\right], \qquad (3.13)
\end{aligned}$$

substituting into (3.10) with $\tilde{\phi}=\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}$, we obtain

$$\begin{aligned}
&\frac{1}{2\Gamma(2-\alpha)} \qquad (3.14)\\
&\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left[\|\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\|_0^2+\sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-1-l}-\boldsymbol{P}_h^{k-2-l}\right)(b_l+b_{l+1}),\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\right)\right]\\
&\qquad+\frac{1}{2}\left(\|\boldsymbol{P}_h^k\|_0^2-\|\boldsymbol{P}_h^{k-1}\|_0^2\right)=\frac{1}{2}\epsilon_0\left(\epsilon_s-\epsilon_\infty\right)\left(\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1},\boldsymbol{E}_h^k+\boldsymbol{E}_h^{k-1}\right).
\end{aligned}$$

Multiplying (3.12) by $\epsilon_0(\epsilon_s-\epsilon_\infty)$, then substituting (3.14) into the resultant, we have

$$\begin{aligned}
&\frac{1}{2}\epsilon_0(\epsilon_s-\epsilon_\infty)\left[\epsilon_0\epsilon_\infty\left(\|\boldsymbol{E}_h^k\|_0^2-\|\boldsymbol{E}_h^{k-1}\|_0^2\right)+\mu_0\left(\|\boldsymbol{H}_h^k\|_0^2-\|\boldsymbol{H}_h^{k-1}\|_0^2\right)\right]\\
&\quad+\frac{1}{2}\left(\|\boldsymbol{P}_h^k\|_0^2-\|\boldsymbol{P}_h^{k-1}\|_0^2\right)+\frac{1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\|\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\|_0^2\\
&\quad=\frac{-1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-1-l}-\boldsymbol{P}_h^{k-2-l}\right)(b_l+b_{l+1}),\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\right). \qquad (3.15)
\end{aligned}$$

When $k=1$, due to the special definition of $\tilde{\partial}_t^\alpha \boldsymbol{P}_h^0$, the last term of (3.15) becomes $\frac{1}{2}(\boldsymbol{P}_h^0-\epsilon_0(\epsilon_s-\epsilon_\infty)\boldsymbol{E}_h^0,\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1})$, in which case (3.15) easily leads to

$$\begin{aligned}
&\epsilon_0(\epsilon_s-\epsilon_\infty)\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^1\|_0^2+\mu_0\|\boldsymbol{H}_h^1\|_0^2\right]+\|\boldsymbol{P}_h^1\|_0^2+\frac{1}{\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\|\boldsymbol{P}_h^1-\boldsymbol{P}_h^0\|_0^2\\
&\quad\le C[\epsilon_0(\epsilon_s-\epsilon_\infty)(\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^0\|_0^2+\mu_0\|\boldsymbol{H}_h^0\|_0^2)+\|\boldsymbol{P}_h^0\|_0^2]. \qquad (3.16)
\end{aligned}$$

When $k=2$, (3.15) becomes

$$\begin{aligned}
&\frac{1}{2}\epsilon_0(\epsilon_s-\epsilon_\infty)[\epsilon_0\epsilon_\infty(\|\boldsymbol{E}_h^2\|_0^2-\|\boldsymbol{E}_h^1\|_0^2)+\mu_0(\|\boldsymbol{H}_h^2\|_0^2-\|\boldsymbol{H}_h^1\|_0^2)]\\
&\quad+\frac{1}{2}\left(\|\boldsymbol{P}_h^2\|_0^2-\|\boldsymbol{P}_h^1\|_0^2\right)+\frac{1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\|\boldsymbol{P}_h^2-\boldsymbol{P}_h^1\|_0^2\\
&\quad=\frac{-1}{2\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left(\left(\boldsymbol{P}_h^l-\boldsymbol{P}_h^0\right)(b_0+b_1),\boldsymbol{P}_h^2-\boldsymbol{P}_h^1\right),
\end{aligned}$$

which, coupling with (3.16), completes the proof of (3.11) when $n=2$.

When $k > 2$, the last term in (3.15) should be estimated as follows:

$$\sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-1-l}-\boldsymbol{P}_h^{k-2-l}\right)(b_l+b_{l+1}),\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\right)$$
$$=\sum_{j=1}^{k-1}\left(\left(\boldsymbol{P}_h^{j}-\boldsymbol{P}_h^{j-1}\right)(b_{k-1-j}+b_{k-j}),\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-1}\right)$$
$$\le\sum_{j=1}^{k-1}\left[\delta_1\|\boldsymbol{P}_h^k-\boldsymbol{P}_h^{k-l}\|_0^2+\frac{1}{4\delta_1}(b_{k-1-j}+b_{k-j})^2\|\boldsymbol{P}_h^j-\boldsymbol{P}_h^{j-1}\|_0^2\right],$$

which can be bounded using the known estimates of $\|\boldsymbol{P}_h^j-\boldsymbol{P}_h^{j-1}\|_0^2, 1\le j\le k-1$. By induction method, we complete the proof. ∎

Next we investigate the error caused by the approximation of partial fractional derivative (3.7).

LEMMA 3.2. *Let $\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k$ be the approximation of $\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha}(t_k)$ given by* (3.7), *then*

$$\left|\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha}(t_k)-\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k\right|\le C\tau^{2-\alpha},\quad k\ge 1. \tag{3.17}$$

Proof. Let $t_{j-\frac{1}{2}}=\frac{1}{2}(t_{j-1}+t_j)$. By Taylor expansion, we can have

$$\frac{\partial \boldsymbol{P}(s)}{\partial s}-\frac{\boldsymbol{P}(t_j)-\boldsymbol{P}(t_{j-1})}{\tau}=(s-t_{j-\frac{1}{2}})\boldsymbol{P}_{ss}(t_{j-\frac{1}{2}})+O(\tau^2),$$

using which we obtain

$$\frac{\partial^\alpha \boldsymbol{P}}{\partial t^\alpha}(t_k)-\tilde{\partial}_t^\alpha \boldsymbol{P}_h^k$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left[\frac{\partial \boldsymbol{P}(s)}{\partial s}-\frac{\boldsymbol{P}(t_j)-\boldsymbol{P}(t_{j-1})}{\tau}\right](t_k-s)^{-\alpha}ds$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(s-t_{j-\frac{1}{2}})(t_k-s)^{-\alpha}ds\cdot\boldsymbol{P}_{ss}(t_{j-\frac{1}{2}})+O(\tau^2)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\left[-\left(s-t_{j-\frac{1}{2}}\right)\frac{(t_k-s)^{1-\alpha}}{1-\alpha}\Big|_{s=t_{j-1}}^{t_j}+\int_{t_{j-1}}^{t_j}\frac{(t_k-s)^{1-\alpha}}{1-\alpha}ds\right]$$
$$\boldsymbol{P}_{ss}(t_{j-\frac{1}{2}})+O(\tau^2)$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\Bigg\{-\frac{\tau^{2-\alpha}}{2(1-\alpha)}[(k-j)^{1-\alpha}+(k+1-j)^{1-\alpha}]$$
$$+\frac{\tau^{2-\alpha}}{(2-\alpha)(1-\alpha)}[(k+1-j)^{2-\alpha}-(k-j)^{2-\alpha}]\Bigg\}\boldsymbol{P}_{ss}(t_{j-\frac{1}{2}})+O(\tau^2)$$
$$=-\frac{\tau^{2-\alpha}}{2\Gamma(2-\alpha)}\left\{k^{1-\alpha}+2[(k-1)^{1-\alpha}+(k-2)^{1-\alpha}+\cdots+1^{1-\alpha}]-\frac{2}{2-\alpha}k^{2-\alpha}\right\}$$
$$\boldsymbol{P}_{ss}(t_{j-\frac{1}{2}})+O(\tau^2),$$

which, coupling with the result [18, Lemma 3.1]

$$\left| k^{1-\alpha} + 2\left[(k-1)^{1-\alpha} + (k-2)^{1-\alpha} + \cdots + 1^{1-\alpha}\right] - \frac{2}{2-\alpha}k^{2-\alpha} \right| \le C,$$

concludes the proof. ∎

3.2. The leap-frog scheme. Similar to (3.7), we can approximate the fractional derivative $\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}$ by

$$\left.\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}\right|_{t=t_{k+\frac{1}{2}}} \approx \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot \sum_{l=0}^{k-1} \left(\boldsymbol{P}^{k+\frac{1}{2}-l} - \boldsymbol{P}^{k-\frac{1}{2}-l}\right) b_l \quad \forall\, k \ge 1. \tag{3.18}$$

