Joint Inventory and Pricing Coordination with Incomplete Demand Information

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Abstract

In retailing operations, retailers face the challenge of incomplete demand information. We develop a new concept named K-approximate convexity, which is shown to be a generalization of K-convexity, to address this challenge. This idea is applied to obtain a base-stock list-price policy for the joint inventory and pricing control problem with incomplete demand information and even non-concave revenue function. A worst-case performance bound of the policy is established. In a numerical study where demand is driven from real sales data, we find that the average gap between the profits of our proposed policy and the optimal policy is 0.27%, and the maximum gap is 4.6%.

Key words: inventory and pricing coordination, incomplete demand information, K-approximate convexity

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1 Introduction

1.1 Motivation

Dynamic pricing is widely accepted by retailers as a powerful tool to better match demand and increase profit. It has become increasingly popular in recent years, mainly due to the advancement of information technology. The new technologies enable not only collecting and analyzing massive amounts of demand data, but also automatically optimizing and adjusting prices in realtime. It should be a perfect time for the joint inventory-pricing models developed in the literature to have an influence in the real world. Unfortunately, their implementation faces two important challenges. First, most existing models assumed completely known mathematical relationship between price and expected demand. In practice, however, a decision maker can only collect a few discrete price points at which the product is sold and the corresponding sales data at each price. Even with the most advanced technology, one cannot completely learn the expected demand as a function of price, because it is simply impossible

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to set the price to its every possible value. Second, most models in the literature also require the revenue function to be concave. Even if a sufficient number of price-demand pairs are available to reasonably well estimate the demand function, there is no guarantee that the demand function or the revenue function has any of the desired properties. The violation of these concave assumptions may result in an optimal policy that is too complicated to be implemented.. These two challenges have created a gap between academic research and practical implementation.

To bridge this gap, we introduce a new tool named K-approximate convexity. A function is K-approximate convex (concave) if it can be approximated by a convex (concave) function whose maximal distance to the original function is K. Hence, K somehow measures the degree of the non-convexity of this function. We show that any K-convex function is also K-approximate convex. Therefore, K-approximate convexity (concavity) is a generalization of K-convexity (concavity). This new methodology is applied to solve our challenges. A piecewise linear concave function that best fits the sales data is used to replace the revenue function in the dynamic programming model. This approach solves the challenges of the unknown demand function and non-concave revenue function simultaneously. The resulting policy is a base-stock list-price policy and hence practically implementable. Under mild conditions, we successfully develop upper bounds on the value K that measures the distance between the true revenue function and its concave approximation. We show the effectiveness of the policy through numerical studies where demand is driven from real sales data.

There are two important questions that determine the complexity and effectiveness of this approach. The first question is how to find a convex function that minimizes the maximal distance between this convex function and the function being approximated. This is shown to be as easy as solving a linear program. The second question is how the policy performs compared to the optimal policy with the complete demand information (the optimal policy may not be practically implementable if the objective function is not concave). We derive a worst-case performance bound that is a linear function of K. Therefore, a slight non-convexity (a small K) of the objective function is acceptable because we can still get a base-stock list-price policy with a good performance by optimizing its convex approximation.

1.2 Literature Review

The benefits of coordinated inventory and pricing decisions have been long recognized in the research community. Whitin (1955), Mills (1959, 1962), and Kalin and Carr (1962) incorporated the pricing decisions into single-period inventory models. The joint inventory and pricing newsvendor models were then developed by, among others, Polatoglu (1991), and Petruzzi and

Dada (1999). Zabel (1972) and Thowsen (1975) were among the first to integrate pricing and inventory in a multi-period stochastic setting. Federgruen and Heching (1999) studied a similar model with more general stochastic demand functions and proved the optimality of the base-stock list-price policy. This pioneering work stimulated a rapidly growing body of research on inventory and pricing coordination. Chen and Simichi-Levi (2004a,b) proved that the (s, S, A, p) and (s, S, p) policies are optimal for the finite and infinite horizon models with fixed ordering cost, respectively. Li and Zheng (2006) extended the work of Federgruen and Heching (1999) by considering random yield. Song et al. (2009) studied lost sales models with fixed ordering cost and established the optimality of the (s, S) type policies. Chen et al. (2010) considered a model with concave ordering cost and showed that a generalized (s, S, p) policy is optimal if the demand distributions are Pólya or uniform. Pang et al. (2012) identified various properties of the optimal policy in the presence of positive leadtimes. Feng et al.(2014) studied the dynamic inventory and pricing control problem under a general demand model. All of these papers assumed completely known mathematical relationship between price and expected demand, and most of them assumed concave revenue function. We relax these two assumptions in this paper.

As our model deals with incomplete demand information, this research is also closely related to the growing literature of dynamic pricing and demand learning. Aviv and Pazgal (2005) and Carvalho and Puterman (2005) were among the the first to incorporate incomplete demand information into the model setting. Subsequent works include Cope(2007), Araman and Caldentey (2009), Besbes and Zeevi (2009), Farias and Van Roy (2010), Harrison et al. (2012), den Boer and Zwart (2014) and Wang et al. (2014) among others. These papers focus on pricing and learning without inventory replenishment decisions. In our model, inventory replenishment is a decision variable in each period while we don't consider demand learning because under a stochastic demand model, a number of data points (periods) are required to estimate the expected demand at a single price point, whereas price is periodically adjusted according to the inventory level in our setting.

2 Model Setting

In this paper, we investigate the joint inventory and pricing control problem in a finite-horizon setting with stochastic demand. The planning horizon consists of T periods. At the beginning of each period, the firm reviews its inventory level and makes pricing and replenishment decisions simultaneously. We assume that the demand in period t as a function of the price

p_t is

$$D_t(p_t) = a_t d(p_t) + b_t, \tag{1}$$

where a_t and b_t are random variables, and $d(p_t)$ is a deterministic function. Note that the demand defined in (1) is more flexible than both the additive and multiplicative demand forms. It reduces to the additive demand form when the coefficient a_t is a constant and to the multiplicative demand form when b_t is a constant. Without loss of generality, we assume that $E[a_t] = 1$ and $E[b_t] = 0$. Hence, the expected demand is $E[D_t(p_t)] = d(p_t)$. We also assume that $d(p_t)$ is a strictly decreasing function of p_t with the inverse function $p^{-1}(d_t)$. The monotonicity of $d(p_t)$ implies that there is a one-to-one correspondence between the expected demand and the selling price. Therefore, we can replace the problem of selecting a price with one of selecting an expected demand.

Define $R(d_t) = p^{-1}(d_t)d_t$, which represents the revenue function in terms of the expected demand. Let $V_t(x_t)$ denote the maximum expected profit given the inventory level x_t at the beginning of period t. Then, the firm faces the following dynamic programming problem:

$$V_t(x_t) = \max_{y_t \geq x_t, d_t \in [\underline{d}_t, \bar{d}_t]} \Big\{ R(d_t) - c_t(y_t - x_t) + E\big[-H_t(y_t - a_t d_t - b_t) + \alpha V_{t+1}(y_t - a_t d_t - b_t)\big] \Big\}, \tag{2}$$

where c_t determines the linear variable ordering cost in period t and $H_t(y_t)$ is the convex inventory holding and shortage cost in that period. Furthermore, we assume that the profit $V_{T+1}(x_{T+1})$ at the end of the planning horizon is concave.

The literature typically assumes that the demand function $d(p_t)$ is already known, which immediately yields the revenue function $R(d_t)$. However, in reality, a complete characterization of the demand or revenue function may not be available or even possible. In most cases, a firm will probably have only exercised several selling prices in the past and observed the corresponding realized demand and hence the expected demand by averaging the realized demand in a certain number of periods. In particular, consider p_0 and p_{N+1} being the lowest and highest possible selling prices. We assume that the firm has already exercised the prices, $p_0 > p_1 > \cdots > p_N > p_{N+1}$, and observed the corresponding expected demand, $d_0 < d_1 < \cdots < d_N < d_{N+1}$. Thus, the firm only knows the expected revenue, $r_0, r_1, ..., r_{N+1}$, where $r_i = p_i d_i$, at the corresponding expected demand, $d_0 < d_1 < ... < d_N < d_{N+1}$. That is, the firm has partial information of the revenue function $R(d_i) = r_i$ at some discrete points d_i, $i = 0, 1, ..., N+1$. Moreover, the revenue function is usually assumed to be concave in the literature. This assumption can also be violated in practice. Furthermore, with the limited information of the revenue function, the firm cannot verify whether the revenue function is concave or not.

Our analysis intends to address the above two issues so that the joint inventory-pricing model can be applied to more practical scenarios. First, we propose an approach in which the joint decisions can be made when the complete information of the revenue function is not available and the revenue function is not concave. Second, we build an upper bound on the potential revenue loss from the proposed policy compared to the optimal policy with complete information of $R(d)$ (the optimal policy may not be practically implementable if $R(d)$ is not concave). Third, we demonstrate the efficiency of our proposed approach using a retail data set.

The proposed policy is a base-stock list-price policy. We focus on this policy because of two reasons. First, it is the optimal policy when the revenue function is completely known and concave. Second, it is an easy-to-implement policy for the joint inventory and pricing control problem. Therefore, this paper studies how to construct a base-stock list-price policy with incomplete demand information and nonconcave revenue function, analyze and test the performance of this policy. For this purpose, we first need to introduce a new methodology of K-approximate Convexity.

3 K-Approximate Convexity

This section introduces the definition and properties of K-approximate convexity and presents the linear programming formulation to find a convex approximation of a piecewise linear function.

As mentioned in Section 1, a function is K-approximate convex if its distance in ℓ_∞ norm to some convex function is no greater than K. This concept is formally defined as follows.

Definition 1. *Let S be a convex set in $\mathbb{R}$. A function $f : S \mapsto \mathbb{R}$, is K-approximate convex (concave) if there exists a convex (concave) function $g : S \mapsto \mathbb{R}$ such that $\|f - g\|_\infty \equiv \sup_{x\in S} |f(x) - g(x)| \le K$.*

Figure 1 illustrates the idea of K-approximate convexity. The function defined by the blue line is non-convex, and is approximated by the convex function defined by the red line. For this case, the maximal distance between the two functions is $\frac{11}{8}$, i.e., $K = \frac{11}{8}$. It is worth mentioning that Neave (1970) used the idea of approximating a nonconvex objective function with convex functions when characterizing the optimal policy of a stochastic cash balance problem. However, he didn't introduce this concept and study its properties.

