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A MAXIMUM PRINCIPLE FOR PARTIAL INFORMATION BACKWARD STOCHASTIC CONTROL PROBLEMS WITH APPLICATIONS*

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Abstract. This paper studies the partial information control problems of backward stochastic systems. There are three major contributions made in this paper: (i) First, we obtain a new stochastic maximum principle for partial information control problems. Our method relies on a direct calculation of the derivative of the cost functional. (ii) Second, we introduce two classes of partial information linear-quadratic backward control problems for the first time and then investigate them using the maximum principle. Complete and explicit solutions are obtained in terms of some forward and backward stochastic differential filtering equations. (iii) Last but not least, we study a class of full information stochastic pension fund optimization problems which can be viewed as a special case of our general partial information ones. Applying the aforementioned maximum principle, we derive the optimal contribution policy in closed-form and present some related economic remarks.

Key words. backward stochastic differential equation, stochastic filtering, linear-quadratic control, maximum principle, partial information, pension fund

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1. Introduction.

This paper is concerned with the dynamic system of backward stochastic differential equations (BSDEs). A BSDE is an Itô's stochastic differential equation (SDE) in which the terminal rather than the initial condition is given. The BSDEs were introduced by Bismut [2] in the linear case and by Pardoux and Peng [12] in the general case. Since their introduction, the BSDEs have received considerable research attention in a large range of domains, especially in mathematical finance (see, e.g., Cvitanić and Ma [4], El Karoui, Peng, and Quenez [6], Ma and Yong [9], Schroder and Skiadas [14], Yong and Zhou [23], etc.). In particular, the celebrated Black–Scholes option pricing formula can be derived from a class of linear BSDEs, where the random terminal condition is just the option's payoff at the maturity.

Since BSDEs are well-defined dynamic systems, it is very natural and appealing to consider the control problems of BSDEs. However, there exist only a few works along this line, including Peng [13], Xu [22], Wu [19], Lim and Zhou [7], and Wang and Yu [17]. Our work distinguishes itself from the above ones in the following aspects: (i) Our work is established in the context of partial information which is rather general than that of the partial observation. In fact, our information can be summarized by any subfiltration and free of specific observation structures; thus it includes the partial observation models (in particular, the white noise observation models) as its special cases (see, e.g., Wu [20], Wang and Wu [15]). (ii) Two important classes of partial

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information linear-quadratic (LQ) backward control problems are first proposed and then completely solved. These problems are totally new in control theory and have considerable impacts in both theoretical analysis and practical applications, although they have intrinsic mathematical difficulties. Meanwhile, the optimal controls are characterized in terms of the forward and backward stochastic differential filtering equations (FBSDFEs) which arise naturally in our setup. To our best knowledge, these FBSDFEs are also new in control theory.

The rest of this paper is organized as follows. In section 2, we formulate the partial information stochastic control problems. In section 3, we obtain a maximum principle for these problems by a direct calculation of the derivative of the cost functional. Our method is essentially different from that of Peng [13], Xu [22], Dokuchaev and Zhou [5], Wu [20], Wang and Wu [15] and Wang and Yu [17], where maximum principles were obtained but in some different setup. Section 4 is concerned with two special classes of partial information backward control problems. The key point to solving them is to get some observable optimal controls by explicitly computing the filtering estimates of the corresponding adjoint equations. Combining the filtering equations for BSDEs with the stochastic control theory, we obtain the explicit and observable controls. In section 5, we focus on some stochastic pension fund problem, which is of full information and arises as a special case of our general ones. Applying the derived maximum principle, we get the closed-form optimal contribution and present some economic explanations afterwards.

2. Problem formulation. We begin with a finite time horizon $[0,T]$ for $T > 0$, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ on which a standard $\mathfrak{R}^{m+d}$-valued Brownian motion $(W(\cdot), \overline{W}(\cdot))$ is defined. Moreover, it is assumed that $(\mathcal{F}_t)_{0\le t\le T}$ is the natural filtration generated by $(W(\cdot), \overline{W}(\cdot))$ and $\mathcal{F}_T = \mathcal{F}$.

Throughout this paper, we denote by $\langle\cdot,\cdot\rangle$ (resp., $|\cdot|$) the scalar product (resp., norm) of the Euclidean space E, by S^n the set of symmetric $n\times n$ matrices with real elements. The superscript τ denotes the transpose of a vector or matrix. If $M(\cdot)\in S^n$ is positive (semi) definite, we write $M(\cdot) > (\ge) 0$; if $M(\cdot) : [0,T] \to \mathfrak{R}^{n\times n}$ is deterministic and uniformly bounded, we write $M(\cdot) \in L^\infty(0,T;\mathfrak{R}^{n\times n})$. If $x(\cdot) : [0,T]\times\Omega \to S$ is an $\mathcal{F}_t$-adapted square-integrable process (i.e., $\mathbb{E}\int_0^T |x(t)|^2 dt < +\infty$), we write $x(\cdot) \in L^2_{\mathcal{F}}(0,T;S)$; if $x : \Omega \to S$ is an $\mathcal{F}_T$-measurable square-integrable random variable, we write $x \in L^2_{\mathcal{F}_T}(\Omega; S)$. Now consider a BSDE

$$
\begin{cases} -dy(t) = f(t, y(t), z(t), \bar{z}(t), v(t))dt - z(t)dW(t) - \bar{z}(t)d\overline{W}(t), \\ \quad y(T) = \xi. \end{cases} \tag{1}
$$