When $k = 0$, from (2.3), we can have the approximation

$$\left.\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}\right|_{t=t_{\frac{1}{2}}} \approx \tau_0^{-\alpha}\left[-\boldsymbol{P}_h^{\frac{1}{2}} + \epsilon_0(\epsilon_s - \epsilon_\infty)\boldsymbol{E}_h^{\frac{1}{2}}\right]. \tag{3.19}$$

Now we can formulate a leap-frog scheme for (3.1)–(3.3): Given initial approximations $\boldsymbol{E}_h^{\frac{1}{2}}, \boldsymbol{H}_h^0, \boldsymbol{P}_h^{\frac{1}{2}}$, for all $k \ge 1$ find $\boldsymbol{E}_h^{k+\frac{1}{2}}, \boldsymbol{P}_h^{k+\frac{1}{2}} \in \mathbf{U}_h^0, \boldsymbol{H}_h^k \in \mathbf{V}_h$ such that

$$\epsilon_0\epsilon_\infty\left(\frac{\boldsymbol{E}_h^{k+\frac{1}{2}} - \boldsymbol{E}_h^{k-\frac{1}{2}}}{\tau}, \phi\right) + \left(\frac{\boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}}{\tau}, \phi\right) - \left(\boldsymbol{H}_h^k, \nabla\times\phi\right) = 0 \ \ \forall\, \phi \in \mathbf{U}_h^0, \tag{3.20}$$

$$\mu_0\left(\frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\tau}, \psi\right) + \left(\nabla\times\boldsymbol{E}_h^{k-\frac{1}{2}}, \psi\right) = 0 \quad \forall\, \psi \in \mathbf{V}_h, \tag{3.21}$$

$$\tau_0^\alpha\left(\tilde{\partial}_t^\alpha \boldsymbol{P}_h^{k+\frac{1}{2}}, \tilde{\phi}\right) + \left(\boldsymbol{P}_h^{k+\frac{1}{2}}, \tilde{\phi}\right) = \epsilon_0\left(\epsilon_s - \epsilon_\infty\right)\left(\boldsymbol{E}_h^{k+\frac{1}{2}}, \tilde{\phi}\right) \quad \forall\, \tilde{\phi} \in \mathbf{U}_h, \tag{3.22}$$

where $\tilde{\partial}_t^\alpha \boldsymbol{P}_h^{k+\frac{1}{2}}$ is the approximation of $\frac{\partial^\alpha \boldsymbol{P}(t)}{\partial t^\alpha}$ at $t = (k+\frac{1}{2})\tau$ given by (3.18).

In practical implementation, the leap-frog scheme (3.20)–(3.22) can be realized as follows: At each time step, we first solve (3.21) for $\boldsymbol{H}_h^k$; then solve (3.20) for $\boldsymbol{E}_h^{k+\frac{1}{2}}$ after substituting $\boldsymbol{P}_h^{k+\frac{1}{2}}$ from (3.22) into (3.20); finally, update $\boldsymbol{P}_h^{k+\frac{1}{2}}$ through (3.22).

THEOREM 3.3. *Let $c_v = 1/\sqrt{\mu_0\epsilon_0}$ be the speed of light, and c_{inv} is the constant in the inverse estimate*

$$\|\nabla\times\psi_h\|_0 \le c_{inv}h^{-1}\|\psi_h\|_0, \quad \psi_h \in \boldsymbol{V}_h. \tag{3.23}$$

Assuming that the time step satisfies the condition

$$\tau = \min\left(\frac{\sqrt{\epsilon_\infty}h}{c_v c_{inv}}, 1\right), \tag{3.24}$$

then for any $n \ge 1$, we have the discrete stability for the solutions $(\boldsymbol{E}_h^{n+\frac{1}{2}}, \boldsymbol{H}_h^n, \boldsymbol{P}_h^{n+\frac{1}{2}})$

of (3.20)–(3.22):

$$\epsilon_0(\epsilon_s-\epsilon_\infty)\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+\frac{1}{2}}\|_0^2+\mu_0\|\boldsymbol{H}_h^n\|_0^2\right] + \|\boldsymbol{P}_h^{n+\frac{1}{2}}\|_0^2+\frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha\sum_{k=1}^{n}\|\boldsymbol{P}_h^{k+\frac{1}{2}}-\boldsymbol{P}_h^{k-\frac{1}{2}}\|_0^2 \le C\left[\epsilon_0(\epsilon_s-\epsilon_\infty)\left(\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{\frac{1}{2}}\|_0^2+\mu_0\|\boldsymbol{H}_h^0\|_0^2\right)+\|\boldsymbol{P}_h^{\frac{1}{2}}\|_0^2+\|\nabla\times\boldsymbol{E}_h^{\frac{1}{2}}\|_0^2\right].$$

Proof. Choosing $\phi=\tau(\boldsymbol{E}_h^{k+\frac{1}{2}}+\boldsymbol{E}_h^{k-\frac{1}{2}})$ in (3.20), $\psi=\tau(\boldsymbol{H}_h^k+\boldsymbol{H}_h^{k-1})$ in (3.21), then adding the results together and using the identity

$$\left(\nabla\times\boldsymbol{E}_h^{k-\frac{1}{2}},\boldsymbol{H}_h^k+\boldsymbol{H}_h^{k-1}\right)-\left(\boldsymbol{H}_h^k,\nabla\times\left(\boldsymbol{E}_h^{k+\frac{1}{2}}+\boldsymbol{E}_h^{k-\frac{1}{2}}\right)\right) = \left(\nabla\times\boldsymbol{E}_h^{k-\frac{1}{2}},\boldsymbol{H}_h^{k-1}\right)-\left(\nabla\times\boldsymbol{E}_h^{k+\frac{1}{2}},\boldsymbol{H}_h^k\right),$$

we obtain

$$\epsilon_0\epsilon_\infty\left(\|\boldsymbol{E}_h^{k+\frac{1}{2}}\|_0^2-\|\boldsymbol{E}_h^{k-\frac{1}{2}}\|_0^2\right)+\mu_0\left(\|\boldsymbol{H}_h^k\|_0^2-\|\boldsymbol{H}_h^{k-1}\|_0^2\right) + \tau\left[\left(\nabla\times\boldsymbol{E}_h^{k-\frac{1}{2}},\boldsymbol{H}_h^{k-1}\right)-\left(\nabla\times\boldsymbol{E}_h^{k+\frac{1}{2}},\boldsymbol{H}_h^k\right)\right] + \left(\boldsymbol{P}_h^{k+\frac{1}{2}}-\boldsymbol{P}_h^{k-\frac{1}{2}},\boldsymbol{E}_h^{k+\frac{1}{2}}+\boldsymbol{E}_h^{k-\frac{1}{2}}\right)=0. \tag{3.25}$$