It is easy to see that the following proposition holds. Its proof is omitted from the paper.

Proposition 1. *(a) Any convex function is 0-approximate convex.*

(b) If $f_1(x)$ is K_1-approximate convex and $f_2(x)$ is K_2-approximate convex, then $\alpha f_1(x) +$

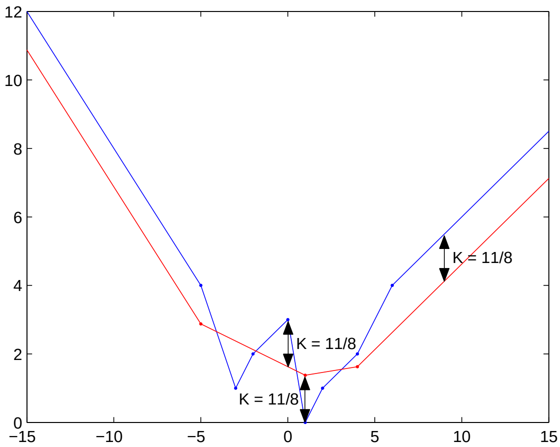

Figure 1: K-approximate Convexity

$\beta f_2(x)$ is $(\alpha K_1 + \beta K_2)$-approximate convex for any $\alpha, \beta \geq 0$.
(c) If $f(x)$ is K-approximate convex, then $E[f(x - D)]$ is K-approximate convex for any random variable D.
(d) If $f(x)$ is K_1-approximate convex, then it must be K_2-approximate convex for any $K_2 \geq K_1$.

The following proposition shows that if a function is K-convex, then it must be $K/2$-approximate convex and hence K-approximate convex. Therefore, K-approximate convexity is a generalization of K-convexity.

Proposition 2. *If $f : \mathbb{R} \mapsto \mathbb{R}$ is a K-convex function, then $f(x)$ is a $K/2$-approximate convex function.*

An important question is how to obtain a convex approximation of a K-approximate convex function such that the distance between the two functions in ℓ_∞ norm is bounded by K. The answer to this question determines the practical value of K-approximate convexity. We will show that if a K-approximate convex function is piecewise linear, then the approximation can be found by solving a linear programming problem.

Consider a piecewise linear function $W(x)$ with m pieces. Let b_j, $j = 0, 1, ..., m-1$, denote the slope for the jth piece and $-\infty = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = +\infty$ denote the breakpoints at which the slopes change. Here, we assume that $b_0 \leq b_{m-1}$. With the additional parameter I_j, $j = 0, 1, ..., m-1$, we can write the function $W(x) = I_j + b_j x$ for any $x \in [x_j, x_{j+1}]$, $j = 0, 1, \ldots, m-1$. Instead of considering all of the convex approximations of $W(x)$, we first limit our choice to continuous piecewise linear convex functions with the same breakpoints as $W(x)$. In other words, we want to find a continuous piecewise linear convex function $\bar{W}(x)$ defined by $\bar{W}(x) = \bar{I}_j + \bar{b}_j x$ for any $x \in [x_j, x_{j+1}]$, $j = 0, 1, \ldots, m-1$, which minimizes the $\max_x |W(x) - \bar{W}(x)|$, the maximum deviation in ℓ_∞ norm.

There are two important observations here. First, in the optimal solution, we must have $\bar{b}_0 = b_0$ and $\bar{b}_{m-1} = b_{m-1}$ to prevent $\max_x |W(x) - \bar{W}(x)|$ from blowing up to infinity when x goes to $-\infty$ or $+\infty$. Second, under $\bar{b}_0 = b_0$ and $\bar{b}_{m-1} = b_{m-1}$, $\max_x |W(x) - \bar{W}(x)|$ must be achieved at some breakpoint x_j, $j = 1, \ldots, m-1$, because both $W(x)$ and $\bar{W}(x)$ are linear for $x \in [x_j, x_{j+1}]$. Therefore, $\min_{\bar{I}_j, \bar{b}_j} \max_x |W(x) - \bar{W}(x)|$ can be reduced to the following optimization problem:

$$
\begin{aligned}
\min_{\bar{I}_j, \bar{b}_j} \quad & \max_{j=1,\ldots,m-1} |W(x_j) - \bar{W}(x_j)| = \min_{\bar{I}_j, \bar{b}_j} \max_{j=1,\ldots,m-1} |I_j + b_j x_j - (\bar{I}_j + \bar{b}_j x_j)| && (3) \\
\text{s.t.} \quad & \bar{I}_j + \bar{b}_j x_{j+1} = \bar{I}_{j+1} + \bar{b}_{j+1} x_{j+1}, \quad j = 0, 1, \ldots, m-2, && (4) \\
& \bar{b}_j \le \bar{b}_{j+1}, \quad j = 0, 1, \ldots, m-2. && (5)
\end{aligned}
$$

Here (4) ensures continuity and (5) ensures convexity. It is easy to see that (3)-(5) can be transformed into the following linear programming problem:

$$
\begin{aligned}
\min_{\bar{I}_j, \bar{b}_j, \zeta} \quad & \zeta && (6) \\
\text{s.t.} \quad & -\zeta \le I_j + b_j x_j - (\bar{I}_j + \bar{b}_j x_j) \le \zeta, \quad j = 1, \ldots, m-1, && (7) \\
& \bar{I}_j + \bar{b}_j x_{j+1} = \bar{I}_{j+1} + \bar{b}_{j+1} x_{j+1}, \quad j = 0, 1, \ldots, m-2, && (8) \\
& \bar{b}_j \le \bar{b}_{j+1}, \quad j = 0, 1, \ldots, m-2. && (9)
\end{aligned}
$$

The construction indicates that $\bar{W}(x)$ is closer to $W(x)$ than any other continuous piecewise linear convex function with the same breakpoints as $W(x)$. Proposition 3 shows that for any K-approximate convex piecewise linear function, a convex approximation within the distance K can be found by solving a linear program.

Proposition 3. *If $W(x)$ is K-approximate convex, then $\|W - \bar{W}\|_\infty \le K$.*

The following lemma provides some properties of K-approximate convexity, which will be useful for proving the worst-case performance bound of the policy proposed in Section 4.

Lemma 1. *For any $f, g : S \mapsto \mathbb{R}$, if $\|f - g\|_\infty \le K$, then*
(a) $|\min_{x \in X} f(x) - \min_{x \in X} g(x)| \le K$ for any $X \subseteq S$;
(b) $g(x_f) - \min_{x \in X} g(x) \le 2K$ for any $X \subseteq S$, where $x_f \in \arg\min_{x \in X} f(x)$.

4 A Base-stock List-price Policy

In this section, we apply the idea of K-approximate convexity to approximate the one-period revenue function by a convex function, which leads us a base-stock list-price policy for the dynamic inventory-pricing control problem. We establish a worst-case performance bound for this policy and test its performance on a retail data set.

4.1 Approximation of the Revenue Function

As aforementioned, the only available information of the revenue function $R(d)$ in the dynamic programming model (2) is $R(d_i) = r_i$ at the discrete points $d_0 < d_1 < \cdots < d_N < d_{N+1}$. A function $\bar{R}(d)$ has to be constructed to approximate $R(d)$ and replace it in (2). The approximate $\bar{R}(d)$ should be concave so that the corresponding policy is well-structured, e.g., a base-stock list-price policy. The approximation must be as close to the true revenue function $R(d)$ as possible, at least at the demand values d_i, $i = 0, ..., N+1$, at which the values of $R(d_i) = r_i$ are known. For any point d in (d_i, d_{i+1}), we have no information about the revenue function $R(d)$. A natural method is to assume that the approximate $\bar{R}(d)$ is linear between d_i and d_{i+1}. Therefore, we consider the function $\bar{R}(d)$, to be a piecewise linear function in the form of

$$\bar{R}(d) = \gamma_i(d - d_i) + \sum_{j=0}^{i-1} \gamma_j(d_{j+1} - d_j) + \theta_0, \quad \text{for any } d \in [d_i, d_{i+1}] \text{ and } i = 0, 1, ..., N, \tag{10}$$

where γ_i represents the slope over $[d_i, d_{i+1}]$, and θ_0 is the function value at $d = d_0$. The concavity of $\bar{R}(d)$ implies that $\gamma_i \geq \gamma_{i+1}$. Furthermore, to find the $\bar{R}(d)$ closest to $R(d)$ at the known pairs (d_i, r_i), we should minimize $\max_{i=0,...,N+1} |\bar{R}(d_i) - r_i|$, which leads to the following linear programming formulation:

$$\begin{aligned} \min_{\gamma_i, \theta_0, \zeta} \quad & \zeta \\ \text{s.t.} \quad & -\zeta \leq \theta_0 - r_0 \leq \zeta, \\ & -\zeta \leq \sum_{j=0}^{i} \gamma_j(d_{j+1} - d_j) + \theta_0 - r_i \leq \zeta, \quad i = 0, 1, \ldots, N, \\ & \gamma_i \geq \gamma_{i+1}, \quad i = 0, \ldots, N-1. \end{aligned} \tag{11}$$

Next, we study how close the approximation $\bar{R}(d)$ is to the true revenue function $R(d)$, which is measured by the distance in ℓ_∞ norm, $\|\bar{R} - R\|_\infty$. To facilitate the analysis, we introduce the piecewise linear function $\hat{R}(d)$ obtained by connecting all of the realized points (d_i, r_i), i.e.,

$$\hat{R}(d) = \beta_i(d - d_i) + \sum_{j=0}^{i-1} \beta_j(d_{j+1} - d_j) + r_0, \quad \text{for and } d \in [d_i, d_{i+1}] \text{ and } i = 0, 1, ..., N,$$

where $\beta_i = \frac{r_{i+1} - r_i}{d_{i+1} - d_i}$ for $i = 0, 1, ..., N$. The following result gives an upper bound on the gap between $\bar{R}(d)$ and $\hat{R}(d)$. Define $\mu_0 = \beta_0$ and $\mu_j = \min\{\mu_{j-1}, \beta_j\} = \min\{\beta_i, i = 0, 1, \ldots, j\}$.

