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An analytic criterion for generalized synchronization in unidirectionally coupled systems based on the auxiliary system approach

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An analytic criterion is developed to investigate generalized synchronization (GS) in unidirectionally coupled systems based on the auxiliary system approach. The criterion is derived by transforming the existence problem of generalized synchronization into an eigenvalue problem. Numerical simulations show that the analytic criterion is almost as accurate as the response Lyapunov exponents method, and may provide an estimation of the threshold of strong generalized synchronization. A significant result can be deduced from our analysis that the more the number of equilibria of the unidirectionally coupled systems, the greater the chance of generalized synchronization, but the harder it may be for strong generalized synchronization to occur. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4748862]

Generalized synchronization (GS) in unidirectionally coupled systems is an important issue both theoretically and practically. It is often understood as a chaotic regime in which the amplitude of the response's state variable correlates with that of the drive's by a generic function. GS has robustness with respect to parameter mismatches in the physical world. Abarbanel *et al.*[10] proposed the well-known auxiliary system approach to detect the occurrence of GS. However, the approach can only be implemented numerically in most cases. On the other hand, the approach provides nothing (such as continuity, smoothness) about the generic function between drive and response dynamical variables if GS occurs. This paper tries to develop an analytic criterion based on the auxiliary system approach to make up part of the scarcity mentioned above. Numerical results obtained by the response Lyapunov exponents (RLEs) method demonstrate the effectiveness of the analytical criterion. An example of application of the criterion is given to numerically illustrate that the criterion can provide an estimation of the threshold of strong GS. Theoretic analysis and numerical simulations show that whether GS occurs easily may be related to the number of equilibria of the unidirectionally coupled systems.

I. INTRODUCTION

Chaos synchronization has been studied with increasing interest over the past few decades due to its numerous potential applications in various fields, such as biology, ecology, sociology, and technology.[1,2] Chaos synchronization, according to Pecora and Carroll,[3] refers to a coincidence of the states of two chaotic systems for $t \to +\infty$, which is called complete synchronization (CS). CS only appears when two interacting systems are identical. However, it is not physically possible to construct two absolutely identical systems. If the parameters of the coupled systems are mismatched, the manifold of CS will probably be destroyed, which limits its practical applications. Consequently, as an extension of CS, GS[4,6,15] has been introduced for drive-response systems, which means there is a functional relation between drive and response systems after the transient is finished. It has been verified by several authors[4,6] that GS is a robust phenomenon. GS between two chaotic systems may remain unchanged even after a change in their parameters, which extends its application ranges.

GS is more intricate because of the complexity of the functional relation between the states of the drive-response system. In the past decade, research on GS between drive-response systems has mostly focused on the following three respects: (1) how to detect the existence of the transformation between drive and response dynamical variables,[4,6,10–13], (2) the complexity and smoothness of the function mapping states of the drive into the state space of the response system and the corresponding GS manifold if GS exists,[5,7,8,14–20] and (3) how to design the coupling scheme so that GS conforms to a synchronization function.[10,21–25] For the first problem, Kocarev and Parlitz[6] proved that GS only occurs when the response system is asymptotically stable. Based on their results, the RLEs can be used as a quantitative criterion for GS detection. GS between drive-response systems will hold if all RLEs are negative. The problem of asymptotic stability can also be discussed by applying the Lyapunov function formalism.[9] As an indirect way of verifying GS in two chaotic systems, the auxiliary system approach[10] was proposed. The idea was to construct an identical copy of the response system (the auxiliary system) that was driven by the same driving signal. GS between drive and response systems would appear if CS appeared between the response and auxiliary systems staring

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from with different initial conditions. In addition, some numerical tools[4,11–13] were developed to discern the presence of GS from time series by reconstruction of attractors and calculation of some statistical quantities.

The existence of GS does not imply that the function which maps the states of the drive into the state space of the response system is smooth. Pyragas[15] revealed that GS in unidirectionally coupled chaotic systems develops into two stages as the coupling strength is increased. When the transfer function is differentiable, GS is strong; when it is not differential, GS is weak. In the case of strong GS, the dimension of the strange attractor in the full phase space of the two systems saturates to the dimension of the driving attractor. The threshold of strong GS can be estimated from the Kaplan-Yorke conjecture.[27] Strong GS occurs if the response system has no effect on the global Lyapunov dimension.[15] Kocarev *et al.*[7] gave several examples of chaos synchronization showing that normal hyperbolicity is a necessary and sufficient condition for the GS manifold to be smooth and persistent under small perturbations. Singh *et al.*[16,17] showed that this transition between strong and weak GS usually has different characteristics depending on the manner in which the parameter space is traversed. Other methods[5,18–20] have also been proposed to detect the differentiability of the transfer function.

Since the characteristics of GS are analyzed in different aspects, different approaches to achieve GS have been proposed.[10,21–25] These approaches can be divided into two classes.[26] One is to design control laws to force coupled systems to satisfy a prescribed functional relation. Based on the linear transform method, Yang and Chua[21] presented linear GS of two chaotic systems and illustrated the efficiency using two Chua's circuits. Nevertheless, their method requires the stability of the linear part of the chaotic system, which limits its applications. Cincotti and Teglio[22] also derived some conditions for linear GS using the idea of linearization. Liu *et al.*[23] and Meng *et al.*[24] investigated nonlinear GS of two chaotic systems via nonlinear control. The other is the auxiliary system approach.[10] He *et al.*[25] proposed a more general model to study GS between drive-response systems via the auxiliary system approach. Some less conservative GS conditions have been obtained by introducing the matrix measure.