From (3.18), we have

$$\begin{aligned}\tau_0^\alpha&\left(\tilde\partial_t^\alpha\boldsymbol{P}_h^{k+\frac{1}{2}}+\tilde\partial_t^\alpha\boldsymbol{P}_h^{k-\frac{1}{2}}\right)\\ &=\frac{1}{\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left[\sum_{l=0}^{k-1}\left(\boldsymbol{P}_h^{k+\frac{1}{2}-l}-\boldsymbol{P}_h^{k-\frac{1}{2}-l}\right)b_l + \sum_{l=0}^{k-2}\left(\boldsymbol{P}_h^{k-\frac{1}{2}-l}-\boldsymbol{P}_h^{k-\frac{3}{2}-l}\right)b_l\right]\\ &=\frac{1}{\Gamma(2-\alpha)}\cdot\left(\frac{\tau_0}{\tau}\right)^\alpha\left[\boldsymbol{P}_h^{k+\frac{1}{2}}-\boldsymbol{P}_h^{k-\frac{1}{2}} + \sum_{l=0}^{k-2}\left(\boldsymbol{P}_h^{k-\frac{1}{2}-l}-\boldsymbol{P}_h^{k-\frac{3}{2}-l}\right)(b_l+b_{l+1})\right].\end{aligned} \tag{3.26}$$

Furthermore, from (3.22), we have

$$\tau_0^\alpha\left(\tilde\partial_t^\alpha\boldsymbol{P}_h^{k+\frac{1}{2}}+\tilde\partial_t^\alpha\boldsymbol{P}_h^{k-\frac{1}{2}},\tilde\phi\right)+\left(\boldsymbol{P}_h^{k+\frac{1}{2}}+\boldsymbol{P}_h^{k-\frac{1}{2}},\tilde\phi\right)=\epsilon_0\left(\epsilon_s-\epsilon_\infty\right)\left(\boldsymbol{E}_h^{k+\frac{1}{2}}+\boldsymbol{E}_h^{k-\frac{1}{2}},\tilde\phi\right),$$

in which we choose $\tilde{\phi} = \boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \Bigg[\left\|\boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right\|_0^2 \\
&\qquad + \sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-\frac{1}{2}-l} - \boldsymbol{P}_h^{k-\frac{3}{2}-l}\right)(b_l + b_{l+1}), \boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right)\Bigg] \qquad (3.27)\\
&\quad + \left(\left\|\boldsymbol{P}_h^{k+\frac{1}{2}}\right\|_0^2 - \left\|\boldsymbol{P}_h^{k-\frac{1}{2}}\right\|_0^2\right) = \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\boldsymbol{E}_h^{k+\frac{1}{2}} + \boldsymbol{E}_h^{k-\frac{1}{2}}, \boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right).
\end{aligned}$$

Multiplying (3.25) by $\epsilon_0(\epsilon_s - \epsilon_\infty)$, then substituting (3.27) into the resultant, we have

$$\begin{aligned}
&\epsilon_0(\epsilon_s - \epsilon_\infty)\left[\epsilon_0\epsilon_\infty\left(\left\|\boldsymbol{E}_h^{k+\frac{1}{2}}\right\|_0^2 - \left\|\boldsymbol{E}_h^{k-\frac{1}{2}}\right\|_0^2\right) + \mu_0\left(\left\|\boldsymbol{H}_h^{k}\right\|_0^2 - \left\|\boldsymbol{H}_h^{k-1}\right\|_0^2\right)\right] \\
&\quad + \left(\left\|\boldsymbol{P}_h^{k+\frac{1}{2}}\right\|_0^2 - \left\|\boldsymbol{P}_h^{k-\frac{1}{2}}\right\|_0^2\right) + \frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \left\|\boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right\|_0^2 \\
&= \epsilon_0(\epsilon_s - \epsilon_\infty)\tau\left[\left(\nabla\times\boldsymbol{E}_h^{k+\frac{1}{2}}, \boldsymbol{H}_h^k\right) - \left(\nabla\times\boldsymbol{E}_h^{k-\frac{1}{2}}, \boldsymbol{H}_h^{k-1}\right)\right] \\
&\quad - \frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-\frac{1}{2}-l} - \boldsymbol{P}_h^{k-\frac{3}{2}-l}\right)(b_l + b_{l+1}), \boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right). \qquad (3.28)
\end{aligned}$$

Summing up (3.28) from $k = 1$ to n, we have

$$\begin{aligned}
&\epsilon_0(\epsilon_s - \epsilon_\infty)\left[\epsilon_0\epsilon_\infty\left(\left\|\boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0^2 - \left\|\boldsymbol{E}_h^{\frac{1}{2}}\right\|_0^2\right) + \mu_0\left(\left\|\boldsymbol{H}_h^{n}\right\|_0^2 - \left\|\boldsymbol{H}_h^{0}\right\|_0^2\right)\right] \\
&\quad + \left(\left\|\boldsymbol{P}_h^{n+\frac{1}{2}}\right\|_0^2 - \left\|\boldsymbol{P}_h^{\frac{1}{2}}\right\|_0^2\right) + \frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \sum_{k=1}^{n}\left\|\boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right\|_0^2 \\
&= \epsilon_0(\epsilon_s - \epsilon_\infty)\tau\left[\left(\nabla\times\boldsymbol{E}_h^{n+\frac{1}{2}}, \boldsymbol{H}_h^n\right) - \left(\nabla\times\boldsymbol{E}_h^{\frac{1}{2}}, \boldsymbol{H}_h^{0}\right)\right] \\
&\quad - \frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \sum_{k=1}^{n}\sum_{l=0}^{k-2}\left(\left(\boldsymbol{P}_h^{k-\frac{1}{2}-l} - \boldsymbol{P}_h^{k-\frac{3}{2}-l}\right)(b_l + b_{l+1}), \boldsymbol{P}_h^{k+\frac{1}{2}} - \boldsymbol{P}_h^{k-\frac{1}{2}}\right). \qquad (3.29)
\end{aligned}$$

By Cauchy–Schwarz inequality and the inverse estimate (3.23), we have

$$\begin{aligned}
\tau\left(\nabla\times\boldsymbol{E}_h^{n+\frac{1}{2}}, \boldsymbol{H}_h^n\right) &\le \tau\cdot c_{inv}h^{-1}\left\|\boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0 \left\|\boldsymbol{H}_h^n\right\|_0 \\
&= \tau\cdot c_{inv}h^{-1}\cdot c_v\sqrt{\epsilon_0}\left\|\boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0 \cdot \sqrt{\mu_0}\left\|\boldsymbol{H}_h^n\right\|_0 \\
&\le \frac{1}{2}\epsilon_0\epsilon_\infty\left\|\boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0^2 + \frac{1}{2}\left(\frac{c_v c_{inv}\tau}{h}\right)^2 \frac{\mu_0}{\epsilon_\infty}\left\|\boldsymbol{H}_h^n\right\|_0^2. \qquad (3.30)
\end{aligned}$$

Substituting (3.30) into (3.29), and using a similar technique as in the proof of Theorem 3.1, we can conclude the proof. ∎

4. The error estimates. To prove the optimal error estimate for the leap-frog scheme, we need the following lemma.

LEMMA 4.1 (see [13, Lemma 2.3]). *Denote $u^k = u(\cdot, k\tau)$. For any $u \in H^2(0,T;L^2(\Omega))$, we have*

$$\text{(i)} \quad \left\| u^k - \frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} u(s)ds \right\|_0^2 \le \frac{\tau^3}{4}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \|u_{tt}(s)\|_0^2 ds,$$

$$\text{(ii)} \quad \left\| u^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} u(s)ds \right\|_0^2 \le \frac{\tau^3}{4}\int_{t_{k-1}}^{t_k} \|u_{tt}(s)\|_0^2 ds,$$

$$\text{(iii)} \quad \left\| \frac{1}{2}\left(u^k + u^{k+1}\right) - \frac{1}{\tau}\int_{t_k}^{t_{k+1}} u(s)ds \right\|_0^2 \le \frac{\tau^3}{4}\int_{t_k}^{t_{k+1}} \|u_{tt}(s)\|_0^2 ds,$$

$$\text{(iv)} \quad \left\| \frac{1}{2}\left(u^{k-\frac{1}{2}} + u^{k+\frac{1}{2}}\right) - \frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} u(s)ds \right\|_0^2 \le \frac{\tau^3}{4}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \|u_{tt}(s)\|_0^2 ds.$$

