Theoretical aspects of edge singularity at surface cracks

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[**Abstract**] The complicated issue of the so-called vertex singularity near the corner point of the intersection of a three-dimensional crack front and the free surface of the solid are reviewed and discussed. It was discovered that classical stress singularity near a crack front of $r^{-0.5}$ (where r is the distance measure from the crack tip) does not hold at the edge of the crack at the free surface, but instead of the form r^{λ}. The exponent λ is a function of mode of cracking (i.e. mode I, II or III), the inclination angle β of the crack front at the free surface, the inclination angle γ of the crack plane with the solid surface and the Poisson's ratio of the solid. Analytical results exist only for very special cases, whereas most existing results were obtained through the use of very fine mesh finite element method, finite difference and hyper-singular integral equation. Although the vertex singularity is of paramount importance in engineering application, these results were not readily available in the handbooks of stress intensity factor. Therefore, the main purpose of this paper is to summarize this importance results in a form that can be easily used by engineers and scientists.

[**Key words**] vertex singularity; surface crack; singularity exponent

裂纹与边界交叉点的奇异性理论研究

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[摘 要] 回顾和探讨了位于三维裂纹前沿和固体自由表面交叉点附近的被称作复杂尖端奇异性的问题. 结果表明, 自由面上的裂纹边界点并不具有传统裂纹尖端应力场 $r^{-0.5}$ 阶的奇异性(其中 r 为从裂纹尖端量起的距离), 而是应该采取 r^{λ} 阶的形式. 其中指数 λ 是关于破裂模式(即模式I、II或III), 裂纹前沿在自由面的倾角 β, 破裂面与固体表面的夹角 γ 以及材料泊松比的函数. 这一问题只对极个别情况存在解析解, 而绝大多数已知结果都是通过高精度有限元、有限差分方法

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和强奇异积分方程得到的. 尽管尖端奇异性问题对工程应用极其重要, 但在应力场强度系数手册中并无现成结果.

[关键词]　尖端奇异性; 表面裂纹; 奇异性分量

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1 INTRODUCTION

There is a major difference between three－dimensional internal and surface cracks. It is a well－known result of linear elastic fracture mechanics that the stress near the crack tip exhibits a singularity of $r^{-0.5}$, where r is the distance measured from the crack tip[1]. However, it was proved by Benthem[2,3] that this classical singularity does not hold at the edge of the crack at the free surface, but instead a singularity of the form r^{λ} exists. The singularity exponent λ is found to be a function of the mode of cracking (i.e. mode I, II or III), the inclination angle β of the crack front at the free surface, the inclination angle γ of the crack plane with the solid surface, and the Poisson's ratio of the solid. The definitions of these angles are given in Figure 1. The surface of the sample is denoted by the $x－y$ surface, and the solid occupies $z \geqslant 0$. The outward normal direction of the free surface is denoted by $\boldsymbol{n}$ whereas the tangential vector on the plane of the three－dimensional crack is denoted by $\boldsymbol{s}$. The thick arrows indicate the uniaxial compression, which forms an angle of α with the crack line on the free surface (i.e. the crack forms an angle of $\pi/2-\alpha$ with the $x－$axis).

We also have conducted some experiments on these kinds of surface crack problem shown in Figure 1 (only the case of $\gamma = 0$ have been tested). Our motivation of studying this problem is on its application to rock mechanics and earthquake mechanics. For such cases, the crack is also subject to compression as shown in Figure 1, instead of subject to tension assumed in the area of metal and material science. The details of these experiments will not be reported here; whereas, only theoretical issue is addressed here.

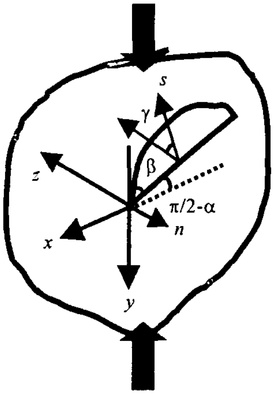

Figure 1　The geometry of a 3－D surface crack under compression

Analytical results exist only for very special cases ($\beta=\pi/2$, $\gamma=0$), whereas most existing results were obtained through the use of very fine mesh finite element method, finite difference and hypersingular integral equation. Although the vertex singularity is of paramount importance in engineering application, these predictions of vertex singular component (or so－called Benthem singularity component) cannot be found on any handbook of stress intensity factors (e.g. [4, 5]). Therefore, there is a need to report these results here. Some approximate formulas are also proposed here for the easy reference by engineers and scientists. We should emphasize that these formulas are original and proposed for the first time.