Here the mapping $f : [0,T]\times\mathfrak{R}^n\times\mathfrak{R}^{n\times m}\times\mathfrak{R}^{n\times d}\times U \to \mathfrak{R}^n$ and $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathfrak{R}^n)$; the nonempty set $U \subseteq \mathfrak{R}^k$ is called the control domain; $v(\cdot) : [0,T]\times\Omega \to U$ is called an admissible control if it satisfies $v(\cdot) \in L^2_{\mathcal{G}}(0,T;U)$, where $\mathcal{G}_t \subseteq \mathcal{F}_t$ is a sub-σ-algebra representing the information available at time t. The set of all admissible controls is denoted by $\mathcal{U}_{ad}$. Now we introduce the following hypothesis.

Hypothesis (H1). The function f is continuously differentiable with respect to $(y, z, \bar{z}, v)$, and the partial derivatives f_y, f_z, $f_{\bar{z}}$, and f_v are uniformly bounded.

Under (H1), the BSDE (1) admits a unique solution for each $v(\cdot) \in \mathcal{U}_{ad}$, which is denoted by the triple $(y^v(\cdot), z^v(\cdot), \bar{z}^v(\cdot))$ (see, e.g., Pardoux and Peng [12], Peng [13], Ma and Yong [9], and Yong and Zhou [23]). The associated cost functional is given by

$$
J(v(\cdot)) = \mathbb{E}\left[\int_0^T l(t, y^v(t), z^v(t), \bar{z}^v(t), v(t))dt + \phi(y^v(0))\right], \tag{2}
$$

where $l : [0,T] \times \mathfrak{R}^n \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{n\times d} \times U \to \mathfrak{R}$ and $\phi : \mathfrak{R}^n \to \mathfrak{R}$ satisfy the following hypothesis.

Hypothesis (H2). There exists a constant K such that

$$\begin{aligned}
&(1+|y|^2+|z|^2+|\bar z|^2+|v|^2)^{-1}|l(t,y,z,\bar z,v)|\\
&+(1+|y|+|z|+|\bar z|+|v|)^{-1}(|l_y(t,y,z,\bar z,v)|\\
&+|l_z(t,y,z,\bar z,v)|+|l_{\bar z}(t,y,z,\bar z,v)|+|l_v(t,y,z,\bar z,v)|)\le K,\\
&(1+|y|^2)^{-1}|\phi|+(1+|y|)^{-1}|\phi_y|\le K, \qquad t\in[0,T].
\end{aligned}$$

The partial information control problem is to seek $u(\cdot)\in\mathcal{U}_{ad}$ such that

$$J(u(\cdot)) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)) \tag{3}$$

subject to (1). If such a $u(\cdot)$ exists, then it is called an optimal control, and the corresponding $(y(\cdot), z(\cdot), \bar z(\cdot))$ in (1) is called the optimal trajectory. Our main goal is to obtain a maximum principle, namely, a necessary condition for the optimal control $u(\cdot)$.

3. A maximum principle. In this section, we will derive a maximum principle of optimality. The method is similar to that of Bensoussan [1]. To start, we need to make the following assumptions.

Assumption (H3). For any t, h such that $t+h\in[t,T]$, and bounded $\mathcal{G}_t$-measurable random variable η, we formulate the control $\zeta(s) = (0,\dots,0,\zeta_i(s),0,\dots,0)\in U$, with

$$\zeta_i(s) = \eta I_{[t,t+h]}(s), \quad s\in[0,T], \quad i=1,2,\dots,k,$$

where $I_{[t,t+h]}(s)$ is the indicator function on the set $[t,t+h]$.

Assumption (H4). For any $\zeta(t)\in\mathcal{G}_t$ with $\zeta(t)$ bounded, $t\in[0,T]$, there exists a $\delta>0$ such that $u(\cdot)+\varepsilon\zeta(\cdot)\in\mathcal{U}_{ad}$ for all $\varepsilon\in(-\delta,\delta)$.