This paper focuses on the auxiliary system approach.[10] Although the approach is very useful for theoretical analysis of the GS occurrence, two problems remain unsolved. One is that in most cases the approach can only be implemented numerically (i.e., it numerically determines the occurrence of CS between response and auxiliary systems), and another is that it does not provide any transformation between drive and response dynamical variables if GS occurs. CS between response and auxiliary systems only means that GS of drive and response systems appears. Whether GS is discontinuous or smooth cannot be determined by the auxiliary system approach. This paper tries to develop an analytic criterion for detecting GS of drive-response systems to solve the two problems mentioned above based on the auxiliary system approach. Through analyzing the evolution of the trajectories near the CS manifold in the phase space of response and auxiliary systems, the stability problem of the GS manifold in the drive-response systems is equivalently transformed into an eigenvalue problem. Then an analytic criterion can be derived directly by applying the Routh-Hurwitz criterion. Numerical simulations demonstrate that the analytic criterion is almost as accurate as the RLEs method, and can provide an estimation of the threshold of strong GS.

The rest of the paper is organized as follows: The model is described in Sec. II. An analytic criterion for detecting GS is developed in Sec. III. Two simulation examples are provided to illustrate the correctness and effectiveness of the criterion in Sec. IV. As an example of application of the criterion, two different chaotic systems are forced to satisfy a prescribed functional relation via nonlinear control in Sec. V. The conclusion is given in Sec. VI.

II. MODEL DESCRIPTION

The behavior of two unidirectionally coupled chaotic oscillators is considered

$$\begin{cases} \dot{x} = f(x), \\ \dot{y} = g(y,x), \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $\mathbf{R}$ denote the field of real numbers; f, g are continuous vector functions. The first and second systems in Eq. (1) are referred to as a drive and response, respectively. System (1) possesses the property of GS between x and y if there exists a transformation $H : \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{t\to\infty} \|y(t) - H(x(t))\| = 0. \tag{2}$$

III. AN ANALYTIC CRITERION FOR DETECTING GS IN SYSTEM (1)

In this section, an analytic criterion is developed for detecting GS in system (1) based on the auxiliary system approach.[10] According to Rulkov *et al.*,[10] the auxiliary system corresponding to system (1) is given by

$$\dot{z} = g(z,x), \tag{3}$$

where $z \in \mathbf{R}^m$.

Condition (2) can be rewritten as

$$\lim_{t\to\infty} \|y(t) - z(t)\| = 0. \tag{4}$$

We define

$$e_1 = \frac{y-z}{2}, \quad e_2 = \frac{y+z}{2},$$

then systems (1) and (3) become

$$\begin{cases} \dot{e}_1 = \frac{1}{2}[g(e_1+e_2,x) - g(e_2-e_1,x)], \\ \dot{e}_2 = \frac{1}{2}[g(e_1+e_2,x) + g(e_2-e_1,x)], \\ \dot{x} = f(x). \end{cases} \tag{5}$$

Condition (4) is equivalent to

$$\lim_{t\to\infty} \|e_1(t)\| = 0. \tag{6}$$

GS of system (1) occurs if condition (6) is satisfied. Consider the phase space of system (5) presented in Fig. 1. Assume that curve $\widehat{AE}$ represents a trajectory from point A to point E, on which points A, C and points B, D have identical e_1 values, respectively. Then condition (6) means trajectory $\widehat{AE}$ infinitely approaches M axis. Since the distance from point A (C) to M axis is equal to that from point B (D) to M axis, trajectories $\widehat{AB}$ and $\widehat{CD}$ are not necessary to be analyzed when we investigate the trend of curve $\widehat{AE}$. Namely, trajectories $\widehat{BC}$ and $\widehat{DE}$ can be used to determine more directly whether trajectory $\widehat{AE}$ infinitely approaches M axis. It is clear that trajectories $\widehat{BC}$ and $\widehat{DE}$ are continuous at point C (D) in the direction of e_1 axis. Then we can use a smooth trajectory $\widehat{A'E'}$, which is obtained by translating the points on $\widehat{BC}$, $\widehat{DE}$ along M axis, instead of trajectories $\widehat{BC}$ and $\widehat{DE}$ to analyze whether $\widehat{AE}$ infinitely approaches M axis. From Fig. 1, trajectory $\widehat{A'E'}$ can be derived through regarding points on trajectory $\widehat{AE}$ with identical e_1 value as one point. Thus, letting

$$\begin{cases} e_2 = u_1(e_1), \\ x = u_2(e_1), \end{cases} \tag{7}$$

where $u_1 : \mathbf{R}^m \to \mathbf{R}^m$, $u_2 : \mathbf{R}^m \to \mathbf{R}^n$ are smooth vector-valued functions, and substituting them into system (5), yields

$$\begin{cases} \dot{e}_1 = \frac{1}{2}[g(e_1 + u_1(e_1), u_2(e_1)) - g(u_1(e_1) - e_1, u_2(e_1))], \\ \dot{u}_1(e_1) = \frac{1}{2}[g(e_1 + u_1(e_1), u_2(e_1)) + g(u_1(e_1) - e_1, u_2(e_1))], \\ \dot{u}_2(e_1) = f(u_2(e_1)). \end{cases} \tag{8}$$

From the analysis above, we can directly investigate whether $e_1 \to 0$ in system (8) to determine whether $e_1 \to 0$ in system (5). It should be pointed out that all synchronization conditions which will be discussed below are only local.

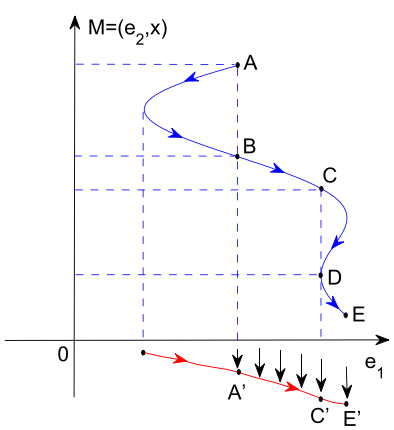

FIG. 1. Analysis for trajectories in the phase space of system (5). Curve $\widehat{AE}$ represents a trajectory from point A to point E. Points A, C and points B, D have identical e_1 values, respectively. Points on trajectories $\widehat{BC}$ and $\widehat{DE}$ are translated along M axis to form a smooth curve $\widehat{A'E'}$, which can reflect more directly whether $\widehat{AE}$ can approach M axis infinitely.