2 VERTEX SINGULARITY

In particular, the crack tip singular stresses are of the following form (for mode I, mode II and mode III respectively):

$$\sigma = \frac{K_{\mathrm{I}}}{\sqrt{2\pi}} r^{\lambda}, \quad \tau_{\mathrm{II}} = \frac{K_{\mathrm{II}}}{\sqrt{2\pi}} r^{\lambda}, \quad \tau_{\mathrm{III}} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi}} r^{\lambda} \tag{1}$$

where σ, τ_{II}, and τ_{III} are the tensile stress, in—plane shear stress and anti—plane shear stress respectively. The exponent λ equals $-1/2$ if the crack tip is not close to the free surface. As remarked earlier, the singularity exponent λ is found to be a function of the mode of cracking (i.e. mode I, II or III), the inclination angle β of the crack front at the free surface, the inclination angle γ of the crack plane with the solid surface, and the Poisson' s ratio of the solid.

2.1 Crack front orthogonal to the free surface

When the free surface is approached, the predominant (or called gravest) singularity component λ is given by both Benthem[2] and Bazant and Estenssoro[6].

A simple curve fitting of their results lead to the following formula for the mode I with $\beta = \pi/2$ and $\gamma = 0$:

$$\lambda = 14.049v^4 - 12.387v^4 + 4.4748v^3 - 0.4409v^2 + 0.1094v - 0.5 \tag{2}$$

with the coefficient of correlation of $R^2 = 0.9999$ and the predictions by (2) for all available data from Benthem[2] and Bazant and Estenssoro[6] are no worse than 0.3% error.

For modes II and III cracks with $\beta = \pi/2$ and $\gamma = 0$:

$$\lambda = 3.111v^4 - 2.882v^3 + 1.4911v^2 - 0.0604v - 0.5 \tag{3}$$

with $R^2 = 1$ (i.e. a perfect of the data given by Bazant and Estenssoro [6]).

2.2 Surface crack intersect the free surface at angle β

When the crack does not intersect the free surface at $\beta = \pi/2$, λ may be estimated from the following formula for the case of mode I with $v = 0.3$ and $\gamma = 0$:

$$\lambda = -0.2417\beta - 0.0731 \tag{4}$$

with the coefficient of correlation of $R^2 = 0.9999$ and β is in radian. These formulas of singularity exponents will be very useful if numerical method will be used further analyses. Note that equations (2 ~4) are originally proposed here and can be used easily by engineers and scientists.

2.3 Critical intersecting angle of crack orthogonal to free surface

Since the internal stress singularity has a singularity of a singularity of $r^{-0.5}$, therefore, if an internal crack propagates toward a free surface, it may has a tendency to preserve such singular. For the singularity exponent λ equals $-1/2$, a critical angle must be intersected between the crack front and the free surface. For mode I and $\gamma = 0$, using the numerical results from Bazant and Estenssoro[6], we propose the following approximate formula:

$$\beta_{\mathrm{crit}} = 0.5376v^2 + 0.4453v + 1.5707 \tag{5}$$

For the same case but modes II and III, we propose the following approximate formula:

$$\beta_{\mathrm{crit}} = 0.8378v^2 - 1.5917v + 1.5708 \tag{6}$$

where v is the Poisson's ratio.

2.4 Critical intersecting angles for crack inclined to free surface

For the case mode I with $\gamma \neq 0$ and $v = 0.3$, we have another approximation for the critical angle:

$$\beta_{crit} = -0.1167\gamma^3 + 0.55\gamma^2 - 0.22\gamma + 1.7813 \tag{7}$$

For modes II and III, the approximate fomula for the critical angle is proposed as:

$$\beta_{crit} = -0.446\gamma^4 + 0.5839\gamma^3 - 0.4508\gamma^2 + 0.026\gamma + 1.2001 \tag{8}$$

A special case of this value has been checked with experiments on aluminum alloy 2219—T581 by Bazant and Estonssoro[6], and the comparison is quite good considering that real materials are not isotropic and homogeneous.

3 CONCLUSION

Stress singularities at the intersection between the surface crack and the free surface are investigated here. In particular, simple approximate formulas were proposed to find the 3—D vertex singularity of the form r^{λ} where the singularity exponent λ is found to be a function of the mode of cracking (i.e. mode I, II or III), the inclination angle β of the crack front at the free surface, the inclination angle γ of the crack plane with the solid surface, and the Poisson's ratio of the solid. These new formulas will be very useful in engineering applications, especially when a crack tip element is needed to be developed at the surface corner point for very accurate finite element analysis.

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