Lemma 2. *The gap between $\bar{R}(d)$ and $\hat{R}(d)$ is bounded by*

$$\|\bar{R} - \hat{R}\|_\infty \leq \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j).$$

With Lemma 2, we can build an upper bound on the gap between $R(d)$ and $\bar{R}(d)$. This gap clearly depends on the properties of $R(d)$. Thus, our analysis will be based on three different assumptions on the revenue function $R(d)$: (i) Lipschitz continuity with the Lipschitz constant L_R, i.e.,

$$|R(d^1) - R(d^2)| \leq L_R|d^1 - d^2|, \quad \text{for any } d^1, d^2 \in [d_0, d_{N+1}], \tag{12}$$

(ii) Lipschitz continuity with the Lipschitz constant L_R and quasi-concavity, and (iii) concavity.

Theorem 1. *(i) Given that $R(d)$ is Lipschitz continuous with the Lipschitz constant L_R, the maximal gap between $R(d)$ and $\bar{R}(d)$ is given by*

$$\|\bar{R} - R\|_\infty \leq \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j) + \frac{L_R}{2}\max_{i=0,\ldots,N}\{d_{i+1} - d_i\};$$

(ii) Given that $R(d)$ is Lipschitz continuous with the Lipschitz constant L_R and quasi-concave, the maximal gap between $R(d)$ and $\bar{R}(d)$ is given by

$$\|\bar{R} - R\|_\infty \leq \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j) + \frac{L_R}{4}\max_{i=0,\ldots,N}\{d_{i+1} - d_i\};$$

(iii) Given that $R(d)$ is concave, the maximal gap between $R(d)$ and $\bar{R}(d)$ is given by

$$\|\bar{R} - R\|_\infty \leq \max_{i=1,\ldots,N}\{(\beta_{i-1} - \beta_i)(d_{i+1} - d_i)\}.$$

Remark 1. *When using a concave function to approximate the unknown function $R(d)$, the approximation gap $\|\bar{R} - R\|_\infty$ is induced by two factors, non-concavity and the unknownness of $R(d)$, which can be measured by $\|\bar{R} - \hat{R}\|_\infty$ and $\|\hat{R} - R\|_\infty$, respectively. Thus, the gaps stated in parts (i) and (ii) consist of two terms, the first from non-concavity (shown in Lemma 2) and the second from unknownness. If $R(d)$ is indeed concave, then $\hat{R}(d)$ must be a concave function and hence the gap resulting from non-concavity vanishes. This is why the gap in part (iii) only has one term resulting from unknownness.*

Similarly, the gap induced by unknownness can be reduced if we know more of the properties of $R(d)$. For instance, the gap from unknownness in part (ii) is only half of that in part (i) because of an additional property – the quasi-concavity of $R(d)$. As Lipschitz continuity is not *required in part (iii), although concavity implies quasi-concavity, we cannot conclude that part (iii) has a smaller gap from unknownness than part (ii). However, for the $R(d)$ that is both Lipschitz continuous and concave, we can show that*

$$\|\bar{R} - R\|_\infty \le \max_{i=1,\dots,N} \left\{ \min \left\{ \frac{L_R}{4}, \beta_{i-1} - \beta_i \right\} \times (d_{i+1} - d_i) \right\}.$$

Furthermore, if the product was sold at more distinct prices, we could observe more data points of the expected demand and hence obtain more pairs of (d_i, r_i), i.e., the value of $d_{i+1} - d_i$ would decrease. In this case, the gap from unknownness in all three cases can be reduced. In particular, it disappears when $R(d)$ is completely known, i.e., when $d_{i+1} - d_i$ goes to zero.

Remark 2. *In parts (i) and (ii), the gap from unknownness is determined by the maximum value of $d_{i+1} - d_i$. This result is of particular importance when we have already collected the price-demand pairs (p_i, d_i) and need to choose more selling prices to learn the revenue curve. Given that $p_0 > p_1 > \cdots > p_N > p_{N+1}$ and $d_0 < d_1 < \cdots < d_N < d_{N+1}$, the next price should be selected in the interval (p_{i^*}, p_{i^*+1}) where $i^* \in \arg\max_i\{d_{i+1} - d_i\}$ such that the approximation gap is minimized.*

Remark 3. *The gap from non-concavity in parts (i) and (ii) is always bounded. To see this, note that Lipschitz continuity of $R(d)$ implies that $\beta_i = \frac{r_{i+1}-r_i}{d_{i+1}-d_i} \in [-L_R, L_R]$. As $\mu_j = \min\{\beta_i, i = 0, 1, \dots, j\} \ge -L_R$. Therefore,*

$$\sum_{j=0}^{N} (\beta_j - \mu_j)(d_{j+1} - d_j) \le \sum_{j=0}^{N} 2L_R(d_{j+1} - d_j) = 2L_R(d_{N+1} - d_0).$$

Remark 4. *Theorem 1 assumes that $R(d_i) = r_i = p_i d_i$ for any i. In practice, we may observe different demands at the same selling price p_i because of demand uncertainty. If a sufficient number of demand realizations are observed, d_i can be estimated by the mean of those demand realizations. Otherwise, we can construct a confidence interval for d_i and hence a confidence interval for $R(d_i)$ (as p_i is given) such that $|R(d_i) - r_i| \le \epsilon$ for any i, we can show that $\|\bar{R} - R\|_\infty$ is no more than the upper bound in Theorem 1 plus the additional term ϵ. certainly, this upper bound holds at a given confidence level.*

4.2 Solution Algorithm and Performance Bound

Once $\bar{R}(d)$ is determined, we replace $R(d)$ with $\bar{R}(d)$ in problem (2) to construct an auxiliary optimization problem

$$W_t(x_t) = \max_{y_t \ge x_t, d_t \in [\underline{d}_t, \bar{d}_t]} \Big\{ \bar{R}(d_t) - c_t(y_t - x_t) + E\big[-H_t(y_t - a_t d_t - b_t) + \alpha W_{t+1}(y_t - a_t d_t - b_t)\big] \Big\}, \quad (13)$$

with the boundary condition $W_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$.

It is well known that a base-stock list-price policy is optimal for this auxiliary problem with a known and concave $\bar{R}(d_t)$ (c.f. Federgruen and Heching 1999). The retailer is able to implement this well-structured policy as a heuristic policy of the original system. Given the starting inventory x_t, let $\bar{V}_t(x_t)$ denote the expected total profit from period t to $T+1$ when this heuristic policy is implemented. By applying K-approximate convexity, we can show the following theorem, which provides a worst-case bound on the performance of the heuristic policy.

Theorem 2. *If* $\|R - \bar{R}\|_\infty \le K$, *then* $\bar{V}_t(x_t) \ge V_t(x_t) - 2K\sum_{i=0}^{T-t}(i+1)\alpha^i$ *for any* $x_t \in \mathbb{R}$ *and* $t \in \{1, ..., T\}$.

Remark 5. *This performance bound only depends on the number of periods* T, *the discount factor* α, *and the parameter* K, *which represents the gap between* $\bar{R}(d)$ *and* $R(d)$ *and is upper bounded in Theorem 1. It is independent of all of the cost parameters and demand distributions. This bound does not blow up when* T *goes to infinity because* $\lim_{T\to\infty}\sum_{i=0}^{T}(i+1)\alpha^i = \frac{1}{(1-\alpha)^2}$.

Remark 6. *If the revenue function* $R_t(d_t)$ *is not stationary, consider an approximate* $\bar{R}_t(d_t)$ *such that* $\|R_t - \bar{R}_t\|_\infty \le K_t$ *for all* $t \in \{1, ..., T\}$. *Similar to Theorem 2, we can show that* $\bar{V}_t(x_t) \ge V_t(x_t) - 2\sum_{i=0}^{T-t}(i+1)\alpha^i K_{t+i}$ *for any* $x_t \in \mathbb{R}$ *and* $t \in \{1, ..., T\}$.

4.3 Numerical Study: Performance Test

In this section, we present a set of numerical experiments. This numerical study has two objectives. The first objective is to assess the performance of the proposed heuristic policy and the effectiveness of the worst case performance bound given by Theorem 2 over a large set of examples. The second is to examine how the system parameters affect the performance of the proposed heuristic policy.

4.3.1 Data

We adopt the data collected from the Dominick's Finer Foods Project(DFFP) conducted by the Kilts Center of the University of Chicago. This database contains store-level pricing and sales data from Dominick's Finer Foods, one of the two biggest supermarket chains in the Chicago area. We select a product, diet orange slice, with sales data at 40 different prices ranging from \$0.79 to \$1.79 collected from 83 stores over 7 years. Because it is hard to confirm whether a zero-sale means a no-trade date or stock-out event, we simply remove samples with zero demand that contain no price information. The remaining data set comprises nearly

20,000 observations from 15 sale zones. We assume that customers in different zones are homogenous, which is an acceptable assumption if all customers are from a single city. We study the aggregate demand in Chicago instead of the demand in each sale zone (the number of selling prices and the corresponding realized demand is much less) for three reasons. First, the aggregate demand has a sufficient number of different selling prices (around 40) such that we can accurately calibrate the demand function.[1] Second, the data set provides a sufficiently large range of prices $[0.79, 1.79]$ to avoid trivial solutions. Third, at each selling price p_i, the number of demand realizations is sufficiently large (at least 20) such that the expected demand d_i can be accurately estimated. However, the data set has a shortcoming, as it does not record some non-price promotions offered by stores. Therefore, our estimated model cannot address those promotion effects in a sophisticated way, except by incorporating the demand induced by those promotions into the random variables b_t.

Figure 2 plots the demand mean under different selling prices in the data set. Based on the distribution of the points, we use a piecewise linear function with two pieces to fit the function $d(p)$ by linear regression.[2] The estimated $d(p)$ (c.f. Figure 2) has the following specific form:

$$d(p) = \begin{cases} -64.2919p + 80.7333, & \text{if } \ 0.79 \le p \le 1.09, \\ -6.8677p + 18.1409, & \text{if } \ 1.09 \le p \le 1.79, \end{cases} \tag{14}$$

where all of the parameter estimates are significant at a p-value of 0.05.