For a sufficiently small e_1, the right hand sides of Eq. (7) can be expanded at zero point as

$$\begin{cases} u_1(e_1) = u_{10} + \dfrac{\partial u_1(0)}{\partial e_1} e_1 + O_1(e_1), \\ u_2(e_1) = u_{20} + \dfrac{\partial u_2(0)}{\partial e_1} e_1 + O_2(e_1), \end{cases} \tag{9}$$

where $u_{10} = u_1(0)$, $u_{20} = u_2(0)$; $O_1(e_1)$, $O_2(e_1)$ represent the higher order terms of e_1.

Then $u_1(e_1)$ and $u_2(e_1)$ can be approximately obtained by substituting Eq. (9) into the last two equations in system (8). System (8) in fact is composed of one differential system (the first equation) and two algebraic equations (the last two equations). To judge whether $e_1 \to 0$, we only need to linearize the first equation in system (8) around $e_1 = 0$

$$\dot{e}_1 = Z(u_{10}, u_{20})\, e_1 + O_3(e_1), \tag{10}$$

where

$$Z(u_{10}, u_{20}) = \left.\frac{\partial g(y,x)}{\partial y}\right|_{y=u_{10}, x=u_{20}}, \tag{11}$$

$O_3(e_1)$ represents the higher order terms of e_1.

According to the well-known Routh-Hurwitz criterion, if all eigenvalues of matrix (11) have negative real parts, the origin of the first equation in system (8) will be asymptotically stable. That is, condition (6) is satisfied, which indicates that GS of system (1) occurs according to the auxiliary system approach. Clearly, we only need calculate u_{10}, u_{20} rather than $u_1(e_1), u_2(e_1)$ to obtain matrix (11). Substituting (9) into the last two equations in system (8), one has

$$\begin{cases} g(u_{10}, u_{20}) = 0, \\ f(u_{20}) = 0. \end{cases} \tag{12}$$

Eq. (12) with real roots means that unidirectionally coupled systems (1) possesses equilibria. For the physical systems in the real world, such condition is very easy to be satisfied. Hence, the criterion (Eqs. (11) and (12)) can be widely used. One thing needs to be pointed out, Eq. (12) may have more than one real root. Assume that (u_{101}, u_{201}), (u_{102}, u_{202}), $\cdots$, (u_{10k}, u_{20k}) are k real roots to Eq. (12). From Fig. 1, it is easy to be understood that there exists trajectories approaching M axis infinitely (GS of system (1) occurs) as long as all eigenvalues of any matrix $Z(u_{10i}, u_{20i})$, $i = 1, 2, \cdots, k$, have negative real parts. The conclusion will be verified through two numerical examples in Sec. IV. In the sense, more equilibria possessed by unidirectionally coupled systems (1) means a higher chance of GS. It was indicated earlier that occurrence of GS does not mean it is strong. In Sec. V, GS of two different chaotic systems with a given transfer function is realized to show that strong GS occurs when all matrixes $Z(u_{101}, u_{201})$, $Z(u_{102}, u_{202})$, $\cdots$, $Z(u_{10k}, u_{20k})$ have no eigenvalue with non-negative real part. In the sense, the more the number of equilibria of the unidirectionally coupled systems, the harder it may be for strong GS to occur.

IV. NUMERICAL EXAMPLES

In this section, two examples are considered numerically to demonstrate the effectiveness of the criterion developed in Sec. III.

A. Example 1

As the first example we select two unidirectionally coupled Rössler oscillators[28]

$$\begin{cases} \dot{x}_d = -\omega_d y_d - z_d, \\ \dot{y}_d = \omega_d x_d + a y_d, \\ \dot{z}_d = p + z_d(x_d - c), \end{cases}$$

$$\begin{cases} \dot{x}_r = -\omega_r y_r - z_r + \epsilon(x_d - x_r), \\ \dot{y}_r = \omega_r x_r + a y_r, \\ \dot{z}_r = p + z_r(x_r - c), \end{cases}$$

where ϵ is a coupling parameter, $a = 0.15$, $p = 0.2$, $c = 10.0$, $\omega_r = 0.95$. Consider a slight mismatch between two coupled Rössler oscillators. Figs. 2(a) and 2(b) demonstrate the dependence of the two largest RLEs of coupled Rössler oscillators on the coupling strength ϵ for slightly ($\omega_d = 0.99$) and greatly ($\omega_d = 1.3$) detuning Rössler systems, respectively. GS is realized when the highest RLEs (corresponding to λ in Fig. 2) is negative. The critical value of ϵ is indicated by a dashed line.

Next, we use our criterion to find the GS region for ϵ. Comparing the coupled Rössler oscillators with system (1), we let

$$f(x) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} -\omega_d x_2 - x_3 \\ \omega_d x_1 + a x_2 \\ p + x_3(x_1 - c) \end{pmatrix},$$

$$g(y,x) : \left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \to \begin{pmatrix} -\omega_r y_2 - y_3 + \epsilon(x_1 - y_1) \\ \omega_r y_1 + a y_2 \\ p + y_3(y_1 - c) \end{pmatrix}.$$

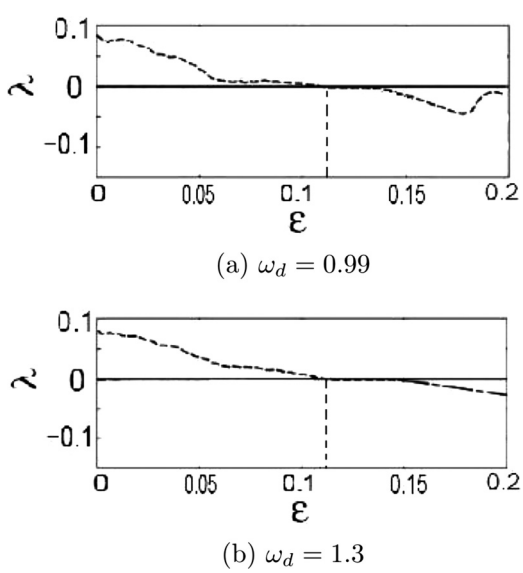

FIG. 2. The dependence of the two RLEs on the parameter ϵ for (a) $\omega_d = 0.99$ (b) $\omega_d = 1.3$. The critical value of ϵ is indicated by a dashed line.