Define a Hamiltonian function by

$$H(t,y,z,\bar z,v,p) = \langle f(t,y,z,\bar z,v),p\rangle + l(t,y,z,\bar z,v), \tag{4}$$

where $H : [0,T]\times\mathfrak{R}^n\times\mathfrak{R}^{n\times m}\times\mathfrak{R}^{n\times d}\times\mathfrak{R}^k\times\mathfrak{R}^n\to\mathfrak{R}$. The adjoint process $p(t)$ is governed by the following SDE:

$$\begin{cases}
dp^v(t) = H_y(t,y^v(t),z^v(t),\bar z^v(t),v(t),p^v(t))dt\\
\qquad\quad + H_z(t,y^v(t),z^v(t),\bar z^v(t),v(t),p^v(t))dW(t)\\
\qquad\quad + H_{\bar z}(t,y^v(t),z^v(t),\bar z^v(t),v(t),p^v(t))d\overline{W}(t),\\
p^v(0) = \phi_y(y^v(0))^\tau.
\end{cases} \tag{5}$$

We will now give the following main result.

THEOREM 3.1. *Let* (H1)–(H4) *hold. Suppose that* $u(\cdot)$ *is a local minimum for* $J(v(\cdot))$*, in the sense that for all bounded* $\zeta(\cdot)\in\mathcal{U}_{ad}$*, there exists a* $\delta>0$ *such that* $u(\cdot)+\varepsilon\zeta(\cdot)\in\mathcal{U}_{ad}$ *for any* $\varepsilon\in(-\delta,\delta)$ *and*

$$\mathcal{J}(\varepsilon) = J(u(\cdot)+\varepsilon\zeta(\cdot)), \qquad \varepsilon\in(-\delta,\delta)$$

gets its minimum at $\varepsilon=0$*. Moreover, suppose that* $p(\cdot)=p^u(\cdot)$ *is a solution of* (5)*. Then* $u(\cdot)$ *is a stationary point for* $\mathbb{E}[H|\mathcal{G}_t]$*, in the sense that for a.s.* $t\in[0,T]$*, we have*

$$\mathbb{E}[H_v(t,y(t),z(t),\bar z(t),u(t),p(t))|\mathcal{G}_t] = 0.$$

Proof. Let $(y^{u+\varepsilon\zeta}(\cdot), z^{u+\varepsilon\zeta}(\cdot), \bar{z}^{u+\varepsilon\zeta}(\cdot))$ be the trajectory under the control $u(\cdot) + \varepsilon\zeta(\cdot)$. If we define

$$\psi(\cdot) = \frac{d}{d\varepsilon}\varphi^{u+\varepsilon\zeta}(\cdot)|_{\varepsilon=0}, \qquad \text{with } \varphi = y, z, \bar{z}, \qquad \text{and } \psi = y_1, z_1, \bar{z}_1,$$

then we have the following variational equation:

$$\begin{cases} -dy_1(t) = [f_y(t, y(t), z(t), \bar{z}(t), u(t))y_1(t) + f_z(t, y(t), z(t), \bar{z}(t), u(t))z_1(t) \\ \qquad\qquad + f_{\bar{z}}(t, y(t), z(t), \bar{z}(t), u(t))\bar{z}_1(t) + f_v(t, y(t), z(t), \bar{z}(t), u(t))\zeta(t)]dt \\ \qquad\qquad - z_1(t)dW(t) - \bar{z}_1(t)d\overline{W}(t), \\ y_1(T) = 0. \end{cases}$$

Applying Itô's formula to $\langle p(\cdot), y_1(\cdot)\rangle$, we derive

$$\begin{aligned} \mathbb{E}\langle p(0), y_1(0)\rangle = \mathbb{E}\int_0^T &\langle p(t), f_v(t, y(t), z(t), \bar{z}(t), u(t))\zeta(t)\rangle dt \\ - \mathbb{E}\int_0^T &\langle l_y(t, y(t), z(t), \bar{z}(t), u(t)), y_1(t)\rangle dt \\ - \mathbb{E}\int_0^T &\langle l_z(t, y(t), z(t), \bar{z}(t), u(t)), z_1(t)\rangle dt \\ - \mathbb{E}\int_0^T &\langle l_{\bar{z}}(t, y(t), z(t), \bar{z}(t), u(t)), \bar{z}_1(t)\rangle dt. \end{aligned} \tag{6}$$

On the other hand, it is easy to check

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathcal{J}(\varepsilon)|_{\varepsilon=0} \\ &= \mathbb{E}\left\{\int_0^T [l_y(t, y(t), z(t), \bar{z}(t), u(t))^\tau y_1(t) + l_z(t, y(t), z(t), \bar{z}(t), u(t))^\tau z_1(t)\right. \\ &\quad + l_{\bar{z}}(t, y(t), z(t), \bar{z}(t), u(t))^\tau \bar{z}_1(t) + l_v(t, y(t), z(t), \bar{z}(t), u(t))^\tau \zeta(t)]dt \\ &\quad \left. + \phi_y(y(0))^\tau y_1(0)\right\}. \end{aligned} \tag{7}$$