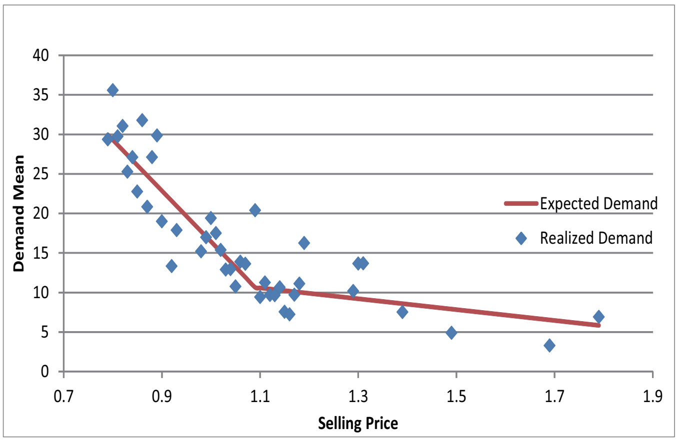

Figure 2: Demand Mean under Various Selling Prices

[1] Note that the heuristic does not require a large number of data points because it can deal with incomplete demand information. However, to evaluate the performance of the heuristic policy, we need the optimal profit under complete demand information as the benchmark, i.e., the true demand function is required. In the numerical study, the demand function calibrated from the data set serves as the true demand function, and hence it needs be accurately estimated.

[2] One could use other function forms, e.g., piecewise linear function with multiple pieces, to better fit the demand function. However, our estimated model seems good enough (all of the parameter estimates are significant at a p-value of 0.05). We also tried the demand model with three linear pieces and obtained similar results.

In the numerical experiment, we treat $d(p)$ stated in (14) as the true expected demand function. Suppose that the demand parameters a_t and b_t in (1) are independent normal random variables with mean and variance pairs $(1, \sigma_a^2)$ and $(0, \sigma_b^2)$, respectively. Given $d(p)$, σ_a and σ_b can be estimated by maximum likelihood, and the results are shown in Table 2. For $p \in [0.79, 1.79]$, the value of the function $d(p)$ is roughly in the interval $[6, 30]$, and hence we set $\underline{d}_t = 6$ and $\overline{d}_t = 30$, i.e., the feasible region for the decision variable d_t is $[6, 30]$. The holding/shortage function has the following specific form: $H_t(x) = h_t \max\{0, x\} + s_t \max\{0, -x\}$ for any period t. Moreover, all system parameters are assumed to be stationary over time and thus we remove the time index of the system parameters. Note that some of the parameters including the ordering, holding, and shortage costs, are not contained in the DFFP data set, and thus their values are set relative to the selling price. However, the tested value covers a large region that is most likely to contain the real ones.

4.3.2 Performance Evaluation of Heuristic Policy

With the complete demand information in (14), we can solve the real system stated in (2). It is worth noting that the expected revenue function $R(d)$ derived from $d(p)$ is not concave. Hence, the optimal policy may not follow the structure of the base-stock and list-price policy. Hence, we solve the real system by enumeration to obtain the optimal solutions, which may not be well-structured.

To evaluate the performance of the heuristic policy under incomplete demand information, we uniformly pick N points from $[6, 30]$, which, together with $\underline{d}_t$ and $\bar{d}_t$, are treated as the values of expected demand at which the expected revenue are observed. Obviously, N measures the degree of the incompleteness of the demand information. The larger N is, the more demand information we have. Given these demand observations, we can construct an approximation of revenue function and solve the auxiliary dynamic programming in (13). After solving the auxiliary problem, we implement its optimal solution as a heuristic policy in the real system. The performance of the heuristic can be measured by $\xi_{HP} = \frac{V_t(x_t) - \bar{V}_t(x_t)}{V_t(x_t)}$, which represents the percentage loss of profit caused by implementing the heuristic policy rather than the optimal one. The worst-case performance bound of the proposed heuristic policy is defined by $\xi_{UP} = \frac{2K \sum_{i=0}^{T-t} (i+1)\alpha^i}{\bar{V}_t(x_t)}$ because Theorem 2 implies that $\frac{\bar{V}_t(x_t)}{V_t(x_t)} \geq \frac{1}{1+\xi_{UP}}$.

We assume that the initial inventory level in the first period x_1 is zero, the number of periods $T = 5$, and the discount factor $\alpha = 0.95$. In designing the experiments, we select the following set of system parameter values: $c \in \{0.1, 0.2, 0.3\}$, $h \in \{0.06, 0.08, 0.10, 0.12\}$, $s \in \{0.08, 0.10, 0.12, 0.14\}$, $N \in \{5, 7, 11\}$, $\sigma_a^2 \in \{1, 3, 5\}$, and $\sigma_b^2 \in \{5, 7, 9\}$. The fitted values of σ_a^2 and σ_b^2 from the real data are 3 and 7, respectively. We try other values because demand variance can affect the optimal decisions and hence the performance of the heuristic policy. All combinations of these system parameters provide $3 \times 4 \times 4 \times 3 \times 3 \times 3 = 1296$ test instances.

Table 1 summarizes the overall performance of ξ_{HP} and ξ_{UP}. We can see that the average gap between the profits generated by the heuristic policy and optimal policy is 0.27% and the maximal gap is 4.6%. Overall, our proposed heuristic policy performs quite well. The worst case performance bound ξ_{UP} implies that the profit returned by the heuristic policy is on average $\frac{1}{1+87.2\%}$ of the optimal profit with a maximum of $\frac{1}{1+45.1\%}$ and a minimum of $\frac{1}{1+180.3\%}$ of the optimal profit. This is a reasonable expectation because we are facing incomplete demand information and a non-concave revenue function, and our problem is a stochastic dynamic programming problem with two decision variables (inventory and pricing). Certainly, the worst-case performance bound can be very bad in terms of the percentage of the optimal profit because the value of the optimal profit can be very small in certain cases. Conversely, it can be very good when the value of the optimal profit is large.

Table 1: Overall Performance of ξ_{HP}, ξ_{RG}, and ξ_{UP} (%)

	ξ_{HP}	ξ_{UP}
Average values	0.27%	87.2%
Minimal values	0	45.1%
Maximal values	4.6%	180.3%

4.3.3 Sensitivity of Model Parameters

Next, we investigate the effect of the model parameters on the performance of the heuristic. To this end, we set a base scenario and vary the parameter values once at a time. Table 2 summarizes the values of all of the parameters in the base scenario.

Table 2: Parameters for Base Scenario

Ordering Cost c_t	Holding Cost h	Shortage Cost s	$\underline{d}_t$	$\bar{d}_t$	Discount Rate α
0.3	0.15	0.2	6	30	0.95
Initial Inventory Level x_1	σ_a^2	σ_b^2	Horizon T	Number of Observed (d_i, r_i) Pairs $N+2$	
0	3.16	7.01	5	7	

The numerical results for sensitivity are presented in Figures 3, 4, and 5 and Table 3. Figure 3 demonstrates the effectiveness of the heuristic under various time horizons. The heuristic performs better as the time horizon increases. Both the total profit of the optimal policy and the profit loss due to the heuristic policy increase with the number of periods T. Figure 3 shows that as the percentage loss of profit decreases in T, the amount of profit loss due to the heuristic policy increases at a lower rate than the total profit.

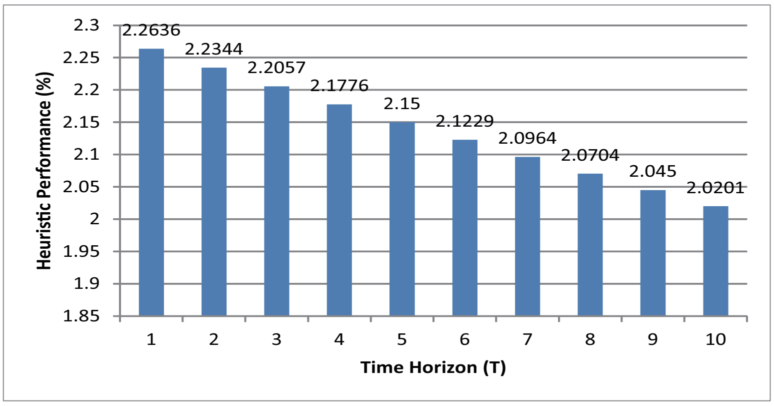

Figure 3: Heuristic Performance under Various Time Horizons

Figure 4 shows how the performance of the heuristic policy depends on the degree of the incompleteness of the demand information. It shows that the heuristic performs better as N increases. This observation is intuitive because more demand information leads to a more accurate decision. Another observation from Figure 4 is that the heuristic policy is not optimal even with more demand information (e.g., N increases from 13 to 25). In this case, the profit loss (0.48%) is due to the non-concavity of $R(d)$.