If $\omega_d = 0.99$, from Eq. (12), $u_{20} = (u_{20}^1, u_{20}^2, u_{20}^3)^T$ can be derived according to the following equations:

$$\begin{cases} -\omega_d u_{20}^2 - u_{20}^3 = 0, \\ \omega_d u_{20}^1 + a u_{20}^2 = 0, \\ p + u_{20}^3 (u_{20}^1 - c) = 0. \end{cases}$$

Then

$$u_{20} = (00031, -0.0202, 002)^T, \text{ or}$$
$$u_{20} = (99969, -65.9798, 6532)^T.$$

$u_{10} = (u_{10}^1, u_{10}^2, u_{10}^3)^T$ can be obtained by solving the following equations:

$$\begin{cases} -\omega_r u_{10}^2 - u_{10}^3 + \epsilon (u_{20}^1 - u_{10}^1) = 0, \\ \omega_r u_{10}^1 + a u_{10}^2 = 0, \\ p + u_{10}^3 (u_{10}^1 - c) = 0. \end{cases}$$

From Eq. (11), we have

$$Z(u_{10}, u_{20}) = \begin{pmatrix} -\epsilon & -0.95 & -1 \\ 0.95 & 0.15 & 0 \\ u_{10}^3 & 0 & u_{10}^1 - 10 \end{pmatrix}.$$

According the analysis results in Sec. III, the GS region for ϵ is $\epsilon > 0.148$. If $\omega_d = 1.3$, the GS range for ϵ also is $\epsilon > 0.148$. From Figs. 2(a) and 2(b), λ varies very gently in the range $\epsilon \in [0.1, 0.15]$, the critical value of ϵ almost does not depend on the parameter mistuning in the coupled Rössler oscillators. Our analytic result and numerical result agree well.

B. Example 2

In the second example, we choose Chua's circuit for the drive system and a one-dimensional system for the response system[6]

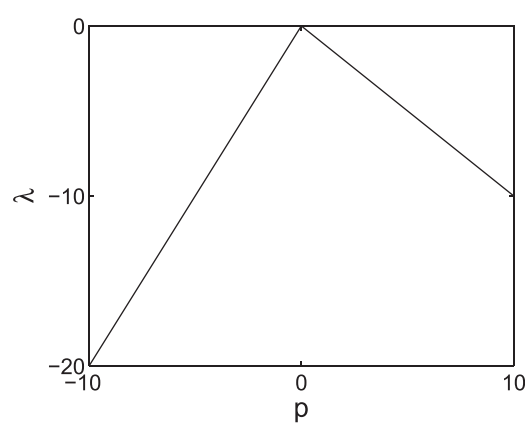

FIG. 3. RLEs of the drive-response system versus parameter p.

$$\begin{cases} \dot{x}_1 = \alpha x_2 - \alpha[m_1 x_1 - m_2(|x_1+1| - |x_1-1|)], \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2, \end{cases}$$

$$\dot{y} = -\frac{y^3}{10} - y(x_1 + p),$$

where $\alpha = 9$, $\beta = 100/7$, $m_1 = 0.2857$ and $m_2 = 0.2143$. Fig. 3 shows that RLEs of the drive-response system are negative for all values of the parameter p and GS always occurs for the drive and the response.

Next, we use our criterion to find the GS region for p. Comparing the drive-response system with system (1), we let

$$f(x): \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} \alpha x_2 - \alpha[m_1 x_1 - m_2(|x_1+1| - |x_1-1|)] \\ x_1 - x_2 + x_3 \\ -\beta x_2 \end{pmatrix},$$

$$g(y,x): \left(y, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \to -\frac{y^3}{10} - y(x_1 + p).$$

From Eqs. (11) and (12), we have

$$(u_{10}, u_{20}) = (0, (0,0,0)^T),\ Z(u_{10}, u_{20}) = -p$$

$$or\ (u_{10}, u_{20}) = (\pm\sqrt{-10p}, (0,0,0)^T),\ Z(u_{10}, u_{20}) = 2p$$

$$or\ (u_{10}, u_{20}) = (0, (\pm 1.5, 0, \mp 15)^T),\ Z(u_{10}, u_{20}) = \mp 1.5 - p$$

$$or\ (u_{10}, u_{20}) = (\pm\sqrt{\mp 5(3-2p)}, (\pm 1.5, 0, \mp 15)^T),$$

$$Z(u_{10}, u_{20}) = \pm 2(15 + p).$$

Obviously, for any $p \in \mathbf{R}$, there always exists $Z(u_{10}, u_{20}) < 0$. Thus, GS occurs for all $p \in \mathbf{R}$, which illustrates that our analytic result agrees with numerical result very well.

V. AN APPLICATION TO DESIGN THE COUPLING SCHEME TO FORCE COUPLED SYSTEMS TO SATISFY A PRESCRIBED FUNCTIONAL RELATION

From the viewpoint of applications, it is an important issue to realize GS between two chaotic oscillators with a designed functional relation. In this section, to demonstrate the use of our criterion, GS between two different chaotic systems is achieved with a prescribed functional relation via