Substituting (6) into (7) and recalling (H3), we get

$$\begin{aligned} 0 &= \mathbb{E}\int_0^T \langle H_v(s, y(s), z(s), \bar{z}(s), u(s), p(s)), \zeta(s)\rangle ds \\ &= \mathbb{E}\int_t^{t+h} \langle H_{v_i}(s, y(s), z(s), \bar{z}(s), u(s), p(s)), \eta\rangle ds, \end{aligned} \tag{8}$$

where $i = 1, 2, \ldots, k$. Differentiating with respect to h at $h = 0$ gives

$$\mathbb{E}\langle H_{v_i}(t, y(t), z(t), \bar{z}(t), u(t), p(t)), \eta\rangle = 0, \quad i = 1, 2, \ldots, k.$$

Since the above equality holds for any bounded $\mathcal{G}_t$-measurable η, we conclude that

$$\mathbb{E}[H_{v_i}(t, y(t), z(t), \bar{z}(t), u(t), p(t))|\mathcal{G}_t] = 0, \quad i = 1, 2, \ldots, k$$

holds as claimed. $\square$

In particular, if we let $\mathcal{G}_t = \mathcal{F}_t$, $t \in [0,T]$, then the following result can be obtained immediately from Theorem 3.1.

COROLLARY 3.1. *Let* (H1)–(H4) *and* $\mathcal{G}_t = \mathcal{F}_t$, $t \in [0,T]$ *hold. Suppose that* $u(\cdot)$ *is a local minimum for* $J(v(\cdot))$, *in the sense that for all bounded* $\zeta(\cdot) \in \mathcal{U}_{ad}$, *there exists* $\delta > 0$ *such that* $u(\cdot) + \varepsilon\zeta(\cdot) \in \mathcal{U}_{ad}$ *for any* $\varepsilon \in (-\delta, \delta)$ *and*

$$\mathcal{J}(\varepsilon) = J(u(\cdot) + \varepsilon\zeta(\cdot)), \qquad \varepsilon \in (-\delta, \delta)$$

has a minimum at $\varepsilon = 0$. *Moreover, suppose that* $p(\cdot) = p^u(\cdot)$ *is a solution of* (5). *Then* $u(\cdot)$ *is a stationary point for the Hamiltonian function* H, *in the sense that for a.s.* $t \in [0,T]$, *we have*

$$H_v(t, y(t), z(t), \bar{z}(t), u(t), p(t)) = 0.$$

4. Application to LQ problems. Theoretically, the maximum principle presented in section 3 characterizes the optimal control through some necessary conditions. However, it is not immediately feasible to implement such a principle, partially due to the difficulty of computing the optimal filter and uncoupling our backward system. In this section, we present two special partial information LQ backward control problems and show how to explicitly solve them using our maximum principle. These problems are still rather general and have substantial applications.

Example 4.1. The partial information LQ optimal control problem of BSDEs is

$$J(u(\cdot)) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)),$$

where

$$J(v(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T [(y^v(t))^\tau Q(t) y^v(t) + v(t)^\tau R(t) v(t)]dt + (y^v(0))^\tau H y^v(0)\right\} \tag{9}$$

subject to

$$\begin{cases} -dy(t) = (A(t)y(t) + B(t)z(t) + C(t)v(t))dt - z(t)dW(t) - \bar{z}(t)d\overline{W}(t), \\ \quad y(T) = \xi. \end{cases} \tag{10}$$

Here,

$$\begin{aligned}
&A(\cdot) \in L^\infty(0,T;\mathfrak{R}^{n\times n}), \quad B(\cdot) \in L^\infty(0,T;\mathfrak{R}^{n\times n}), \quad C(\cdot) \in L^\infty(0,T;\mathfrak{R}^{n\times k}),\\
&Q(\cdot) \in L^\infty(0,T;S^n), \quad Q(\cdot) \geq 0, \quad R(\cdot) \in L^\infty(0,T;S^k), \quad R(\cdot) > 0,\\
&H \in S^n, \quad H > 0, \quad \xi \in L^2_{\mathcal{F}_T}(\Omega;\mathfrak{R}^n).
\end{aligned}$$

It is well known that Wonham's separation theorem [18] is an important tool to solve partial information LQ problems for forward stochastic control systems. Since the running cost of (9) is quadratic with respect to the trajectory $y(\cdot)$, Wonham's separation theorem does not work in this situation. However, the maximum principle developed in section 3 provides an alternative technique. In the following, we will use it to solve our problem in three steps.

Step 1 (Optimal control).