For error estimates, we also need Nédélec interpolation operator Π_h and the standard L^2 projection operator Q_h. For any $\boldsymbol{v} \in H^\gamma(curl;\Omega)$, $\frac{1}{2} < \gamma \le 1$, the Nédélec interpolant [23] $\Pi_h \boldsymbol{v} \in \mathbf{U}_h$ on the lowest-order Raviart–Thomas–Nédélec (RTN) space $\mathbf{U}_h$ can be defined on each tetrahedron $K \in T^h$ by the degrees of freedom $\int_e \boldsymbol{v}\cdot\hat{\tau}$ on each edge e of K, where $\hat{\tau}$ is the unit vector along the edge e. Furthermore, we have the following interpolation error estimate (e.g., [22, eq. (5.42)]):

$$\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_0 + \|\nabla\times(\boldsymbol{v}-\Pi_h\boldsymbol{v})\|_0 \le Ch^\gamma\|\boldsymbol{v}\|_{\gamma,\mathrm{curl}} \ \ \forall\, \boldsymbol{v} \in H^\gamma(\mathrm{curl};\Omega)\ \gamma \in \left(\frac{1}{2}, 1\right]. \tag{4.1}$$

On the other hand, for any $\boldsymbol{u} \in H^\gamma(\Omega)$, the $(L^2(\Omega))^3$ projection $Q_h\boldsymbol{u} \in \mathbf{V}_h$ satisfies

$$(Q_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{\phi}_h) = 0 \quad \forall\, \phi_h \in \mathbf{V}_h.$$

Furthermore, we have the projection error estimate

$$\|\boldsymbol{u} - Q_h\boldsymbol{u}\|_0 \le Ch^\gamma\|\boldsymbol{u}\|_\gamma \quad \forall \boldsymbol{u} \in H^\gamma(\Omega),\ \gamma \in \left(\frac{1}{2}, 1\right]. \tag{4.2}$$

THEOREM 4.2. *Let $(\boldsymbol{E}_h^{n+\frac{1}{2}}, \boldsymbol{H}_h^n, \boldsymbol{P}_h^{n+\frac{1}{2}})$ be the finite element solutions of* (3.20)–(3.22), *and let $(\boldsymbol{E}^{n+\frac{1}{2}}, \boldsymbol{H}^n, \boldsymbol{P}^{n+\frac{1}{2}})$ be the analytic solutions of* (2.6)–(2.7) *and* (2.3). *Then under proper regularity assumption of the solutions, there exists a constant $C > 0$, independent of both the time step size τ and the mesh size h, such that*

$$\begin{aligned}
&\max_{n\ge1}\left(\left\|\boldsymbol{E}^{n+\frac{1}{2}} - \boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0 + \|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 + \left\|\boldsymbol{P}^{n+\frac{1}{2}} - \boldsymbol{P}_h^{n+\frac{1}{2}}\right\|_0\right)\\
&\le C(h^\gamma + \tau^{2-\alpha}) + C\left(\left\|\boldsymbol{E}^{\frac{1}{2}} - \boldsymbol{E}_h^{\frac{1}{2}}\right\|_{0,\mathrm{curl}} + \|\boldsymbol{H}^0 - \boldsymbol{H}_h^0\|_0 + \left\|\boldsymbol{P}^{\frac{1}{2}} - \boldsymbol{P}_h^{\frac{1}{2}}\right\|_{0,\mathrm{curl}}\right),
\end{aligned}$$

where γ is the regularity constant from (4.1).

Remark 4.1. When the initial errors $\|\boldsymbol{E}^{\frac{1}{2}} - \boldsymbol{E}_h^{\frac{1}{2}}\|_{0,\mathrm{curl}} = \|\boldsymbol{H}^0 - \boldsymbol{H}_h^0\|_0 = \|\boldsymbol{P}^{\frac{1}{2}} - \boldsymbol{P}_h^{\frac{1}{2}}\|_{0,\mathrm{curl}} = O(h^\gamma + \tau^{2-\alpha})$, we have the optimal error estimate

$$\max_{n\ge1}\left(\|\boldsymbol{E}^{n+\frac{1}{2}} - \boldsymbol{E}_h^{n+\frac{1}{2}}\|_0 + \|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 + \|\boldsymbol{P}^{n+\frac{1}{2}} - \boldsymbol{P}_h^{n+\frac{1}{2}}\|_0\right) \le C\left(h^\gamma + \tau^{2-\alpha}\right).$$

Proof. Integrating (3.1), (3.2), and (3.3) with respect to t over $I_k \equiv [t_{k-\frac{1}{2}}, t_{k+\frac{1}{2}}]$, $I_{k-\frac{1}{2}}$ and $I_{k+\frac{1}{2}}$, respectively, we have

$$\epsilon_0\epsilon_\infty \left(\frac{\boldsymbol{E}^{k+\frac{1}{2}} - \boldsymbol{E}^{k-\frac{1}{2}}}{\tau}, \phi\right) + \left(\frac{\boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k-\frac{1}{2}}}{\tau}, \phi\right) - \left(\frac{1}{\tau}\int_{I_k} \boldsymbol{H}(s)ds, \nabla \times \phi\right)$$
$$= 0 \ \forall \phi \in \mathbf{U}_h^0, \tag{4.3}$$
$$\mu_0 \left(\frac{\boldsymbol{H}^k - \boldsymbol{H}^{k-1}}{\tau}, \psi\right) + \left(\frac{1}{\tau}\int_{I_{k-\frac{1}{2}}} \nabla \times \boldsymbol{E}(s)ds, \psi\right) = 0 \ \forall\ \psi \in \mathbf{V}_h, \tag{4.4}$$
$$\frac{\tau_0^\alpha}{\tau}\left(\int_{t_k}^{t_{k+1}} \frac{\partial^\alpha \boldsymbol{P}(s)}{\partial s^\alpha} ds, \tilde{\phi}\right) + \left(\frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{P}(s)ds, \tilde{\phi}\right)$$
$$= \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{E}(s)ds, \tilde{\phi}\right) \quad \forall\ \tilde{\phi} \in \mathbf{U}_h. \tag{4.5}$$

Let us denote $\xi_h^k = \Pi_h \boldsymbol{E}^k - \boldsymbol{E}_h^k, \eta_h^k = Q_h \boldsymbol{H}^k - \boldsymbol{H}_h^k, \chi_h^k = \Pi_h \boldsymbol{P}^k - \boldsymbol{P}_h^k$. Subtracting (3.20)–(3.22) from (4.3)–(4.5), we obtain

$$\epsilon_0\epsilon_\infty \left(\frac{\xi_h^{k+\frac{1}{2}} - \xi_h^{k-\frac{1}{2}}}{\tau}, \phi\right) + \left(\frac{\chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}}{\tau}, \phi\right) - \left(\eta_h^k, \nabla \times \phi\right)$$
$$= \epsilon_0\epsilon_\infty \left(\frac{(\Pi_h \boldsymbol{E}^{k+\frac{1}{2}} - \boldsymbol{E}^{k+\frac{1}{2}}) - (\Pi_h \boldsymbol{E}^{k-\frac{1}{2}} - \boldsymbol{E}^{k-\frac{1}{2}})}{\tau}, \phi\right)$$
$$+ \left(\frac{(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k+\frac{1}{2}}) - (\Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \boldsymbol{P}^{k-\frac{1}{2}})}{\tau}, \phi\right)$$
$$- \left(Q_h \boldsymbol{H}^k - \frac{1}{\tau}\int_{I_k} \boldsymbol{H}(s)ds, \nabla \times \phi\right), \mu_0 \left(\frac{\eta_h^k - \eta_h^{k-1}}{\tau}, \psi\right) \tag{4.6}$$
$$+ \left(\nabla \times \xi_h^{k-\frac{1}{2}}, \psi\right) = \mu_0 \left(\frac{(Q_h \boldsymbol{H}^k - \boldsymbol{H}^k) - (Q_h \boldsymbol{H}^{k-1} - \boldsymbol{H}^{k-1})}{\tau}, \psi\right)$$
$$+ \left(\nabla \times \Pi_h \boldsymbol{E}^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k-\frac{1}{2}}} \nabla \times \boldsymbol{E}(s)ds, \psi\right), \tag{4.7}$$
$$\tau_0^\alpha \left(\tilde{\partial}_t^\alpha \chi_h^{k+\frac{1}{2}}, \tilde{\phi}\right) + \left(\chi_h^{k+\frac{1}{2}}, \tilde{\phi}\right) - \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\xi_h^{k+\frac{1}{2}}, \tilde{\phi}\right)$$
$$= \left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{P}(s)ds, \tilde{\phi}\right)$$
$$- \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\Pi_h \boldsymbol{E}^{k+\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{E}(s)ds, \tilde{\phi}\right)$$
$$+ \tau_0^\alpha \left(\tilde{\partial}_t^\alpha \Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \tilde{\partial}_t^\alpha \boldsymbol{P}^{k+\frac{1}{2}} + Loc^{k+1}, \tilde{\phi}\right), \tag{4.8}$$