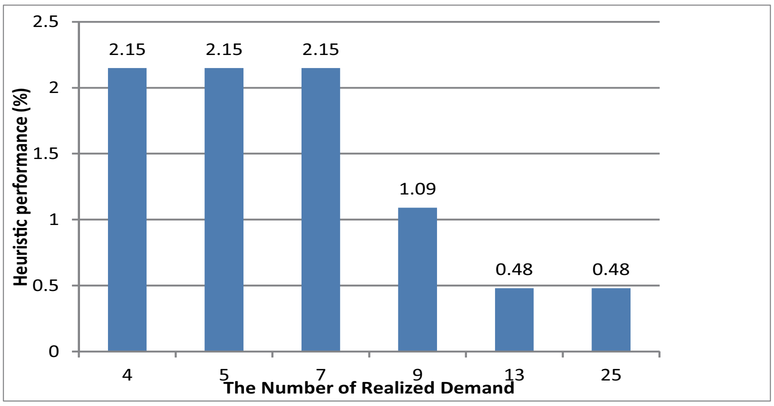

Figure 4: Heuristic Performance with Respect to the Number of Observed (p_i, r_i) Pairs

Table 3: The Impact of Holding and Shortage Costs on Heuristic Performance (%)

Shortage	Holding Cost h				
Cost s	0.05	0.1	0.2	0.3	0.4
0.1	0	0	0	0	0
0.2	0	0	0.01	0.17	0.18
0.3	0	0	2.15	0.59	1.47
0.4	0	0.14	0.97	3.08	1.96
0.5	0	0.56	0	1.56	0.30

Table 3 illustrates the effectiveness of the heuristic policy under different inventory holding and shortage costs. To understand the effect, consider two extreme cases: zero holding cost and zero shortage cost. Clearly, in the former case, the optimal policy, i.e., the optimal solution to (2), is to only order once at the start of the first period to satisfy all of the demands in the planning horizon, and in every period set the price that maximizes the true revenue function $R(d)$. For the heuristic policy obtained by solving (13), we should also order enough inventory at the start of the first period, and in each period quote the price that maximizes the approximate revenue function $\bar{R}(d)$. As $R(d)$ and $\bar{R}(d)$ have the same maximizer, the heuristic policy is optimal. In the latter case, for both the optimal and heuristic policies, the price quoted in every period is the maximizer of both $R(d)$ and $\bar{R}(d)$, and the order should only be placed at the end of the planning horizon to fulfill all of the backlogs. This explains the observation in Table 3, that the heuristic policy performs well when the ratio, s/h, is small or large. However, we do not observe any other monotonicity properties.

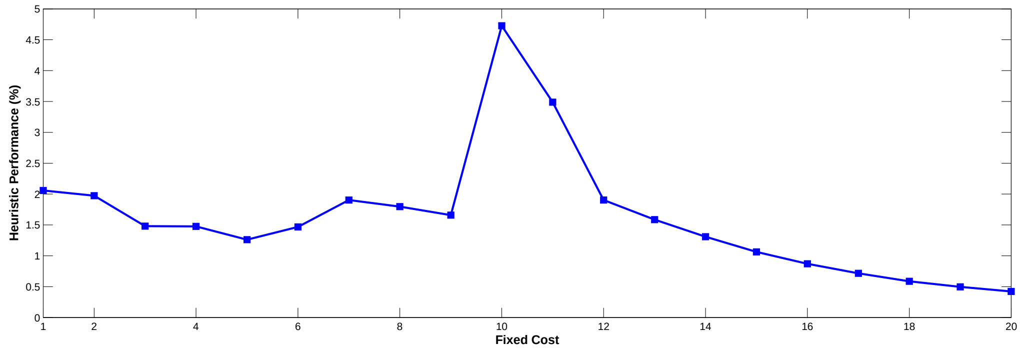

Figure 5: Heuristic Performance with Respect to Fixed Costs

So far, we have assumed that the fixed ordering cost is zero in our problem. If there exists a fixed ordering cost, the heuristic policy becomes a (s, S, A, p) policy (see Chen and Simchi-Levi 2004a). Figure 5 demonstrates the effect of fixed costs on the heuristic performance. We can see that the heuristic policy performs well over a large range of fixed costs. We further observe that its performance is extremely good when the fixed cost is very small or very large. This is because in the extreme cases, both the optimal policy and the heuristic policy become smoother, which makes the heuristic policy match the optimal policy better.

5 Conclusion

In this paper, we develop a new concept of K-approximate convexity, which is a generalization of K-convexity. Based on this concept, we solve the challenges of incomplete demand information and the non-concave revenue function by approximating the one-period cost function, which significantly enhances the practical value of the joint inventory-pricing coordination in a dynamic setting.

The methodology based on K-approximate convexity is general enough to find many other applications. For example, in marketing, the sales as a function of advertising effort can be an S-shaped increasing function, i.e, first convex and then concave (c.f. Danaher 2008). The advertising effort has a carryover effect to the next period (c.f. Little 1979), which naturally leads to a dynamic problem with a non-convex and non-concave objective function. Therefore, the analysis in this paper can readily lend itself to such a marketing problem. In some inventory control problems, the ordering cost can be a non-convex and non-concave function of ordering quantity (c.f. Chan et al. 2002 and Zhang et al. 2012). K-approximate convexity can help us construct and analyze well-structured heuristic policies for these inventory problems.

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A Appendix

To show Proposition 2, we first need the following lemmas. Our proof involves the concept of convex envelope.

Definition 2. *For any function $f : \mathcal{S} \mapsto \mathbb{R}$ where $\mathcal{S} \subseteq \mathbb{R}$, the function $\underline{f}^* : \mathcal{S} \mapsto \mathbb{R}$ is the convex envelope of f if*

$$\underline{f}^* = \sup\{\underline{f} : \mathcal{S} \mapsto \mathbb{R}, \underline{f} \textit{ convex and } \underline{f} \leq f\}.$$

The following lemma provides a useful tool to characterize a function's convex envelope.

Lemma 3. *Suppose that $f : \mathcal{S} \mapsto \mathbb{R}$ and $g : (a, b) \mapsto \mathbb{R}$, where $(a, b) \subseteq \mathcal{S}$, are both convex functions. If $\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(x)$, $\lim_{x\uparrow b} f(x) = \lim_{x\uparrow b} g(x)$, and $f(x) \leq g(x)$ for any $x \in (a, b)$, then $h : \mathcal{S} \mapsto \mathbb{R}$ where*

$$h(x) = \begin{cases} f(x) & \forall\ x \in \mathcal{S} \setminus (a, b) \\ g(x) & \forall\ x \in (a, b) \end{cases}$$

is convex.

Proof. Let $\delta_{f,a}$, $\delta_{f,b}$, $\delta_{g,a}$, and $\delta_{g,b}$ denote the semi-derivatives of f and g at a and b, respectively, i.e.,

$$\delta_{f,a} = \lim_{x\downarrow a} \frac{f(x) - \lim_{y\downarrow a} f(y)}{x - a}, \quad \delta_{f,b} = \lim_{x\uparrow b} \frac{f(x) - \lim_{y\uparrow b} f(y)}{x - b},$$
$$\delta_{g,a} = \lim_{x\downarrow a} \frac{g(x) - \lim_{y\downarrow a} g(y)}{x - a}, \quad \delta_{g,b} = \lim_{x\uparrow b} \frac{g(x) - \lim_{y\uparrow b} g(y)}{x - b}.$$

Note that $\delta_{f,a} \leq \delta_{g,a}$ and $\delta_{f,b} \geq \delta_{g,b}$ as $\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(x)$, $\lim_{x\uparrow b} f(x) = \lim_{x\uparrow b} g(x)$, and $f(x) \leq g(x)$ for any $x \in (a, b)$.

Consider the convex function $\tilde{g} : \mathcal{S} \mapsto \mathbb{R}$ that

$$\tilde{g}(x) = \begin{cases} \lim_{x\downarrow a} g(x) + \delta_{g,a}(x - a) & \forall\ x \in \mathcal{S} \text{ and } x \leq a \\ g(x) & \forall\ x \in (a, b) \\ \lim_{x\uparrow b} g(x) + \delta_{g,b}(x - b) & \forall\ x \in \mathcal{S} \text{ and } x \geq b. \end{cases}$$

For any $x \in \mathcal{S}$ and $x \leq a$, the convexity of f yields $f(x) \geq \lim_{x\downarrow a} f(x) + \delta_{f,a}(x - a) \geq \lim_{x\downarrow a} g(x) + \delta_{g,a}(x - a) = \tilde{g}(x)$, where the second inequality follows from $\delta_{f,a} \leq \delta_{g,a}$. Similarly, we can show $f(x) \geq \lim_{x\uparrow b} f(x) + \delta_{f,b}(x - b) \geq \lim_{x\uparrow b} g(x) + \delta_{g,b}(x - b) = \tilde{g}(x)$ for any $x \in \mathcal{S}$ and $x \geq b$. Therefore, we have $h = \max\{f, \tilde{g}\}$, which is convex as both f and $\tilde{g}$ are convex. □

The following lemma gives a characterization of a function's convex envelope.

Lemma 4. *Consider a function $f : \mathcal{S} \mapsto \mathbb{R}$ where $\mathcal{S} \subseteq \mathbb{R}$ and its convex envelope $\underline{f}^*$. Suppose that there exists an interval $(a, b) \subseteq \mathcal{S}$ such that*

$$\inf_{x\in[c,d]} \left\{f(x) - \underline{f}^*(x)\right\} > 0 \qquad \forall\ [c, d] \subset (a, b).$$

Then

$$\underline{f}^*(x) = \frac{b - x}{b - a} \lim_{y\downarrow a} \underline{f}^*(y) + \frac{x - a}{b - a} \lim_{y\uparrow b} \underline{f}^*(y) \qquad \forall\ x \in (a, b).$$

Proof. To simplify the notation, define the linear function

$$l(x) = \frac{b-x}{b-a}\lim_{y\downarrow a}\underline{f}^*(y) + \frac{x-a}{b-a}\lim_{y\uparrow b}\underline{f}^*(y) \qquad \forall\, x\in(a,b).$$

Assume for contradiction that there exists $x_0 \in (a,b)$ such that $\underline{f}^*(x_0) \neq l(x_0)$. As $\underline{f}^*$ is convex, we have $\underline{f}^*(x) \le l(x)$ for any $x\in(a,b)$ and hence $\underline{f}^*(x_0) < l(x_0)$. Consider the following two cases:

- Suppose that $f(x) \ge l(x)$ for any $x \in (a,b)$. Define

$$\underline{f}(x) = \begin{cases} \underline{f}^*(x) & \forall\, x\in\mathcal{S}\setminus(a,b) \\ l(x) & \forall\, x\in(a,b). \end{cases}$$

  Obviously, $\underline{f}^* \le \underline{f} \le f$ and $\underline{f}^*(x_0) < l(x_0) = \underline{f}(x_0)$. Lemma 3 shows $\underline{f}(x)$ is convex, which contradicts that $\underline{f}^*$ is the convex envelope of f.