The corresponding Hamiltonian function is given by

$$H(t,y,z,v,p) = \langle A(t)y + B(t)z + C(t)v, p\rangle + \frac{1}{2}(y^\tau Q(t)y + v^\tau R(t)v),$$

where the adjoint process $p(\cdot)$ satisfies

$$\begin{cases} dp(t) = (Q(t)y(t) + A(t)^\tau p(t))dt + B(t)^\tau p(t)dW(t), \\ p(0) = Hy(0). \end{cases} \tag{11}$$

Note that the Hamiltonian function H is quadratic with respect to v whose coefficient $R(\cdot) > 0$; we claim that there is an optimal control. From Theorem 3.1, if $u(\cdot)$ is optimal, then

$$\mathbb{E}[R(t)u(t) + C(t)^\tau p(t)|\mathcal{G}_t] = 0,$$

i.e., the optimal control is of the form

$$u(t) = -R(t)^{-1}C(t)^\tau \mathbb{E}[p(t)|\mathcal{G}_t], \tag{12}$$

where $p(\cdot)$ is the solution of (11).

Step 2 (Optimal filtering with $\mathcal{G}_t = \sigma\{W(s); 0 \leq s \leq t\}$).

Now we aim to give a more explicit expression of $u(\cdot)$ for the special case of $\mathcal{G}_t = \sigma\{W(s); 0 \leq s \leq t\}$. We must compute the optimal filter of $(p(t), y(t))$ based on the observable filtration $\mathcal{G}_t$ at time $t \in [0, T]$. Note that $(p(\cdot), y(\cdot))$ satisfies the generalized Hamiltonian system (10) and (11); thus $(p(\cdot), y(\cdot))$ becomes a coupled system which is impossible to be separated in the sense of Wonham [18]. Meanwhile, to our best knowledge, there is no general filtering result for such kind of Hamiltonian system except that of Wang and Wu [15], where a filtering problem for linear forward and backward stochastic differential equations (FBSDEs) was studied by a "four-step scheme" (see, e.g., Ma and Yong [9], Yong and Zhou [23]). Unfortunately, their setup is more restrictive, and the result derived there is not readily suitable to our problem. Nevertheless, it is remarkable that there do exist rich literatures of filtering theory to forward SDEs (for more details, see Liptser and Shiryayev [8], Xiong [21], and the references therein); thus it is natural to use these classical filtering equations to solve our problem. To get it, first let

$$\hat{x}(t) = \mathbb{E}[x(t)|\mathcal{G}_t], \qquad \text{with } x = y, z, p.$$

Recall in the adjoint equation (11), $p(\cdot)$ depends on the trajectory $y(\cdot)$. However, if we fix the trajectory, then $p(\cdot)$ actually satisfies some forward SDE which is well-posed if we note that $y(\cdot) \in L^2_{\mathcal{F}}(0, T; \Re^n)$. Now fix $y(\cdot)$ in (11), then from Liptser and Shiryayev [8] or Xiong (see [21], Lemma 5.4), we have

$$\begin{cases} d\hat{p}(t) = (Q(t)\hat{y}(t) + A(t)^\tau \hat{p}(t))dt + B(t)^\tau \hat{p}(t)dW(t), \\ \hat{p}(0) = H\hat{y}(0). \end{cases} \tag{13}$$

The remainder of this step is to compute $\hat{y}(\cdot)$. Recall (10), and note that the observable filtration is $\mathcal{G}_t$, then apply Lemma 5.4 in [21] to $y(t)$, and we obtain

$$\begin{cases} -d\hat{y}(t) = (A(t)\hat{y}(t) + B(t)\hat{z}(t) - C(t)R(t)^{-1}C(t)^\tau \hat{p}(t))dt - \hat{z}(t)dW(t), \\ \hat{y}(T) = \mathbb{E}[\xi|\mathcal{G}_T]. \end{cases} \tag{14}$$

Now it is noted that (14) is a *backward* stochastic differential filtering equation (BSDFE), which is different from the classical filtering equations. The filtering estimate $(\hat{p}(\cdot), \hat{y}(\cdot), \hat{z}(\cdot))$ satisfies (13) and (14), which is a coupled FBSDE and admits a unique solution. We call it a kind of coupled FBSDFE. To our best knowledge, this is a kind of new filtering equation. We emphasize that the FBSDFEs arise naturally from our derivations to (13) and (14), thus they cannot really be viewed as an "artificial" ones.

Step 3 (Optimal feedback).