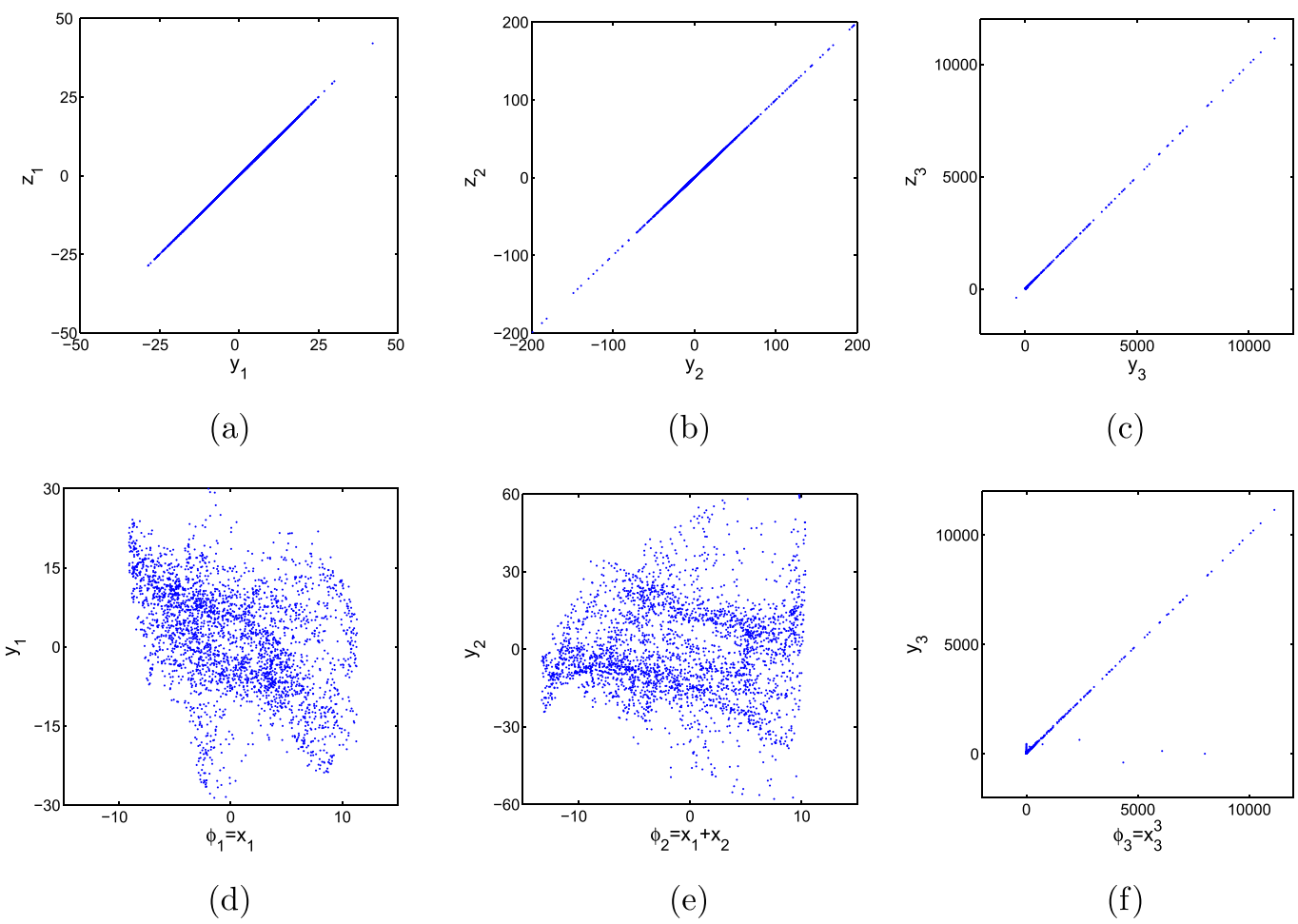

FIG. 4. The projection of the chaotic attractor generated by systems (14) and (16) with $c = -2$ onto the plane (a) (y_1, z_1), (b) (y_2, z_2), (c) (y_3, z_3), which means that GS between systems (13) and (14) occurs. The projection of the chaotic attractor generated by systems (13) and (14) with $c = -2$ onto the plane (d) (ϕ_1, y_1), (e) (ϕ_2, y_2), (f) (ϕ_3, y_3), which means that the GS state between systems (13) and (14) does not completely satisfy the transfer function $y = \phi(x)$.

nonlinear control. By demonstrating how strong GS between two chaotic oscillators occurs with the coupling strength increasing, we numerically show that our criterion can provide an estimation of the threshold of strong GS.

Consider a Rössler system

$$\begin{cases} \dot{x}_1 = 6(-x_2 - x_3), \\ \dot{x}_2 = 6(x_1 + 0.2x_2), \\ \dot{x}_3 = 6(0.2 + x_3(x_1 - 5.7)), \end{cases} \tag{13}$$

drives a Lorenz system

$$\begin{cases} \dot{y}_1 = 10(-y_1 + y_2) + k_1, \\ \dot{y}_2 = 28y_1 - y_2 - y_1y_3 + k_2, \\ \dot{y}_3 = y_1y_2 - 2.666y_3 + k_3, \end{cases} \tag{14}$$

where $k = (k_1, k_2, k_3)^T$ is the controller.

The prescribed functional relation $\phi = (\phi_1, \phi_2, \phi_3)^T$ is given by

$$\phi : (x_1, x_2, x_3)^T \to (x_1, x_1 + x_2, x_3^3)^T.$$

For clarity, denote systems (13) and (14) by $\dot{x} = f_1(x)$ and $\dot{y} = f_2(y) + k$, respectively. The following equation is necessary for occurrence of GS between systems (13) and (14) with the functional relation $y = \phi(x)$:

$$D\phi(x) \cdot f_1(x) = f_2(\phi(x)) + k,$$

where $D\phi(x)$ is the Jacobian matrix of $\phi(x)$. Then controller k can be designed by

$$k = D\phi(x) \cdot f_1(x) - f_2(\phi(x)) + c\,(y - \phi(x)), \tag{15}$$

where $c \in \mathbf{R}$ is the control parameter. That is,

$$\begin{cases} k_1 = -16x_2 - 6x_3 - c\,(y_1 - x_1), \\ k_2 = -21x_1 - 3.8x_2 - 6x_3 + x_1x_3^3 - c\,(y_2 - x_1 - x_2), \\ k_3 = -x_1(x_1 + x_2) + 2.666x_3^3 + 18x_3^2(0.2 + x_3(x_1 - 5.7)) \\ \qquad -c\,(y_3 - x_3^3). \end{cases}$$