where Loc^{k+1} is defined as

$$Loc^{k+1} = \tilde{\partial}_t^{\alpha} \boldsymbol{P}^{k+\frac{1}{2}} - \frac{1}{\tau}\bigg(\int_{t_k}^{t_{k+1}} \frac{\partial^{\alpha} \boldsymbol{P}(s)}{\partial s^{\alpha}} ds \bigg). \tag{4.9}$$

Choosing $\phi = \tau(\xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}})$ in (4.6), $\psi = \tau(\eta_h^k + \eta_h^{k-1})$ in (4.7), and adding the results together, we have

$$\begin{aligned}
&\epsilon_0\epsilon_\infty \left(\left\| \xi_h^{k+\frac{1}{2}} \right\|_0^2 - \left\| \xi_h^{k-\frac{1}{2}} \right\|_0^2 \right) + \mu_0 \left(\left\| \eta_h^k \right\|_0^2 - \left\| \eta_h^{k-1} \right\|_0^2 \right) \\
&\quad + \left(\chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}, \xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}} \right) \\
&\ = \tau \left[\left(\nabla \times \xi_h^{k+\frac{1}{2}}, \eta_h^k \right) - \left(\nabla \times \xi_h^{k-\frac{1}{2}}, \eta_h^{k-1} \right) \right] \\
&\quad + \epsilon_0\epsilon_\infty \left(\left(\Pi_h \boldsymbol{E}^{k+\frac{1}{2}} - \boldsymbol{E}^{k+\frac{1}{2}} \right) - \left(\Pi_h \boldsymbol{E}^{k-\frac{1}{2}} - \boldsymbol{E}^{k-\frac{1}{2}} \right), \xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}} \right) \\
&\quad + \left(\left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k+\frac{1}{2}} \right) - \left(\Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \boldsymbol{P}^{k-\frac{1}{2}} \right), \xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}} \right) \\
&\quad - \tau \left(\boldsymbol{H}^k - \frac{1}{\tau} \int_{I_k} \boldsymbol{H}(s) ds, \nabla \times \left(\xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}} \right) \right) \\
&\quad + \tau \bigg(\nabla \times \left(\Pi_h \boldsymbol{E}^{k-\frac{1}{2}} - \boldsymbol{E}^{k-\frac{1}{2}} \right) + \nabla \times \boldsymbol{E}^{k-\frac{1}{2}} \\
&\qquad - \frac{1}{\tau} \int_{I_{k-\frac{1}{2}}} \nabla \times \boldsymbol{E}(s) ds, \eta_h^k + \eta_h^{k-1} \bigg).
\end{aligned} \tag{4.10}$$

Adding (4.8) with different indices $k + \frac{1}{2}$ and $k - \frac{1}{2}$, we have

$$\begin{aligned}
&\tau_0^{\alpha} \left(\tilde{\partial}_t^{\alpha} \chi_h^{k+\frac{1}{2}} + \tilde{\partial}_t^{\alpha} \chi_h^{k-\frac{1}{2}}, \tilde{\phi} \right) + \left(\chi_h^{k+\frac{1}{2}} + \chi_h^{k-\frac{1}{2}}, \tilde{\phi} \right) - \epsilon_0(\epsilon_s - \epsilon_\infty) \left(\xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}}, \tilde{\phi} \right) \\
&\ = \left(\left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \frac{1}{\tau} \int_{I_{k+\frac{1}{2}}} \boldsymbol{P}(s) ds \right) + \left(\Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \frac{1}{\tau} \int_{I_{k-\frac{1}{2}}} \boldsymbol{P}(s) ds \right), \tilde{\phi} \right) \\
&\qquad - \epsilon_0(\epsilon_s - \epsilon_\infty) \left(\Pi_h \boldsymbol{E}^{k+\frac{1}{2}} - \frac{1}{\tau} \int_{I_{k+\frac{1}{2}}} \boldsymbol{E}(s) ds + \Pi_h \boldsymbol{E}^{k-\frac{1}{2}} - \frac{1}{\tau} \int_{I_{k-\frac{1}{2}}} \boldsymbol{E}(s) ds, \tilde{\phi} \right) \\
&\qquad + \tau_0^{\alpha} \bigg(\tilde{\partial}_t^{\alpha} \left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k+\frac{1}{2}} \right) \\
&\qquad + \tilde{\partial}_t^{\alpha} \left(\Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \boldsymbol{P}^{k-\frac{1}{2}} \right) + Loc^{k+1} + Loc^{k-1}, \tilde{\phi} \bigg).
\end{aligned} \tag{4.11}$$

Choosing $\tilde{\phi} = \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}$ in (4.11) and using (3.26), we have

$$
\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \left\|\chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\right\|_0^2 + \left\|\chi_h^{k+\frac{1}{2}}\right\|_0^2 - \left\|\chi_h^{k-\frac{1}{2}}\right\|_0^2 \\
&\quad - \epsilon_0(\epsilon_s - \epsilon_\infty)\left(\xi_h^{k+\frac{1}{2}} + \xi_h^{k-\frac{1}{2}}, \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\right) \\
&= -\frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha \left(\sum_{l=0}^{k-2}\left(\chi_h^{k-\frac{1}{2}-l} - \chi_h^{k-\frac{3}{2}-l}\right)(b_l + b_{l+1}), \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\right) \\
&\quad + \left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{P}(s)ds + \Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k-\frac{1}{2}}} \boldsymbol{P}(s)ds, \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\right) \\
&\quad - \epsilon_0(\epsilon_s - \epsilon_\infty)\Bigg(\Pi_h \boldsymbol{E}^{k+\frac{1}{2}} - \frac{1}{\tau}\int_{I_{k+\frac{1}{2}}} \boldsymbol{E}(s)ds + \Pi_h \boldsymbol{E}^{k-\frac{1}{2}} \\
&\qquad\qquad - \frac{1}{\tau}\int_{I_{k-\frac{1}{2}}} \boldsymbol{E}(s)ds, \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\Bigg) \\
&\quad + \tau_0^\alpha\Big(\tilde{\partial}_t^\alpha\left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k+\frac{1}{2}}\right) + \tilde{\partial}_t^\alpha\left(\Pi_h \boldsymbol{P}^{k-\frac{1}{2}} - \boldsymbol{P}^{k-\frac{1}{2}}\right) \\
&\qquad + Loc^{k+1} + Loc^{k-1}, \chi_h^{k+\frac{1}{2}} - \chi_h^{k-\frac{1}{2}}\Big).
\end{aligned} \tag{4.12}
$$