- Suppose that $\Delta = \sup\{l(x) - f(x) : x\in(a,b)\} > 0$. Consider the function $l(x) - \underline{f}^*(x)$. Note that $l(x) - \underline{f}^*(x)$ is concave in (a,b). The definition of $l(x)$ implies $\lim_{x\downarrow a}\left\{l(x) - \underline{f}^*(x)\right\} = \lim_{x\uparrow b}\left\{l(x) - \underline{f}^*(x)\right\} = 0$. As $\underline{f}^*(x_0) < l(x_0)$ for some $x_0\in(a,b)$, we have $l(x) - \underline{f}^*(x) > 0$ for any $x\in(a,b)$. Therefore, we can define

$$\delta_a = \lim_{x\downarrow a}\frac{l(x)-\underline{f}^*(x)}{x-a} > 0 \qquad \text{and} \qquad \delta_b = \lim_{x\uparrow b}\frac{l(x)-\underline{f}^*(x)}{x-b} < 0,$$

  which correspond to the semi-derivatives of $l(x) - \underline{f}^*(x)$ at a and b, respectively. Choose a sufficiently large $n > 1$ such that $\frac{\Delta}{n\delta_a} + \frac{\Delta}{-n\delta_b} < b-a$. For any $x\in(a, a+\frac{\Delta}{n\delta_a})$, the concavity of $l(x) - \underline{f}^*(x)$ implies $l(x) - f(x) \le l(x) - \underline{f}^*(x) < \frac{\Delta}{n}$. Similarly, $l(x) - f(x) \le l(x) - \underline{f}^*(x) < \frac{\Delta}{n}$ for any $x\in(b-\frac{\Delta}{-n\delta_b}, b)$. Hence, we obtain

$$\sup\left\{l(x) - f(x) : x\in\left[a+\frac{\Delta}{n\delta_a}, b-\frac{\Delta}{-n\delta_b}\right]\right\} = \Delta. \tag{15}$$

  Let

$$\epsilon = \inf\left\{f(x) - \underline{f}^*(x) : x\in\left[a+\frac{\Delta}{n\delta_a}, b-\frac{\Delta}{-n\delta_b}\right]\right\} > 0.$$

  According to (15), there exists $x^*\in[a+\frac{\Delta}{n\delta_a}, b-\frac{\Delta}{-n\delta_b}]$ such that $l(x^*) - f(x^*) > \Delta - \epsilon$. Also note that $f(x^*) - \underline{f}^*(x^*) \ge \epsilon$. We have

$$l(x^*) - \Delta > f(x^*) - \epsilon \ge \underline{f}^*(x^*).$$

  Define

$$\underline{f}(x) = \begin{cases} \underline{f}^*(x) & \forall\, x\in\mathcal{S}\setminus(a,b) \\ \max\left\{l(x)-\Delta, \underline{f}^*(x)\right\} & \forall\, x\in(a,b). \end{cases}$$

Lemma 3 yields that $\underline{f}(x)$ is convex. The definition of Δ implies $l(x)-\Delta \le f(x)$ for any $x \in (a,b)$, and hence $\underline{f}^* \le \underline{f} \le f$. Also note that $l(x^*) - \Delta > \underline{f}^*(x^*)$, where $x^* \in (a,b)$. Therefore, $\underline{f}(x^*) = l(x^*) - \Delta > \underline{f}^*(x^*)$, which is a contradiction to that $\underline{f}^*$ is the convex envelope of f. $\square$

We now show that any K-convex function over a bounded interval is $K/2$-approximate convex.

Lemma 5. *If $f : (a,b) \mapsto \mathbb{R}$ is a K-convex function, then $f(x)$ is a $K/2$-approximate convex function.*

Proof. Let $\underline{f}^*$ denote the convex envelope of f. It suffices to show that $\|f - \underline{f}^*\|_\infty \le K$.

First, we prove that $\inf_{x\in(a,a+\delta]}\{f(x) - \underline{f}^*(x)\} = 0$ for any $\delta \in (0, b-a)$. Assume for contradiction that $\inf_{x\in(a,a+\delta]}\{f(x) - \underline{f}^*(x)\} = \epsilon > 0$ for some $\delta \in (0, b-a)$. Define the function

$$\phi(x) = \begin{cases} \epsilon + \frac{\epsilon}{\delta}(a-x) \le \epsilon & \text{if } x \in (a, a+\delta], \\ 0 & \text{if } x \in (a+\delta, b). \end{cases}$$

Obviously, ϕ is convex and so is $\underline{f}^*+\phi$. The definition of ϵ implies $f(x) \ge \underline{f}^*(x)+\epsilon$ for any $x \in (a, a+\delta]$, and hence $\underline{f}^* \le \underline{f}^*+\phi \le f$. As $\underline{f}^*(x)+\phi(x) > \underline{f}^*(x)$ for any $x \in (a, a+\delta)$, we obtain a contradiction to that $\underline{f}^*$ is the convex envelope of f. As a result, $\inf_{x\in(a,a+\delta]}\{f(x) - \underline{f}^*(x)\} = 0$ for any $\delta \in (0, b-a)$. Similarly, we can show that $\inf_{x\in[b-\delta,b)}\{f(x) - \underline{f}^*(x)\} = 0$ for any $\delta \in (0, b-a)$.

Consider any $x_0 \in (a,b)$ such that $f(x_0) > \underline{f}^*(x_0)$. Recall that $\inf_{x\in(a,a+\delta]}\{f(x) - \underline{f}^*(x)\} = \inf_{x\in[b-\delta,b)}\{f(x) - \underline{f}^*(x)\} = 0$ for any $\delta \in (0, b-a)$. There exist some $a_0 \in [a, x_0]$ and $b_0 \in [x_0, b]$ such that

$$\inf_{x\in[a_0-\delta,a_0+\delta]\cap(a,b)} \left\{f(x) - \underline{f}^*(x)\right\} = \inf_{x\in[b_0-\delta,b_0+\delta]\cap(a,b)} \left\{f(x) - \underline{f}^*(x)\right\} = 0 \qquad \forall\, \delta > 0 \tag{16}$$

and

$$\inf_{x\in[c,d]} \left\{f(x) - \underline{f}^*(x)\right\} > 0 \qquad \forall\, [c,d] \subset (a_0, b_0).$$

According to the definitions of a_0 and b_0, $a_0 = x_0$ implies

$$\inf_{x\in[x_0-\delta,x_0)\cap(a,b)} \left\{f(x) - \underline{f}^*(x)\right\} = 0 \qquad \forall\, \delta > 0 \tag{17}$$

and $b_0 = x_0$ implies

$$\inf_{x\in(x_0,x_0+\delta]\cap(a,b)} \left\{f(x) - \underline{f}^*(x)\right\} = 0 \qquad \forall\, \delta > 0. \tag{18}$$

Consider the following two cases:

Case 1. Suppose that $a_0 = b_0 = x_0$. Consider any arbitrary $\epsilon > 0$. As $\underline{f}^*$ is continuous in (a, b), there exists $\delta_\epsilon > 0$ such that

$$|\underline{f}^*(x) - \underline{f}^*(x_0)| < \frac{\epsilon}{2} \qquad \forall\ x \in [x_0 - \delta_\epsilon, x_0 + \delta_\epsilon] \subset (a, b). \tag{19}$$

(17) and (18) yield that there exist some $a_\epsilon \in [x_0 - \delta_\epsilon, x_0)$ and $b_\epsilon \in (x_0, x_0 + \delta_\epsilon]$ such that

$$f(a_\epsilon) - \underline{f}^*(a_\epsilon) < \frac{\epsilon}{2} \qquad \text{and} \qquad f(b_\epsilon) - \underline{f}^*(b_\epsilon) < \frac{\epsilon}{2}.$$

Combining with (19) and $f \geq \underline{f}^*$, we have

$$|f(a_\epsilon) - \underline{f}^*(x_0)| < \epsilon \qquad \text{and} \qquad |f(b_\epsilon) - \underline{f}^*(x_0)| < \epsilon.$$

Applying the K-convexity of f, we obtain

$$\begin{aligned} f(x_0) &\leq \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} f(a_\epsilon) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} f(b_\epsilon) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} K \\ &< \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} \big(\underline{f}^*(x_0) + \epsilon\big) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} \big(\underline{f}^*(x_0) + \epsilon\big) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} K < \underline{f}^*(x_0) + \epsilon + K. \end{aligned}$$

The arbitrary of ϵ yields $f(x_0) - \underline{f}^*(x_0) \leq K$.

Case 2. Suppose that $a_0 < b_0$. Lemma 4 shows that

$$\underline{f}^*(x_0) = \frac{b_0 - x_0}{b_0 - a_0} \lim_{x \downarrow a_0} \underline{f}^*(x) + \frac{x_0 - a_0}{b_0 - a_0} \lim_{x \uparrow b_0} \underline{f}^*(x). \tag{20}$$

Consider any $\epsilon > 0$. There exists some $\delta_{\lim} > 0$ such that

$$\begin{aligned} &\left|\underline{f}^*(x) - \lim_{x \downarrow a_0} \underline{f}^*(x)\right| < \frac{\epsilon}{4} \qquad \forall x \in [a_0 - \delta_{\lim}, a_0 + \delta_{\lim}] \cap (a, b) \\ &\left|\underline{f}^*(x) - \lim_{x \uparrow b_0} \underline{f}^*(x)\right| < \frac{\epsilon}{4} \qquad \forall x \in [b_0 - \delta_{\lim}, b_0 + \delta_{\lim}] \cap (a, b). \end{aligned} \tag{21}$$

Define

$$\epsilon_\lambda = \frac{\epsilon/2}{\left|\lim_{x \downarrow a_0} \underline{f}^*(x) - \lim_{x \uparrow b_0} \underline{f}^*(x)\right| + 1} > 0 \qquad \text{and} \qquad \delta_\lambda = \frac{\epsilon_\lambda(b_0 - a_0)}{3 + 2\epsilon_\lambda} > 0.$$

Next, we show that there exist a_ϵ and b_ϵ such that

$$\begin{gathered} a < a_\epsilon < x_0 < b_\epsilon < b, \qquad |a_\epsilon - a_0| \leq \min\{\delta_{\lim}, \delta_\lambda\}, \qquad |b_\epsilon - b_0| \leq \min\{\delta_{\lim}, \delta_\lambda\}, \\ f(a_\epsilon) - \underline{f}^*(a_\epsilon) < \frac{\epsilon}{4} \qquad \text{and} \qquad f(b_\epsilon) - \underline{f}^*(b_\epsilon) < \frac{\epsilon}{4}. \end{gathered} \tag{22}$$

- Suppose that $a_0 < x_0 < b_0$. Let $\delta_\epsilon = \min\{\delta_{\lim}, \delta_\lambda, \frac{x_0 - a_0}{2}, \frac{b_0 - x_0}{2}\} > 0$. According to (16), there exist some $a_\epsilon \in [a_0 - \delta_\epsilon, a_0 + \delta_\epsilon] \cap (a, b)$ and $b_\epsilon \in [b_0 - \delta_\epsilon, b_0 + \delta_\epsilon] \cap (a, b)$ satisfying (22).