We first construct the auxiliary system

$$\begin{cases} \dot{z}_1 = 10(-z_1 + z_2) + k_1, \\ \dot{z}_2 = 28z_1 - z_2 - z_1z_3 + k_2, \\ \dot{z}_3 = z_1z_2 - 2.666z_3 + k_3. \end{cases} \tag{16}$$

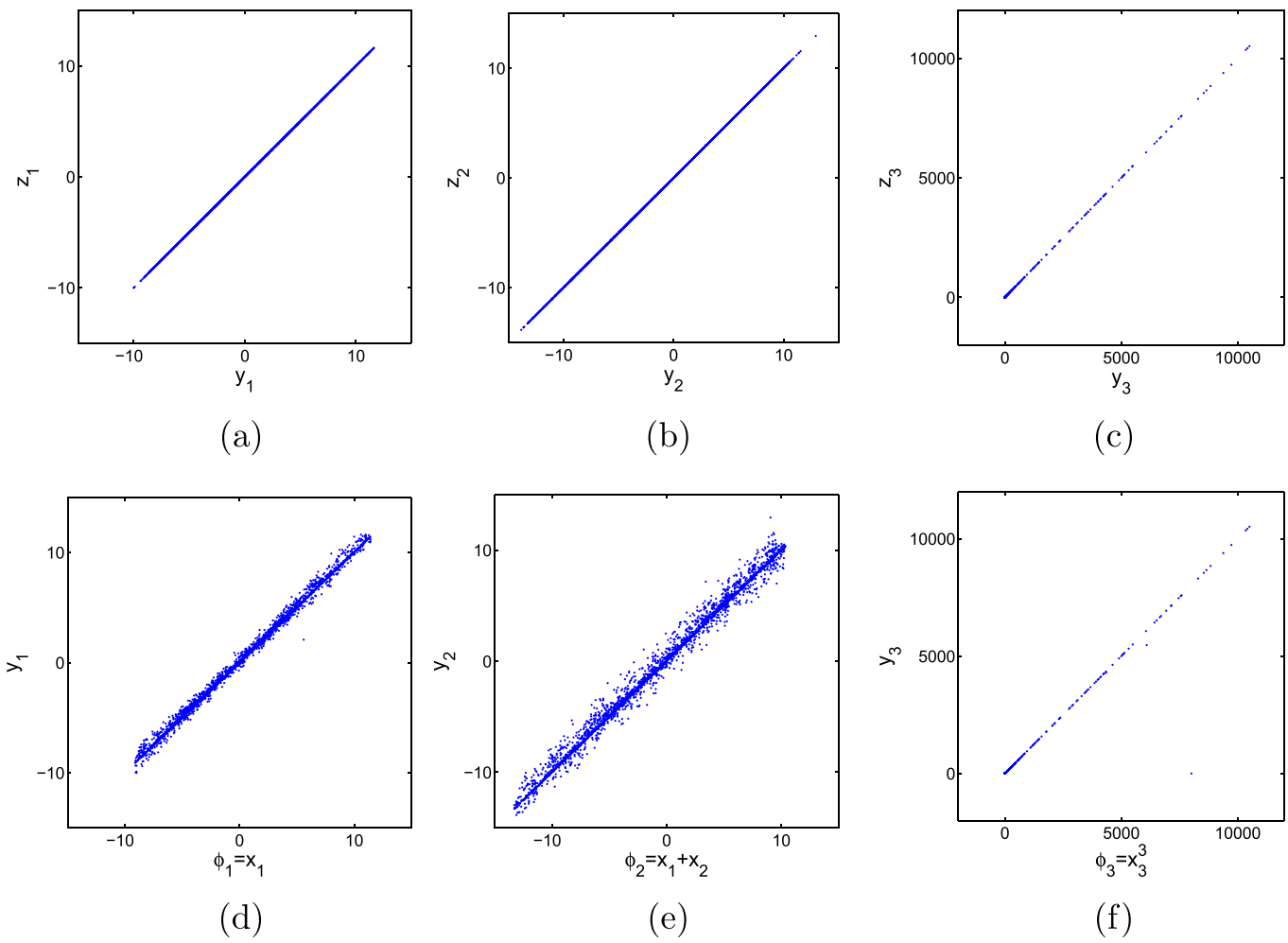

FIG. 5. The projection of the chaotic attractor generated by systems (14) and (16) with $c = 10$ onto the plane (a) (y_1, z_1), (b) (y_2, z_2), (c) (y_3, z_3), which that means GS between systems (13) and (14) occurs. The projection of the chaotic attractor generated by systems (13) and (14) with $c = 10$ onto the plane (d) (ϕ_1, y_1), (e) (ϕ_2, y_2), (f) (ϕ_3, y_3), which means that the GS state between systems (13) and (14) does not completely satisfy the transfer function $y = \phi(x)$.