By Lemma 3.2, we obtain

$$
\begin{aligned}
Loc^{k+1} &= \tilde{\partial}_t^\alpha \boldsymbol{P}^{k+\frac{1}{2}} - \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \frac{\partial^\alpha \boldsymbol{P}(s)}{\partial s^\alpha} ds \\
&= \tilde{\partial}_t^\alpha \boldsymbol{P}^{k+\frac{1}{2}} - \frac{\partial^\alpha \boldsymbol{P}(t_{k+\frac{1}{2}})}{\partial s^\alpha} - \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \left[\frac{\partial^\alpha \boldsymbol{P}(s)}{\partial s^\alpha} - \frac{\partial^\alpha \boldsymbol{P}(t_{k+\frac{1}{2}})}{\partial s^\alpha}\right] ds \\
&= O(\tau^{2-\alpha}) + O(\tau^2) = O(\tau^{2-\alpha}),
\end{aligned}
$$

where we used the fact that

$$
\frac{1}{\tau}\int_{t_k}^{t_{k+1}} (f(s) - f(t_{k+\frac{1}{2}}))ds = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \left[(s - t_{k+\frac{1}{2}})f_t(t_{k+\frac{1}{2}})) + O(\tau^2)\right] ds = O(\tau^2).
$$

By (3.18), (3.7), and (4.1), and the facts that $k\tau \le T$ and $b_l \le 1$, we have

$$
\begin{aligned}
\left\|\tilde{\partial}_t^\alpha\left(\Pi_h \boldsymbol{P}^{k+\frac{1}{2}} - \boldsymbol{P}^{k+\frac{1}{2}}\right)\right\|_0 &\le \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1} \|(\Pi_h \boldsymbol{P} - \boldsymbol{P})_t\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\, \tau b_l \\
&\le \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} C h^\gamma \|\boldsymbol{P}_t\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}.
\end{aligned}
$$

Multiplying (4.10) by $\epsilon_0(\epsilon_s - \epsilon_\infty)$, adding the result to (4.12), then using Lemma 4.1

and estimates (4.1) and (4.2), we obtain

$$
\begin{aligned}
&\epsilon_0(\epsilon_s-\epsilon_\infty)\left[\epsilon_0\epsilon_\infty\left(\left\|\xi_h^{k+\frac{1}{2}}\right\|_0^2-\left\|\xi_h^{k-\frac{1}{2}}\right\|_0^2\right)+\mu_0\left(\|\eta_h^k\|_0^2-\|\eta_h^{k-1}\|_0^2\right)\right]\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha\left\|\chi_h^{k+\frac{1}{2}}-\chi_h^{k-\frac{1}{2}}\right\|_0^2+\left\|\chi_h^{k+\frac{1}{2}}\right\|_0^2-\left\|\chi_h^{k-\frac{1}{2}}\right\|_0^2\\
&\ \le\tau\left[\left(\nabla\times\xi_h^{k+\frac{1}{2}},\eta_h^k\right)-\left(\nabla\times\xi_h^{k-\frac{1}{2}},\eta_h^{k-1}\right)\right]\cdot\epsilon_0(\epsilon_s-\epsilon_\infty)\\
&\quad+\tau\cdot Ch^\gamma\left(\|\boldsymbol{E}_t\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}+\|\boldsymbol{P}_t\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}\right)\left\|\xi_h^{k+\frac{1}{2}}+\xi_h^{k-\frac{1}{2}}\right\|_0\\
&\quad+C\tau^3\|\nabla\times\boldsymbol{H}\|_{C^2(0,T;\boldsymbol{L}^2(\Omega))}\left\|\xi_h^{k+\frac{1}{2}}+\xi_h^{k-\frac{1}{2}}\right\|_0\\
&\quad+\left(\tau\cdot Ch^\gamma\|\nabla\times\boldsymbol{E}\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}+C\tau^3\|\nabla\times\boldsymbol{E}\|_{C^2(0,T;\boldsymbol{L}^2(\Omega))}\right)\|\eta_h^k+\eta_h^{k-1}\|_0\\
&\quad+\frac{2}{\Gamma(2-\alpha)}\left(\frac{\tau_0}{\tau}\right)^\alpha\sum_{l=0}^{k-2}\left\|\chi_h^{k-\frac{1}{2}-l}-\chi_h^{k-\frac{3}{2}-l}\right\|_0\left\|\chi_h^{k+\frac{1}{2}}-\chi_h^{k-\frac{1}{2}}\right\|_0\\
&\quad+\left(C\tau^2\|\boldsymbol{P}\|_{C^2(0,T;\boldsymbol{L}^2(\Omega))}+Ch^\gamma\|\boldsymbol{P}\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}\right)\left\|\chi_h^{k+\frac{1}{2}}-\chi_h^{k-\frac{1}{2}}\right\|_0\\
&\quad+\left(C\tau^2\|\boldsymbol{E}\|_{C^2(0,T;\boldsymbol{L}^2(\Omega))}+Ch^\gamma\|\boldsymbol{E}\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}\right)\left\|\chi_h^{k+\frac{1}{2}}-\chi_h^{k-\frac{1}{2}}\right\|_0\\
&\quad+\left(\frac{\tau_0}{\tau}\right)^\alpha\cdot\frac{1}{\Gamma(2-\alpha)}[Ch^\gamma\|\boldsymbol{P}_t\|_{L^\infty(0,T;H^\gamma(\mathrm{curl};\Omega))}\\
&\qquad\qquad+\tau^{2-\alpha}\|\boldsymbol{P}\|_{C^2(0,T;L^\infty(\Omega))}]\left\|\chi_h^{k+\frac{1}{2}}-\chi_h^{k-\frac{1}{2}}\right\|_0.
\end{aligned}
$$

The rest of the proof is similar to the stability analysis carried out in Theorem 3.3. ◻

Remark 4.2. By similar techniques, we can prove the optimal error estimates for the Crank–Nicolson scheme. More specifically, let $(\boldsymbol{E}_h^n,\boldsymbol{H}_h^n,\boldsymbol{P}_h^n)$ be the finite element solutions of (3.8)–(3.10), and let $(\boldsymbol{E}^n,\boldsymbol{H}^n,\boldsymbol{P}^n)$ be the analytic solutions of (2.6)–(2.7) and (2.3). Then under proper regularity assumption of the solutions, there exists a constant $C>0$, independent of both the time step size τ and the mesh size h, such that

$$
\begin{aligned}
&\max_{n\ge1}(\|\boldsymbol{E}^n-\boldsymbol{E}_h^n\|_0+\|\boldsymbol{H}^n-\boldsymbol{H}_h^n\|_0+\|\boldsymbol{P}^n-\boldsymbol{P}_h^n\|_0)\\
&\qquad\le C(h^\gamma+\tau^{2-\alpha})+C(\|\boldsymbol{E}^0-\boldsymbol{E}_h^0\|_{0,\mathrm{curl}}+\|\boldsymbol{H}^0-\boldsymbol{H}_h^0\|_0+\|\boldsymbol{P}^0-\boldsymbol{P}_h^0\|_{0,\mathrm{curl}}),
\end{aligned}
$$

where γ is the regularity constant from (4.1).

If we choose the initial approximations for the Crank–Nicolson scheme (3.8)–(3.10) as follows:

$$\boldsymbol{E}_h^0=\Pi_h\boldsymbol{E}^0,\qquad\boldsymbol{P}_h^0=\Pi_h\boldsymbol{P}^0,\quad\boldsymbol{H}_h^0=Q_h\boldsymbol{H}^0,$$

then we have the optimal error estimate

$$\max_{n\ge1}(\|\boldsymbol{E}^n-\boldsymbol{E}_h^n\|_0+\|\boldsymbol{H}^n-\boldsymbol{H}_h^n\|_0+\|\boldsymbol{P}^n-\boldsymbol{P}_h^n\|_0)\le C(h^\gamma+\tau^{2-\alpha}).$$

Remark 4.3. We like to remark that the above error analysis holds true for other RTN mixed finite element spaces. For example, when the underlying solutions of (2.6)–(2.7) and (2.3) have enough regularity, we can use the following RTN cubic elements [22, 23]: for any $k \geq 1$,

(4.13)
$$\mathbf{V}_h = \{\boldsymbol{v}_h \in H(\mathrm{div};\Omega): \ \boldsymbol{v}_h|_K \in Q_{k,k-1,k-1} \times Q_{k-1,k,k-1} \times Q_{k-1,k-1,k} \ \ \forall\, K \in T^h\},$$
(4.14)
$$\mathbf{U}_h = \{\boldsymbol{u}_h \in H(\mathrm{curl};\Omega): \ \boldsymbol{u}_h|_K \in Q_{k-1,k,k} \times Q_{k,k-1,k} \times Q_{k,k,k-1} \ \ \forall\, K \in T^h\},$$

in which case, the optimal error estimates become

$$\max_{n\geq 1}\left(\left\|\boldsymbol{E}^{n+\frac{1}{2}} - \boldsymbol{E}_h^{n+\frac{1}{2}}\right\|_0 + \|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 + \left\|\boldsymbol{P}^{n+\frac{1}{2}} - \boldsymbol{P}_h^{n+\frac{1}{2}}\right\|_0\right) \leq C\left(h^k + \tau^{2-\alpha}\right),$$

and

$$\max_{n\geq 1}(\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 + \|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 + \|\boldsymbol{P}^n - \boldsymbol{P}_h^n\|_0) \leq C(h^k + \tau^{2-\alpha})$$

for the leap-frog and Crank–Nicolson schemes, respectively.