Recall the initial condition of (13). We put

$$\hat{p}(t) = \psi(t)\hat{y}(t), \quad \psi(0) = H,$$

where $\psi(\cdot)$ is a deterministic function defined later on. Apply Itô's formula to $\hat{p}(\cdot)$,

$$\begin{aligned} d\hat{p}(t) &= \dot{\psi}(t)\hat{y}(t)dt + \psi(t)d\hat{y}(t) \\ &= \{\dot{\psi}(t)\hat{y}(t) + \psi(t)[C(t)R(t)^{-1}C(t)^\tau \hat{p}(t) - A(t)\hat{y}(t) - B(t)\hat{z}(t)]\}dt \\ &\quad + \psi(t)\hat{z}(t)dW(t). \end{aligned} \tag{15}$$

Comparing the drift and diffusion terms of (13) and (15), we have

$$\begin{aligned} &\dot{\psi}(t)\hat{y}(t) + \psi(t)[C(t)R(t)^{-1}C(t)^\tau \hat{p}(t) - A(t)\hat{y}(t) - B(t)\hat{z}(t)] = Q(t)\hat{y}(t) + A(t)^\tau \hat{p}(t), \\ &\psi(t)\hat{z}(t) = B(t)^\tau \hat{p}(t). \end{aligned}$$

Then it follows that

$$\begin{cases} \dot{\psi}(t) - A(t)^\tau \psi(t) - \psi(t)A(t) - \psi(t)B(t)\psi(t)^{-1}B(t)^\tau \psi(t) \\ \qquad\qquad + \psi(t)C(t)R(t)^{-1}C(t)^\tau \psi(t) - Q(t) = 0, \\ \psi(0) = H. \end{cases} \tag{16}$$

PROPOSITION 4.1. *If all the hypotheses hold, then the optimal control* $u(\cdot)$ *can be rewritten as*

$$u(t) = -R(t)^{-1}C(t)^\tau \psi(t)\hat{y}(t),$$

where $\hat{y}(\cdot)$ *and* $\psi(\cdot)$ *are given by* (14) *and* (16).

Remark 4.1. We can consider a more general state equation

$$\begin{cases} -dy(t) = (A(t)y(t) + B(t)z(t) + \bar{B}(t)\bar{z}(t) + C(t)v(t))dt - z(t)dW(t) - \bar{z}(t)d\overline{W}(t), \\ \quad y(T) = \xi. \end{cases} \tag{17}$$

In this case, the optimal control is still given by the formula

$$u(t) = -R(t)^{-1}C(t)\hat{p}(t),$$

where the optimal filtering state process satisfies

$$\begin{cases} -d\hat{y}(t) = (A(t)\hat{y}(t) + B(t)\hat{z}(t) + \bar{B}(t)\hat{\bar{z}}(t) - C(t)R(t)^{-1}C(t)^\tau \hat{p}(t))dt - \hat{z}(t)dW(t), \\ \quad \hat{y}(T) = \mathbb{E}[\xi|\mathcal{G}_T], \end{cases} \tag{18}$$

with the adjoint process

$$\begin{cases} d\hat{p}(t) = (Q(t)\hat{y}(t) + A(t)^\tau \hat{p}(t))dt + B(t)^\tau \hat{p}(t)dW(t), \\ \hat{p}(0) = H\hat{y}(0). \end{cases} \tag{19}$$

Note that, in this case the solution of FBSDFEs (18), (19) is not unique. However, if we fix $\hat{\bar{z}}(t)$, then the corresponding FBSDFEs determine a unique solution $(\hat{y}^{\hat{\bar{z}}}, \hat{z}^{\hat{\bar{z}}}, \hat{p}^{\hat{\bar{z}}})$

where the superscripts emphasize the dependence on $\hat{\hat{z}}(t)$. These correspond to a family of stationary points of the original control problem (17), (9). To solve the original problem, we need to seek the best solution within the family $(\hat{y}^{\hat{\hat{z}}}, \hat{z}^{\hat{\hat{z}}}, \hat{p}^{\hat{\hat{z}}})$. Via this, we convert the original problem to the following optimization problem with $\hat{\hat{z}}$ as control and $(\hat{y}, \hat{z}, \hat{p})$ as state. The state equations are (18), (19), and the cost function $\hat{J}$ is then

$$\begin{aligned}\hat{J}(\hat{\hat{z}}) &= J(u(\cdot)) = J(-R(\cdot)^{-1}C(\cdot)\hat{p}^{\hat{\hat{z}}}(\cdot))\\ &= \frac{1}{2}\mathbb{E}\left\{\int_0^T [(\hat{y}^{\hat{\hat{z}}}(t))^\tau Q(t)\hat{y}^{\hat{\hat{z}}}(t) + (\hat{p}^{\hat{\hat{z}}}(t))^\tau C(t)R(t)^{-1}C(t)^\tau \hat{p}^{\hat{\hat{z}}}(t)]dt\right.\\ &\qquad\qquad \left. + (\hat{y}^{\hat{\hat{z}}}(0))^\tau H\hat{y}^{\hat{\hat{z}}}(0)\right\}.\end{aligned} \tag{20}$$

This is a full information control problem with state given by FBSDE, and it can be solved using the maximum principle of an FBSDE system.

Another interesting example is as follows.