- Suppose that $a_0 = x_0 < b_0$. Let $\delta_\epsilon = \min\{\delta_{\lim}, \delta_\lambda, \frac{b_0-x_0}{2}\} > 0$. According to (16) and (17), there exist some $a_\epsilon \in [a_0-\delta_\epsilon, a_0)\cap(a,b)$ and $b_\epsilon \in [b_0-\delta_\epsilon, b_0+\delta_\epsilon]\cap(a,b)$ satisfying (22).

- Suppose that $a_0 < x_0 = b_0$. Let $\delta_\epsilon = \min\{\delta_{\lim}, \delta_\lambda, \frac{x_0-a_0}{2}\} > 0$. According to (16) and (18), there exist some $a_\epsilon \in [a_0-\delta_\epsilon, a_0+\delta_\epsilon]\cap(a,b)$ and $b_\epsilon \in (b_0, b_0+\delta_\epsilon]\cap(a,b)$ satisfying (22).

Applying (21), (22) and $f \geq \underline{f}^*$, we have

$$\left|f(a_\epsilon) - \lim_{x\downarrow a_0}\underline{f}^*(x)\right| < \frac{\epsilon}{2} \qquad \text{and} \qquad \left|f(b_\epsilon) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right| < \frac{\epsilon}{2}. \tag{23}$$

As $a < a_\epsilon < x_0 < b_\epsilon < b$, the K-convexity of f implies

$$\begin{aligned}
f(x_0) &\leq \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} f(a_\epsilon) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} f(b_\epsilon) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} K \\
&< \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon}\left(\lim_{x\downarrow a_0}\underline{f}^*(x) + \frac{\epsilon}{2}\right) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon}\left(\lim_{x\uparrow b_0}\underline{f}^*(x) + \frac{\epsilon}{2}\right) + K \\
&= \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon}\lim_{x\downarrow a_0}\underline{f}^*(x) + \frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon}\lim_{x\uparrow b_0}\underline{f}^*(x) + K + \frac{\epsilon}{2} \\
&= \underline{f}^*(x_0) + K + \frac{\epsilon}{2} + \left(\frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} - \frac{b_0 - x_0}{b_0 - a_0}\right)\lim_{x\downarrow a_0}\underline{f}^*(x) + \left(\frac{x_0 - a_\epsilon}{b_\epsilon - a_\epsilon} - \frac{x_0 - a_0}{b_0 - a_0}\right)\lim_{x\uparrow b_0}\underline{f}^*(x) \\
&\leq \underline{f}^*(x_0) + K + \frac{\epsilon}{2} + \left|\frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} - \frac{b_0 - x_0}{b_0 - a_0}\right|\left|\lim_{x\downarrow a_0}\underline{f}^*(x) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right|
\end{aligned}$$

where the second inequality and the second equality follow from (23) and (20), respectively. Note that

$$\frac{b_0 - \delta_\lambda - x_0}{b_0 - a_0 + 2\delta_\lambda} \leq \frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} \leq \frac{b_0 + \delta_\lambda - x_0}{b_0 - a_0 - 2\delta_\lambda}.$$

Furthermore,

$$\begin{aligned}
\frac{b_0 - \delta_\lambda - x_0}{b_0 - a_0 + 2\delta_\lambda} &= \frac{b_0 - \frac{\epsilon_\lambda(b_0-a_0)}{3+2\epsilon_\lambda} - x_0}{b_0 - a_0 + 2\times\frac{\epsilon_\lambda(b_0-a_0)}{3+2\epsilon_\lambda}} = \frac{(3+4\epsilon_\lambda)(b_0-x_0) - 2\epsilon_\lambda(b_0-x_0) - \epsilon_\lambda(b_0-a_0)}{(3+4\epsilon_\lambda)(b_0-a_0)} \\
&\geq \frac{(3+4\epsilon_\lambda)(b_0-x_0) - 3\epsilon_\lambda(b_0-x_0)}{(3+4\epsilon_\lambda)(b_0-a_0)} = \frac{b_0-x_0}{b_0-a_0} - \frac{3}{3+4\epsilon_\lambda}\epsilon_\lambda \geq \frac{b_0-x_0}{b_0-a_0} - \epsilon_\lambda \\
\frac{b_0 + \delta_\lambda - x_0}{b_0 - a_0 - 2\delta_\lambda} &= \frac{b_0 + \frac{\epsilon_\lambda(b_0-a_0)}{3+2\epsilon_\lambda} - x_0}{b_0 - a_0 - 2\times\frac{\epsilon_\lambda(b_0-a_0)}{3+2\epsilon_\lambda}} = \frac{3(b_0-x_0) + 2\epsilon_\lambda(b_0-x_0) + \epsilon_\lambda(b_0-a_0)}{3(b_0-a_0)} \\
&\leq \frac{3(b_0-x_0) + 3\epsilon_\lambda(b_0-a_0)}{3(b_0-a_0)} = \frac{b_0-x_0}{b_0-a_0} + \epsilon_\lambda.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{b_\epsilon - x_0}{b_\epsilon - a_\epsilon} - \frac{b_0 - x_0}{b_0 - a_0}\right|\left|\lim_{x\downarrow a_0}\underline{f}^*(x) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right| \leq \epsilon_\lambda\left|\lim_{x\downarrow a_0}\underline{f}^*(x) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right| \\
&= \frac{\epsilon/2}{\left|\lim_{x\downarrow a_0}\underline{f}^*(x) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right| + 1} \times \left|\lim_{x\downarrow a_0}\underline{f}^*(x) - \lim_{x\uparrow b_0}\underline{f}^*(x)\right| < \frac{\epsilon}{2},
\end{aligned}$$

which yields $f(x_0) < \underline{f}^*(x_0) + K + \epsilon$. As this inequality holds for any arbitrary $\epsilon > 0$, we have $f(x_0) - \underline{f}^*(x_0) \leq K$. $\square$

Finally, we can show Proposition 2, which states that any K-convex over $\mathbb{R}$ is $K/2$-approximate convex.

Proof of Proposition 2. For any $M \in \mathbb{Z}^+$, define $\underline{f}^*_M$ as the convex envelop of the function f in $(-M, +M)$. Consider any $M_1, M_2 \in \mathbb{Z}^+$ and $M_1 \leq M_2$. $\underline{f}^*_{M_2}$ is convex in $(-M_1, M_1)$ and $\underline{f}^*_{M_2} \leq f$ in $(-M_1, M_1)$. As $\underline{f}^*_{M_1}$ is the convex envelop of f in $(-M_1, M_1)$, we have $\underline{f}^*_{M_2}(x) \leq \underline{f}^*_{M_1}(x)$ for any $x \in (-M_1, M_1)$. Therefore, for any given x, $\underline{f}^*_M(x)$ decreases in M for any $M \in \mathbb{Z}^+$ and $M > |x|$. Furthermore, given x, the proof of Lemma 5 shows that $\underline{f}^*_M(x) \geq f(x) - K$ for any $M \in \mathbb{Z}^+$ and $M > |x|$.

As the sequence $\underline{f}^*_M(x)$ decreases in M and is bounded, we can define the function $\underline{f} : \mathbb{R} \mapsto \mathbb{R}$ such that $\underline{f}(x) = \lim_{M\to\infty} \underline{f}^*_M(x)$ for any $x \in \mathbb{R}$. Note that $\underline{f}^*_M(x) \in [f(x) - K, f(x)]$ for any $M \in \mathbb{Z}^+$ and $M > |x|$, which yields $\underline{f} \leq f$ and $\|\underline{f} - f\| \leq K$. Therefore, we can complete the proof by establishing the convexity of $\underline{f}$.

Consider any $x_1, x_2 \in \mathbb{R}$ and $x = \lambda x_1 + (1-\lambda)x_2$ where $\lambda \in [0,1]$. For any $M \in \mathbb{Z}^+$ and $M > \max\{|x_1|, |x_2|, |x|\}$, the convexity of $\underline{f}^*_M$ implies $\underline{f}^*_M(x) \leq \lambda \underline{f}^*_M(x_1) + (1-\lambda)\underline{f}^*_M(x_2)$. Taking the limit on both sides, we obtain

$$\underline{f}(x) = \lim_{M\to\infty} \underline{f}^*_M(x) \leq \lambda \lim_{M\to\infty} \underline{f}^*_M(x_1) + (1-\lambda) \lim_{M\to\infty} \underline{f}^*_M(x_2) = \lambda \underline{f}(x_1) + (1-\lambda)\underline{f}(x_2). \quad \square$$

Proof of Proposition 3. As $W(x)$ is K-approximate convex, there exists a convex function $w : \mathbb{R} \mapsto \mathbb{R}$ such that $\|W - w\|_\infty \leq K$. We define

$$\begin{aligned} &\bar{b}_0 = b_0, && \bar{b}_j = \frac{w(x_{j+1}) - w(x_j)}{x_{j+1} - x_j} \text{ for all } j = 1, ..., m-2, && \bar{b}_{m-1} = b_{m-1}, \\ &\bar{I}_0 = w(x_1) - \bar{b}_0 x_1, && \bar{I}_j = w(x_j) - \bar{b}_j x_j \text{ for all } j = 1, ..., m-1, && \zeta = K. \end{aligned} \tag{24}$$

It is sufficient to show that the solution in (24) satisfies the constraints (7)-(9).