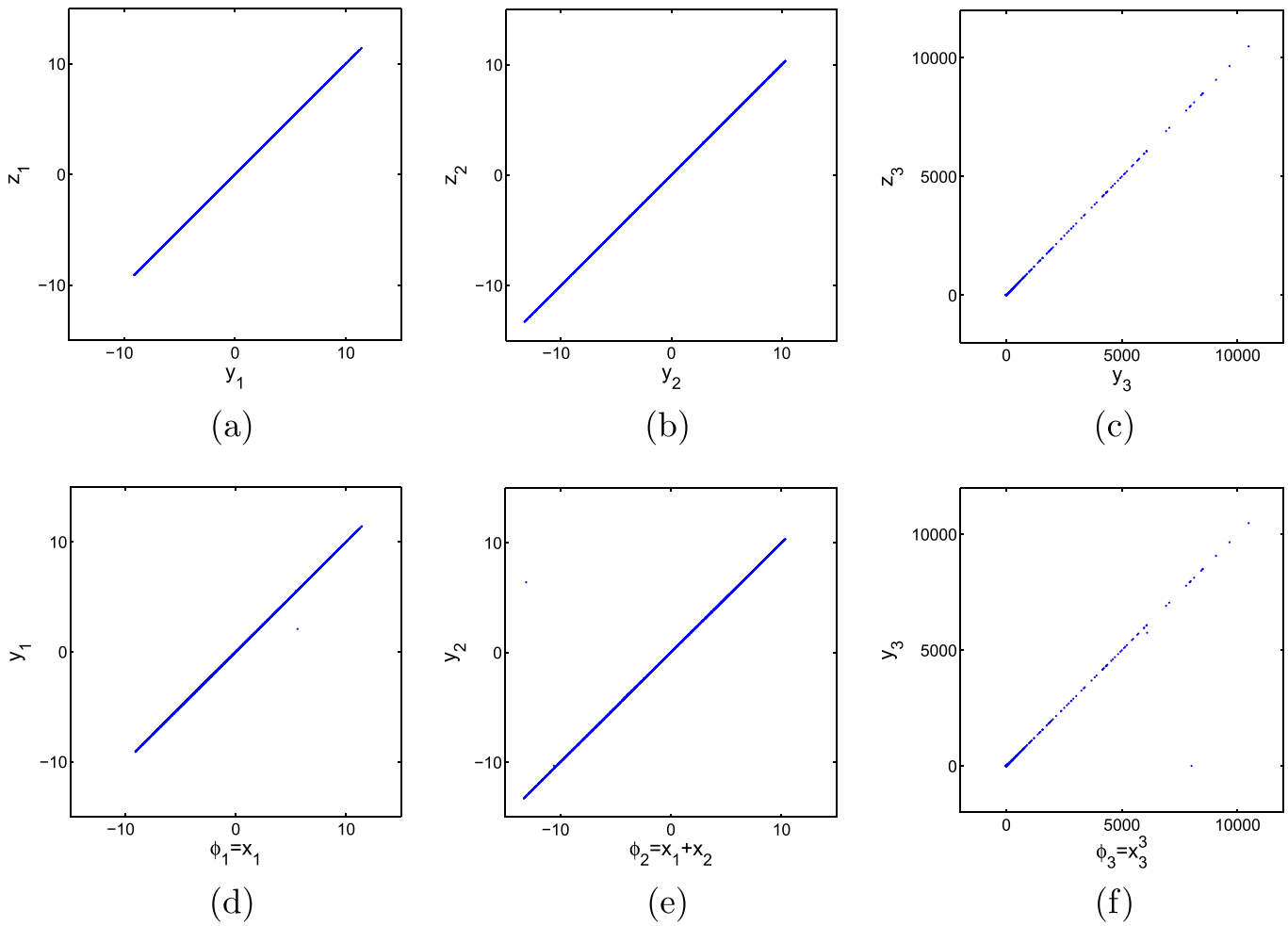

FIG. 6. The projection of the chaotic attractor generated by systems (14) and (16) with $c = 13$ onto the plane (a) (y_1, z_1), (b) (y_2, z_2), (c) (y_3, z_3), which means that GS between systems (13) and (14) occurs. The projection of the chaotic attractor generated by systems (13) and (14) with $c = 13$ onto the plane (d) (ϕ_1, y_1), (e) (ϕ_2, y_2), (f) (ϕ_3, y_3), which means that the GS state between systems (13) and (14) completely satisfies the transfer function $y = \phi(x)$.

Comparing systems (13) and (14) with system (1), we let

$$f(x) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} 6(-x_2 - x_3) \\ 6(x_1 + 0.2x_2) \\ 6(0.2 + x_3(x_1 - 5.7)) \end{pmatrix},$$

$$g(y,x) : \left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \to \begin{pmatrix} 10(-y_1 + y_2) + k_1 \\ 28y_1 - y_2 - y_1y_3 + k_2 \\ y_1y_2 - 2.666y_3 + k_3 \end{pmatrix}.$$

From Eq. (12), we have

$$u_{20} = (u_{20}^1, u_{20}^2, u_{20}^3)^T = (0.007, -0.0351, 0.0351)^T,$$
$$or, u_{20} = (u_{20}^1, u_{20}^2, u_{20}^3)^T = (5.6930, -28.4649, 28.4649)^T.$$

$u_{10} = (u_{10}^1, u_{10}^2, u_{10}^3)^T$ can be obtained by solving the following equations:

$$\begin{cases} -(10+c)u_{10}^1 + 10\,u_{10}^2 + c\,u_{20}^1 - 16\,u_{20}^2 - 6\,u_{20}^3 = 0, \\ (-u_{10}^3 + 28)u_{10}^1 - (1+c)u_{10}^2 + \left(-21 + c + (u_{20}^3)^3\right)u_{20}^1 \\ \quad + (c - 3.8)u_{20}^2 - 6\,u_{20}^3 = 0, \\ u_{10}^1 u_{10}^2 - (2.666 + c)u_{10}^3 + \left(-u_{20}^2 + 18\,(u_{20}^3)^3\right)u_{20}^1 - (u_{20}^1)^2 \\ \quad + 3.6\,(u_{20}^3)^2 + (c - 99.934)(u_{20}^3)^3 = 0. \end{cases} \tag{17}$$

From Eq. (11)

$$Z(u_{10}, u_{20}) = \begin{pmatrix} -10 - c & 10 & 0 \\ 28 - u_{10}^3 & -1 - c & -u_{10}^1 \\ u_{10}^2 & u_{10}^1 & -2.666 - c \end{pmatrix}. \tag{18}$$

When $c > -2.88$, there exists at least one matrix $Z(u_{10}, u_{20})$ (Eq. (18)) whose eigenvalues have no non-negative real part. Then, according to our criterion GS between systems (13) and (14) occurs when $c > -2.88$. But this does not mean that the GS state certainly satisfies the given functional relation $y = \phi(x)$. Fig. 4 shows that though GS between systems (13) and (14) appears when $c = -2$, the transfer function is not continuous. In the numerical simulations, the initial values of the drive system (13), response system (14), and the auxiliary system (16) are always chosen as $(x_1(0), x_2(0), x_3(0)) = (5.6, -20.6, 20.01)$, $(y_1(0), y_2(0), y_3(0)) = (2.08, -2.9, 3.12)$, $(z_1(0), z_2(0), z_3(0)) = (2.08, 2.98, 3.12)$.