Example 4.2. Suppose that all the hypotheses used in Example 4.1 hold except that the cost functional (9) is replaced by

$$J(v(\cdot)) = \frac{1}{2}\mathbb{E}\left[\int_0^T v(t)^\tau R(t)v(t)dt + y^v(0)\right]. \tag{21}$$

Since the running cost of (21) does not contain the trajectory $y^v(\cdot)$ and $y^v(0)$ is a constant, Wonham's separation theorem holds in this situation. According to the theorem, we need first to compute the optimal filtering estimate $(\hat{y}(\cdot), \hat{z}(\cdot))$ and then to solve a full information optimization problem. Similar to Example 4.1, we derive the following BSDFE:

$$\begin{cases} -d\hat{y}(t) = (A(t)\hat{y}(t) + B(t)\hat{z}(t) + C(t)v(t))dt - \hat{z}(t)dW(t),\\ \hat{y}(T) = \mathbb{E}[\xi|\mathcal{G}_T]. \end{cases}$$

In addition, the cost functional (21) is equivalent to

$$J(v(\cdot)) = \frac{1}{2}\mathbb{E}\left[\int_0^T v(t)^\tau R(t)v(t)dt + \widehat{y^v}(0)\right].$$

Therefore, the original problem is equivalent to some full information optimization one. We write down the Hamiltonian function

$$H(t,y,z,v,p) = \langle A(t)y + B(t)z + C(t)v, p\rangle + \frac{1}{2}v^\tau R(t)v,$$

where the adjoint process $p(t)$ is $\mathcal{G}_t$-adapted and satisfies

$$\begin{cases} dp(t) = A(t)^\tau p(t)dt + B(t)^\tau p(t)dW(t),\\ p(0) = I_{n\times 1}. \end{cases} \tag{22}$$

Thus if $u(\cdot)$ is optimal, then we have

$$R(t)u(t) + C(t)^\tau p(t) = 0,$$

i.e., the optimal control is

$$u(t) = -R(t)^{-1}C(t)^{\tau}p(t), \tag{23}$$

where $p(\cdot)$ is the solution of (22).

PROPOSITION 4.2. *If all the hypotheses hold, then the optimal control* $u(\cdot)$ *is given by* (23).

5. Application to pension fund problems. In this section, we present an LQ backward control problem of the defined benefit (DB) pension fund. It is well known that a pension fund can be classified into two main categories: Defined benefit (DB) pension scheme and defined contribution (DC) pension scheme. In a DB scheme, the benefits are fixed in advance by the sponsor, and the contributions are designed to assure the future payments to claim holders in their retirement period. We consider a continuous-time setup, and the dynamics of pension fund is given by

$$dF(t) = F(t)d\Delta(t) + (C(t) - DB)dt,$$

where $F(t)$ is the pension fund at time t, $d\Delta(t)$ is the instantaneous return during the time interval $(t, t+dt)$, $C(t)$ is the contribution rate which acts as our control variable, and DB is the pension scheme benefit outgo which is assumed to be a constant for sake of simplicity. Suppose that the pension fund is invested in a risk-free asset (bond) and a risky asset (stock). The dynamics of the bond is

$$dS_0(t) = r(t)S_0(t)dt,$$

where $r(t)$ is the interest rate at time t. Meanwhile, the dynamics of the stock follows

$$dS_1(t) = \mu(t)S_1(t)dt + \sigma(t)S_1(t)dW(t), \tag{24}$$

where $(W(\cdot))$ is an $\mathfrak{R}$-valued standard Brownian motion, $\mu(\cdot)$ is its instantaneous rate of return, and $\sigma(\cdot)$ is its instantaneous volatility. Suppose that the proportion $\pi(t)$ of the pension fund is to be allocated in the stock, while $1 - \pi(t)$ is to be allocated in the bond. Thus the instantaneous return becomes

$$d\Delta(t) = [r(t) + (\mu(t) - r(t))\pi(t)]dt + \sigma(t)\pi(t)dW(t).$$

Hence the pension fund dynamics can be written as the following form:

$$dF(t) = [r(t)F(t) + (\mu(t) - r(t))\pi(t)F(t) + C(t) - DB]dt + \sigma(t)\pi(t)F(t)dW(t).$$

Let $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathfrak{R}_+)$. If the pension fund manager wants to achieve the wealth level ξ at the terminal time T to fulfill his/her obligations, then the dynamics of the fund is

$$\begin{cases} dF(t) = [r(t)F(t) + (\mu(t) - r(t))\pi(t)F(t) + C(t) - DB]dt + \sigma(t)\pi(t)F(t)dW(t), \\ F(T) = \xi. \end{cases}$$

On the other hand, if we set $\sigma(\cdot)\pi(\cdot)F(t) = Z(\cdot)$, then the above equation is equivalent to

$$\begin{cases} -dF(t) = -\left[r(t)F(t) + \dfrac{\mu(t) - r(t)}{\sigma(t)}Z(t) + C(t) - DB\right]dt - Z(t)dW(t), \\ F(T) = \xi. \end{cases}$$

This is a standard BSDE. To guarantee the existence and uniqueness of its solution, we assume the following.