5. Numerical results. In this section, we will present some numerical results obtained by both the leap-frog scheme (3.20)–(3.22) and the Crank–Nicolson scheme (3.8)–(3.10). For simplicity, we perform the computation for a two-dimensional (2-D) problem by assuming that the electrical field $\boldsymbol{E}$ is a vector, while the magnetic field H is a scalar. Note that proofs of our error estimates still hold true by introducing 2-D curl operators:

$$\nabla \times H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right)', \quad \nabla \times \boldsymbol{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \quad \forall\, \boldsymbol{E} \equiv (E_1, E_2). \tag{5.1}$$

Consider a 2-D domain $\Omega = [0,1]^2$, we partition it by a family of regular rectangular meshes T^h with maximum mesh size h. In our test, we use the following mixed spaces:

$$\mathbf{V}_h = \{\psi_h \in L^2(\Omega): \quad \psi_h|_K \in Q_{0,0} \ \ \forall\, K \in T^h\},$$
$$\mathbf{U}_h = \{\boldsymbol{\phi}_h \in H(\mathrm{curl};\Omega): \quad \boldsymbol{\phi}_h|_K \in Q_{0,1} \times Q_{1,0} \ \ \forall\, K \in T^h\}.$$

Here $Q_{i,j}$ denotes the space of polynomials whose degrees are less than or equal to i, j in variables x, y, respectively. It is easy to see that $\nabla \times \mathbf{U}_h \subset \mathbf{V}_h$ still holds.

For clarity, we choose all physical coefficients in (3.20)–(3.22) to be one (i.e., $\epsilon_0\epsilon_\infty = \mu_0 = \tau_0 = \epsilon_0(\epsilon_s - \epsilon_\infty) = 1$) and add a source term $(\boldsymbol{f}^k, \phi)$ to (3.20) so that we have the following analytical solutions for testing the convergence rate:

$$H(x,y,t) = -\left[\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}t^{3-\alpha} + \frac{1}{3}t^3\right] \cdot 2\pi\cos\pi x\cos\pi y,$$
$$\boldsymbol{P}(x,y,t) = t^2\boldsymbol{w}(x,y), \quad \boldsymbol{E}(x,y,t) = \left[\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)}t^{2-\alpha} + t^2\right]\boldsymbol{w}(x,y),$$

where $\boldsymbol{w} = \left(\begin{smallmatrix} -\cos\pi x\sin\pi y \\ \sin\pi x\cos\pi y \end{smallmatrix}\right)$. The corresponding source function

$$\begin{aligned}\boldsymbol{f}(x,y,t) &= \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t} - \nabla \times H \\ &= \Bigg\{\frac{2}{\Gamma(1-\alpha)(1-\alpha)}t^{1-\alpha} + 4t \\ &\qquad + 2\pi^2\left[\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}t^{3-\alpha} + \frac{1}{3}t^3\right]\Bigg\}\boldsymbol{w}(x,y).\end{aligned}$$

Solving (3.22), we obtain

$$\boldsymbol{P}_h^{k+\frac{1}{2}} = \frac{\tau^\alpha \Gamma(2-\alpha)}{\tau^\alpha \Gamma(2-\alpha)+1} \boldsymbol{E}_h^{k+\frac{1}{2}} + \frac{1}{\tau^\alpha \Gamma(2-\alpha)+1} \left[\boldsymbol{P}_h^{k-\frac{1}{2}} - \sum_{l=1}^{k} \left(\boldsymbol{P}_h^{k+\frac{1}{2}-l} - \boldsymbol{P}_h^{k-\frac{1}{2}-l} \right) b_l \right], \tag{5.2}$$

where constant $b_l = (l+1)^{1-\alpha} - l^{1-\alpha}$.

Substituting (5.2) into (3.20) with added source term, we have

$$\begin{aligned} &\left(1 + \frac{\tau^\alpha \Gamma(2-\alpha)}{\tau^\alpha \Gamma(2-\alpha)+1}\right) \left(\boldsymbol{E}_h^{k+\frac{1}{2}}, \phi\right) \\ &\quad = \left(\boldsymbol{E}_h^{k-\frac{1}{2}}, \phi\right) + \tau \left(H_h^k, \nabla \times \phi\right) + \tau \left(\boldsymbol{f}^k, \phi\right) + \left(\frac{\tau^\alpha \Gamma(2-\alpha)}{\tau^\alpha \Gamma(2-\alpha)+1}\right) \left(\boldsymbol{P}_h^{k-\frac{1}{2}}, \phi\right) \\ &\quad + \frac{1}{\tau^\alpha \Gamma(2-\alpha)+1} \left(\sum_{l=1}^{k} \left(\boldsymbol{P}_h^{k+\frac{1}{2}-l} - \boldsymbol{P}_h^{k-\frac{1}{2}-l} \right) b_l, \phi \right). \end{aligned} \tag{5.3}$$

In summary, in our implementation, the leap-frog scheme (3.20)–(3.22) is realized as follows: Choose initial values

$$\boldsymbol{E}_h^{\frac{1}{2}} = \Pi_h \boldsymbol{E}(0.5\tau), \quad \boldsymbol{P}_h^{\frac{1}{2}} = \Pi_h \boldsymbol{P}(0.5\tau), \quad H_h^0 = Q_h H(0),$$

then at each time step,

Step 1: Solve $(H_h^k, \psi) = (H_h^{k-1}, \psi_h) - \tau(\nabla \times \boldsymbol{E}_h^{k-\frac{1}{2}}, \psi) \ \ \forall\, \psi \in \mathbf{V}_h$, for H_h^k.

Step 2: Solve (5.3) for $\boldsymbol{E}_h^{k+\frac{1}{2}}$.

Step 3: Update $\boldsymbol{P}_h^{k+\frac{1}{2}}$ using (5.2).

We solved this problem using a fixed $\tau = 0.005$ on various meshes. The obtained error estimates are presented in Tables 5.1 and 5.2 for the fractional-order parameter $\alpha = 0.5$ and $\alpha = 0.7$, respectively. Our results confirm the theoretical convergence rate

$$\max_{n \geq 1} \left(\left\| \boldsymbol{E}^{n+\frac{1}{2}} - \boldsymbol{E}_h^{n+\frac{1}{2}} \right\|_0 + \| \boldsymbol{H}^n - \boldsymbol{H}_h^n \|_0 + \left\| \boldsymbol{P}^{n+\frac{1}{2}} - \boldsymbol{P}_h^{n+\frac{1}{2}} \right\|_0 \right) \leq C(h + \tau^{2-\alpha}),$$

which is $O(h)$ for sufficiently small τ. The time convergence rate $O(\tau^{2-\alpha})$ cannot be validated due to the CFL constraint (3.24), which makes $O(h + \tau^{2-\alpha})$ always dominated by the spatial error $O(h)$. In Tables 5.1 and 5.2, the convergence rate is calculated as $r = \ln(Err_{2h}/Err_h)/\ln 2$, where Err_{2h} and Err_h are errors obtained on a coarse mesh and a fine mesh (with half of the coarse mesh size), respectively.