In the following, we will show that the GS state completely satisfies $y = \phi(x)$ only when every matrix $Z(u_{10}, u_{20})$ (Eq. (18)) has no eigenvalue with non-negative real part. When $c = -2$, systems (13) and (14) have four equilibria:

$$(u_{101}, u_{201}) = ((0.007, -0.0281, 0)^T, (0.007, -0.0351, 0.0351)^T),$$
$$(u_{102}, u_{202}) = ((4.9141, 3.8976, 28.7588)^T, (0.007, -0.0351, 0.0351)^T),$$
$$(u_{103}, u_{203}) = ((-4.879, -3.9369, 28.7588)^T, (0.007, -0.0351, 0.0351)^T),$$
$$(u_{104}, u_{204}) = ((5.693, -22.7719, 23063.6264)^T, (5.693, -28.4649, 28.4649)^T).$$

Only the eigenvalues of matrix $Z(u_{104}, u_{204})$ have no positive real part when $c = -2$. When c increases to 10, all eigenvalues of matrixes $Z(u_{102}, u_{202})$, $Z(u_{103}, u_{203})$, $Z(u_{104}, u_{204})$ have no positive real part. Fig. 5 shows that the transfer function still is not continuous when $c = 10$. As c goes on increasing, equilibria (u_{102}, u_{202}) and (u_{103}, u_{203}) disappear. All eigenvalues of matrixes $Z(u_{101}, u_{201})$, $Z(u_{104}, u_{204})$ have no positive real part when $c > 11.8278$. Fig. 6 shows that the GS state completely satisfies $y = \phi(x)$ when $c = 13$.

VI. CONCLUSION

In this paper, an analytic criterion is developed to detect the occurrence of GS in unidirectionally coupled systems based on the auxiliary system approach. The criterion works as long as the unidirectionally coupled systems possess equilibria. For the physical systems in the real world, such condition is very easy to be satisfied. Therefore, the criterion can be widely used. How drive and response systems affect the occurrence of GS can be clearly reflected in the criterion. Two numerical examples demonstrate that the analytic criterion has almost the same accuracy as the RLEs method. Through forcing two different chaotic systems to satisfy a given functional relation, we numerically show that the analytic criterion can provide an estimation of the threshold of strong GS.

Theoretic analysis and numerical simulations indicate that, on the one hand, more equilibria possessed by the unidirectionally coupled systems means a higher chance of GS, on the other hand, the more the number of equilibria of the unidirectionally coupled systems, the harder it may be for strong GS to occur. Since strongly nonlinear oscillators, which are general in nature, often have many equilibria, the analysis criterion may explain why GS phenomena exist so widely but strong GS phenomena occur so rarely in the real world.

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[1]A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, 2001).
[2]G. V. Osipov, J. Kurths, and C. Zhou, *Synchronization in Oscillatory Networks* (Springer, Berlin, 2007).
[3]L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[4]N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
[5]N. F. Rulkov and V. S. Afraimovich, Phys. Rev. E **67**, 066218 (2003).
[6]L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[7]L. Kocarev, U. Parlitz, and R. Brown, Phys. Rev. E **61**, 3716 (2000).
[8]A. H. Hu, Z. Y. Xu, and L. X. Guo, Chaos **20**, 013112 (2010).
[9]S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, Reading, MA, 1994).
[10]H. D. I. Abarbanel, N. F. Rulkov, and M. M. Suschik, Phys. Rev. E **53**, 4528 (1996).
[11]J. Arnhold, P. Grassberger, K. Lehnertz, and C. E. Elger, Physica D **134**, 419 (1999).
[12]R. Q. Quian, J. Arnhold, and P. Grassberger, Phys. Rev. E **61**, 5142 (2000).
[13]Z. H. Liu, J. Zhou, and T. Munakata, Europhys. Lett. **87**, 50002 (2009).
[14]K. Josic, Phys. Rev. Lett. **80**, 3053 (1998).
[15]K. Pyragas, Phys. Rev. E **54**, R4508 (1996).
[16]T. U. Singh, A. Nandi, and R. Ramaswamy, Phys. Rev. E **78**, 025205 (2008).
[17]T. U. Singh, H. H. Jafri, and R. Ramaswamy, Phys. Rev. E **81**, 016208 (2010).
[18]B. R. Hunt, E. Ott, and J. A. Yorke, Phys. Rev. E **55**, 4029 (1997).
[19]L. Guo and Z. Xu, Chaos **18**, 033134 (2008).
[20]D. He, Z. Zheng, and L. Stone, Phys. Rev. E **67**, 026223 (2003).
[21]T. Yang and L. O. Chua, Int. J. Bifurcation Chaos **9**, 215 (1999).
[22]S. Cincotti and A. Teglio, *Proc. IEEE International Symposium on Circuits and Systems 2002* (ISCAS 2002), Vol. 3, pp. 61.
[23]S. T. Liu and G. Chen, Chaos, Solitons Fractals **22**, 35 (2004).
[24]J. Meng and X. Y. Wang, Chaos **18**, 023108 (2008).
[25]W. L. He and J. D. Cao, Chaos **19**, 013118 (2009).
[26]H. Liu, J. Chen, J. A. Lu, and M. Cao, Physica A **389**, 1759 (2010).
[27]J. L. Kaplan and J. A. Yorke, *Functional Differential Equations and Approximation of Fixed Points* (Springer-Verlag, Berlin, 1979).
[28]A. E. Hramov and A. A. Koronovskii, Phys. Rev. E **71**, 067201 (2005).