Assumption (H5). The market coefficients $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$, and $\sigma^{-1}(\cdot)$ are uniformly bounded and deterministic in $[0, T]$.

Set $\mathcal{U}_{ad} = L^2_{\mathcal{F}}(0, T; \mathfrak{R}_+)$. An element of $\mathcal{U}_{ad}$ is called admissible. For any $C(\cdot) \in \mathcal{U}_{ad}$, it is easy to see that the above BSDE admits a unique solution under (H5). Let us introduce a cost functional

$$J(C(\cdot)) = \mathbb{E}\left[\int_0^T \frac{1}{2}e^{-\beta t}(C(t) - NC)^2 dt + F(0)\right], \tag{25}$$

where β is a discount factor and NC is a preset target, say, the normal cost.

The aim of the fund manager is to minimize the cost function $J(C(\cdot))$ over $\mathcal{U}_{ad}$. Recall that the first term of $J(C(\cdot))$ is the running cost due to the deviation of the contribution rate from the preset target level. This term is introduced here to measure the stability of our DB pension scheme. On the other hand, the second term $F(0)$ is just the initial reserve to operate the scheme. There is much literature to study the stochastic optimization of pension funds, such as Chang, Tzeng, and Miao [3], Owadally and Haberman [11], Ngwira and Gerrard [10], etc. However, our problems are essentially different in that we study the optimal pension fund problem in the framework of LQ backward controls. Therefore, our work may be regarded as a contribution to this research domain but from a rather different viewpoint (*backward, linear quadratic*). To solve this problem, we write down the Hamiltonian function

$$H(t, F, Z, C, p) = \frac{1}{2}e^{-\beta t}(C - NC)^2 - \left(r(t)F + \frac{\mu(t) - r(t)}{\sigma(t)}Z + C - DB\right)p,$$

where the adjoint process $p(\cdot)$ satisfies

$$\begin{cases} dp(t) = -r(t)p(t)dt - \dfrac{\mu(t) - r(t)}{\sigma(t)}p(t)dW(t), \\ p(0) = 1. \end{cases} \tag{26}$$

According to Corollary 3.1, the optimal contribution rate $C^*(\cdot)$ should satisfy

$$C^*(t) = e^{\beta t}p(t) + NC, \tag{27}$$

where $p(\cdot)$ is given by (26).

PROPOSITION 5.1. *Let* (H5) *hold. If* $p(\cdot)$ *satisfies the adjoint equation* (26)*, then the optimal contribution rate is given by* (27).

Now introducing the *risk premium*

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \tag{28}$$

we can rewrite (26) as

$$\begin{cases} -\dfrac{dp(t)}{p(t)} = r(t)dt + \theta(t)dW(t), \\ p(0) = 1. \end{cases} \tag{29}$$

It follows that

$$p^{-1}(t) = \exp\left\{\int_0^t r(s)ds - \frac{1}{2}\int_0^t \theta^2(s)ds + \int_0^t \theta(s)dW(s)\right\}. \tag{30}$$

Remark 5.1. The process $p(t)$ is similar to the "shadow price" discussed in Yong and Zhou [23]. Note that $p(\cdot) > 0$, thus $C^*(\cdot) > NC$; that is, the optimal contribution rate is always more than the preset target, say, NC. Meanwhile, the optimal contribution rate does not depend on the benefit rate DB.

Remark 5.2. From (30) and (24), we conclude that $p(\cdot)$ decreases as the stock price $S_1(\cdot)$ increases and vice versa under some suitable conditions. In other words, the higher the stock price, the less the contribution rate we need to achieve the optimality. This is reasonable because if the return in a risky asset is higher, then the fund managers prefer to charge the claim holder less so as to make the pension scheme more attractive.

Remark 5.3. Note that we allow for the possibility of $\beta < 0$, and if we let $\beta \to -\infty$, then in this case, the weight of term $(C(t) - NC)^2$ becomes very large; hence the cost functional takes more account of the first term than the second term. In other words, we care more about the deviations of our contribution rate from NC. Consequently, we need only set the optimal contribution rate to be NC because in this situation, the deviation is always zero.

Remark 5.4. If we assume $\mu(\cdot) = r(\cdot)$, then $\theta(\cdot) \equiv 0$. And then,

$$-dp(t) = r(t)p(t)dt.$$

It follows that

$$p(t) = S_0^{-1}(t).$$

Thus if the price of the riskless asset is higher, then we need only set a lower contribution rate to the claim holder. This is similar to Remark 5.2. In addition, if ξ is deterministic, then all investments should be on the riskless asset S_0.

To end this paper, it is remarkable that the cost functional (25) is introduced here primarily for illustration purposes. In fact, it can be the starting point to the study of much more challenging and complicated cost functionals. We leave this as our future research work.

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