Example 1. In this example, we present our tests for the Crank–Nicolson scheme (3.8)–(3.10). Substituting (3.7) into (3.10) and solving for $\boldsymbol{P}_h^k$, we obtain

$$\boldsymbol{P}_h^k = \frac{C_\alpha - 1}{C_\alpha + 1} \boldsymbol{P}_h^{k-1} - \frac{C_\alpha}{C_\alpha + 1} \sum_{l=1}^{k-1} (b_l + b_{l-1}) \left(\boldsymbol{P}_h^{k-l} - \boldsymbol{P}_h^{k-l-1} \right) + \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{C_\alpha + 1} \left(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1} \right), \tag{5.4}$$

where we denote $C_\alpha = (\frac{\tau_0}{\tau})^\alpha \cdot \frac{1}{\Gamma(2-\alpha)}$.

TABLE 5.1
L^2 errors obtained with fixed $\tau = 0.005$ and $\alpha = 0.5$.

h	$\boldsymbol{H}$ error	Rate	$\boldsymbol{E}$ error	Rate	$\boldsymbol{P}$ error	Rate
$\frac{1}{4}$	0.922304190560348	—	0.288689879688438	—	0.114946505563886	—
$\frac{1}{8}$	0.468519608152406	0.9771	0.143027284829818	1.0132	0.057092467484029	1.0096
$\frac{1}{16}$	0.235184727611048	0.9943	0.071339302014101	1.0035	0.028496902545996	1.0025
$\frac{1}{32}$	0.117708098349557	0.9986	0.035647572089714	1.0009	0.014242303383554	1.0006
$\frac{1}{64}$	0.058868519678121	0.9996	0.017821021910717	1.0002	0.007120518397762	1.0001

TABLE 5.2
L^2 errors obtained with fixed $\tau = 0.005$ and $\alpha = 0.7$.

h	$\boldsymbol{H}$ error	Rate	$\boldsymbol{E}$ error	Rate	$\boldsymbol{P}$ error	Rate
$\frac{1}{4}$	1.063336516637811	—	0.313468283366440	—	0.114974024401720	—
$\frac{1}{8}$	0.540351999003493	0.9766	0.155045896248337	1.0156	0.057086537054800	1.0101
$\frac{1}{16}$	0.271268268741503	0.9942	0.077296371374641	1.0042	0.028491619022295	1.0026
$\frac{1}{32}$	0.135770875194541	0.9985	0.038619417038869	1.0011	0.014240420530257	1.0005
$\frac{1}{64}$	0.067902540859465	0.9996	0.019306160204807	1.0003	0.007121793125438	0.9997

Substituting (5.4) into (3.8) with an added source term $(\boldsymbol{f}^{k-\frac{1}{2}}, \phi)$, we have

$$\begin{aligned}
&\left(\epsilon_0\epsilon_\infty + \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{C_\alpha + 1}\right)(\boldsymbol{E}_h^k, \phi) - \frac{\tau}{2}(\boldsymbol{H}_h^k, \nabla \times \phi) \\
&\quad = \left(\epsilon_0\epsilon_\infty - \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{C_\alpha + 1}\right)\left(\boldsymbol{E}_h^{k-1}, \phi\right) + \frac{\tau}{2}\left(\boldsymbol{H}_h^{k-1}, \nabla \times \phi\right) + \tau\left(\boldsymbol{f}^{k-\frac{1}{2}}, \phi\right) \\
&\qquad + \left(\frac{2}{C_\alpha + 1}\boldsymbol{P}_h^{k-1} + \frac{C_\alpha}{C_\alpha + 1}\sum_{l=1}^{k-1}(b_l + b_{l-1})\left(\boldsymbol{P}_h^{k-l} - \boldsymbol{P}_h^{k-l-1}\right), \phi\right).
\end{aligned} \tag{5.5}$$

For implementation, we rewrite (3.9) as

$$\frac{\tau}{2}\left(\nabla \times \boldsymbol{E}_h^k, \psi\right) + \mu_0\left(\boldsymbol{H}_h^k, \psi\right) = \mu_0\left(\boldsymbol{H}_h^{k-1}, \psi\right) - \frac{\tau}{2}\left(\nabla \times \boldsymbol{E}_h^{k-1}, \psi\right). \tag{5.6}$$

In summary, the Crank–Nicolson scheme (3.8)–(3.10) with an added source term is implemented as follows: At each time step,

Step 1: Solve a system formed by (5.5) and (5.6) for $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$.

Step 2: Update $\boldsymbol{P}_h^k$ using (5.4).

We solved the same problem as Example 1 using the Crank–Nicolson scheme with various time step and mesh sizes. Selected numerical results are presented in Table 5.3 for the fractional-order parameter $\alpha = 0.7$ with a fixed time step size $\tau = 0.001$ running 1000 time steps on various meshes. Results of Table 5.3 confirm the theoretical convergence rate

$$\max_{n\geq 1}(\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 + \|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 + \|\boldsymbol{P}^n - \boldsymbol{P}_h^n\|_0) \leq C\left(h + \tau^{2-\alpha}\right),$$

which is $O(h)$ for sufficiently small τ. The time convergence rate $O(\tau^{2-\alpha})$ is not that clear due to the mesh size limitation on our computer; see our results listed in Table 5.4 for errors of $\boldsymbol{P}$ obtained with $h = \frac{1}{256}$ and various time steps.

Example 2. Since our numerical results from Examples 1 and 1 couldn't confirm the time convergence rate $O(\tau^{2-\alpha})$ due to the mesh size limitation, in this example,

Table 5.3
L^2 errors obtained with fixed $\tau = 0.001$, $\alpha = 0.7$, and various meshes.

h	H error	Rate	$\boldsymbol{E}$ error	Rate	$\boldsymbol{P}$ error	Rate
$\frac{1}{4}$	1.0618	—	0.4426	—	0.1639	—
$\frac{1}{8}$	0.5401	0.9752	0.2186	1.0177	0.0810	1.0168
$\frac{1}{16}$	0.2712	0.9939	0.1089	1.0053	0.0405	1.0
$\frac{1}{32}$	0.1358	0.9979	0.0544	1.0013	0.0203	0.9964
$\frac{1}{64}$	0.0679	1.0	0.0272	1.0	0.0102	0.9929

Table 5.4
L^2 errors of $\boldsymbol{P}$ obtained with fixed mesh $h = 1/256$, $\alpha = 0.7$, and various time step τ.

τ	1/2	1/4	1/8	1/16	1/32	1/64
$\boldsymbol{P}$ error	0.0910	0.0414	0.0177	0.0077	0.0039	0.0019

Table 5.5
L^2 errors of P obtained for $\alpha = 0.7$ with different τ.

τ	1/8	1/16	1/32	1/64	1/128	1/256	1/512	1/1024
P error	0.0249	0.0104	0.0043	0.0017	7.1170e-4	2.8980e-4	1.1788e-4	4.7919e-5
Rate $O(\tau^r)$	—	1.2596	1.2742	1.3388	1.2562	1.2962	1.2977	1.2986

we solve the problem

$$\frac{\partial^\alpha P(t)}{\partial t^\alpha} + P(t) = E(t), \tag{5.7}$$

which is a simplified version of (2.3) and no spatial error is involved. Here we assume that $E(t)$ is a fixed source term, where $E(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + t^2$ such that the exact solution of (5.7) is $P(t) = t^2$.

We use the same scheme (5.4) for solving P. The errors obtained using fixed $\alpha = 0.7$ and various time step size τ are given in Table 5.5, which clearly shows the rate $O(\tau^{2-\alpha})$, especially when τ becomes quite small.

6. Conclusions. In this paper, we developed two fully discrete finite element schemes for solving the Cole–Cole dispersive medium model, which is described by the Maxwell's equations plus a fractional time derivative term. The stability and optimal error estimates are then proved for both schemes. Finally, we implemented the proposed algorithms and presented many numerical results justifying our theoretical analysis.

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