Consider any $j = 1, ..., m-1$. The fifth equation in (24) implies that $w(x_j) = \bar{I}_j + \bar{b}_j x_j$. According to the definition of $\{x_j, I_j, b_j\}$, $W(x_j) = I_j + b_j x_j$. As $\|W - w\|_\infty \leq K$, we have $|W(x_j) - w(x_j)| \leq K = \zeta$, i.e., the solution in (24) satisfies the constraint (7).

The fifth equation in (24) also yields $w(x_{j+1}) = \bar{I}_{j+1} + \bar{b}_{j+1} x_{j+1}$ for any $j = 0, 1, ..., m-2$. When $j = 0$, the forth equation in (24) shows $w(x_1) = \bar{I}_0 + \bar{b}_0 x_1$. For any $j = 1, ..., m-2$,

$$\bar{I}_j + \bar{b}_j x_{j+1} = \bar{I}_j + \bar{b}_j x_j + \bar{b}_j (x_{j+1} - x_j) = w(x_j) + \frac{w(x_{j+1}) - w(x_j)}{x_{j+1} - x_j}(x_{j+1} - x_j) = w(x_{j+1}),$$

where the second inequality is obtained from the second and fifth equations in (24). Therefore, the constraint (8) is also satisfied by the solution in (24).

For any $j = 1, ..., m-3$, as $-\infty < x_j < x_{j+1} < x_{j+2} < +\infty$, the convexity of w implies

$$\begin{aligned} w(x_{j+1}) &\leq \frac{x_{j+2} - x_{j+1}}{x_{j+2} - x_j} w(x_j) + \frac{x_{j+1} - x_j}{x_{j+2} - x_j} w(x_{j+2}) \\ \Big((x_{j+2} - x_{j+1}) + (x_{j+1} - x_j) \Big) w(x_{j+1}) &\leq (x_{j+2} - x_{j+1}) w(x_j) + (x_{j+1} - x_j) w(x_{j+2}) \\ \frac{w(x_{j+1}) - w(x_j)}{x_{j+1} - x_j} &\leq \frac{w(x_{j+2}) - w(x_{j+1})}{x_{j+2} - x_{j+1}}. \end{aligned} \tag{25}$$

Hence, by the second equation in (24), the constraint (9) is satisfied for any $j = 1, ..., m-3$.

Consider any $x < x_1$. Similar to (25), the convexity of w yields

$$\frac{w(x_1) - w(x)}{x_1 - x} \leq \frac{w(x_2) - w(x_1)}{x_2 - x_1} = \bar{b}_1, \text{ i.e., } w(x) \geq w(x_1) - \bar{b}_1 (x_1 - x).$$

The definition of $\{x_j, I_j, b_j\}$ shows $W(x) = I_0 + b_0 x$ and $W(x_1) = I_0 + b_0 x_1$, which imply $W(x) = W(x_1) - b_0(x_1 - x)$. Therefore, $w(x) - W(x) \geq w(x_1) - W(x_1) + (b_0 - \bar{b}_1)(x_1 - x)$. If $b_0 > \bar{b}_1$, then $w(x_1) - W(x_1) + (b_0 - \bar{b}_1)(x_1 - x) > K$ for sufficiently small x, which contradicts $\|W - w\|_\infty \leq K$. As a result, we have $\bar{b}_0 = b_0 \leq \bar{b}_1$, i.e., the constraint (9) is satisfied when $j = 0$.

Similarly, for any $x > x_{m-1}$, the convexity of w implies

$$\frac{w(x) - w(x_{m-1})}{x - x_{m-1}} \geq \frac{w(x_{m-1}) - w(x_{m-2})}{x_{m-1} - x_{m-2}} = \bar{b}_{m-2}, \text{ i.e., } w(x) \geq w(x_{m-1}) + \bar{b}_{m-2}(x - x_{m-1}).$$

The piecewise linear property of W yields $W(x) = W(x_{m_1-1}) + b_{m-1}(x - x_{m_1-1})$. Thus, $w(x) - W(x) \geq w(x_{m-1}) - W(x_{m-1}) + (\bar{b}_{m-2} - b_{m-1})(x - x_{m_1-1})$. If $\bar{b}_{m-2} > b_{m-1}$, then $w(x_{m-1}) - W(x_{m-1}) + (\bar{b}_{m-2} - b_{m-1})(x - x_{m_1-1}) > K$ for sufficiently large x, which contradicts $\|W - w\|_\infty \leq K$. Therefore, we obtain $\bar{b}_{m-2} \leq b_{m-1} = \bar{b}_{m-1}$, i.e., the constraint (9) is satisfied when $j = m-2$. □

Proof of Lemma 1. (a) Consider $x_f \in \arg\min_{x \in X} f(x)$ and $x_g \in \arg\min_{x \in X} f(x)$. Then

$$\min_{x \in X} f(x) = f(x_f) \leq f(x_g) \leq g(x_g) + K = \min_{x \in X} g(x) + K,$$

where the first inequality follows from the optimality of x_f and the second inequality follows from $|f(x) - g(x)| \leq K$. Symmetrically, we have $\min_{x \in X} g(x) \leq \min_{x \in X} f(x) + K$, which completes the proof of part (a).

(b) According to part (a), $f(x_f) = \min_{x \in X} f(x) \leq \min_{x \in X} g(x) + K$. $\|f - g\|_\infty \leq K$ yields $f(x_f) \geq g(x_f) - K$. Therefore, we obtain $g(x_f) - \min_{x \in X} g(x) \leq 2K$. □

Proof of Lemma 2. First, consider a piecewise linear function $\bar{R}^*(d)$ defined as

$$\bar{R}^*(d) = \mu_i(d - d_i) + \sum_{j=0}^{i-1} \mu_j (d_{j+1} - d_j) + r_0 + \frac{1}{2} \sum_{j=0}^{N} (\beta_j - \mu_j)(d_{j+1} - d_j)$$

for any $d \in [d_i, d_{i+1}]$ and $i = 0, 1, ..., N$. The definition of $\bar{R}(d)$ yields

$$\bar{R}^*(d) - \hat{R}(d) = (\mu_i - \beta_i)(d - d_i) + \sum_{j=0}^{i-1}(\mu_j - \beta_j)(d_{j+1} - d_j) + \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j)$$

for any $d \in [d_i, d_{i+1}]$ and $i = 0, 1, ..., N$. Note that $\mu_i \leq \beta_i$ and thus $\bar{R}^*(d) - \hat{R}(d)$ is a decreasing function over $[d_0, d_{N+1}]$. Therefore,

$$\begin{aligned}\|\bar{R}^* - \hat{R}\|_\infty &= \max_{d\in[d_0,d_{N+1}]} |\bar{R}^*(d) - \hat{R}(d)| = \max\left\{|\bar{R}^*(d_0) - \hat{R}(d_0)|, |\bar{R}^*(d_{N+1}) - \hat{R}(d_{N+1})|\right\} \\ &= \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j).\end{aligned}$$

Consider $\bar{R}(d)$, which corresponds to an optimal solution to (11). As $\mu_i \geq \mu_{i+1}$ for any $i = 0, 1, ...N-1$, $\bar{R}^*(d)$ corresponds to a feasible solution to (11). The optimality of $\bar{R}(d)$ implies

$$\max_{i=0,...,N+1} |\bar{R}(d) - r_i| \leq \max_{i=0,...,N+1} |\bar{R}^*(d) - r_i|.$$

The piecewise linear properties of both $\bar{R}^*(d)$ and $\bar{R}(d)$ yields

$$\|\bar{R}^* - \hat{R}\|_\infty = \max_{i=0,...,N+1} |\bar{R}^*(d) - r_i| = \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j) \text{ and } \|\bar{R} - \hat{R}\|_\infty = \max_{i=0,...,N+1} |\bar{R}(d) - r_i|.$$

Hence, we obtain

$$\|\bar{R} - \hat{R}\|_\infty \leq \|\bar{R}^* - \hat{R}\|_\infty = \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j). \qquad \square$$

Proof of Theorem 1. (i) Suppose that $R(d)$ is Lipschitz continuous with the Lipschitz constant L_R. For $i = 0, ..., N$, define

$$\widehat{d}_i = \frac{r_i - r_{i+1}}{2L_R} + \frac{d_i + d_{i+1}}{2} \quad \text{and} \quad \widetilde{d}_i = \frac{r_{i+1} - r_i}{2L_R} + \frac{d_i + d_{i+1}}{2}.$$

Moreover, define

$$R^L(d) = \begin{cases} -L_R(d - d_i) + r_i & \text{if } d \in [d_i, \widehat{d}_i], \quad i = 0, ..., N, \\ L_R(d - d_{i+1}) + r_{i+1} & \text{if } d \in [\widehat{d}_i, d_{i+1}], \quad i = 0, ..., N. \end{cases}$$

and

$$R^U(d) = \begin{cases} L_R(d - d_i) + r_i & \text{if } d \in [d_i, \widetilde{d}_i], \quad i = 0, ..., N, \\ -L_R(d - d_{i+1}) + r_{i+1} & \text{if } d \in [\widetilde{d}_i, d_{i+1}], \quad i = 0, ..., N. \end{cases}$$

First, we prove that $R^L(d)$ and $R^U(d)$ are lower and upper bounds on the real revenue function, respectively, i.e., $R^L(d) \leq R(d) \leq R^U(d)$. We start with showing that $R^L(d)$ is a lower bound of $R(d)$ by contradiction. Suppose there exists $\nu \in [d_i, d_{i+1}]$ such that $R(\nu) < R^L(\nu)$. If $\nu \in [d_i, \widehat{d}_i]$, then we have

$$\frac{R(d_i) - R(\nu)}{d_i - \nu} = \frac{r_i - R(\nu)}{d_i - \nu} < \frac{r_i - R^L(\nu)}{d_i - \nu} = -L_R,$$

which contradicts to (12). If $\nu \in [\widehat{d}_i, d_{i+1}]$, then we have

$$\frac{R(d_{i+1}) - R(\nu)}{d_{i+1} - \nu} = \frac{r_{i+1} - R(\nu)}{d_{i+1} - \nu} > \frac{r_{i+1} - R^L(\nu)}{d_{i+1} - \nu} = L_R,$$

which again contradicts to (12). Hence, $R^L(d)$ is a lower bound of $R(d)$. The proof that $R^U(d)$ is an upper bound is similar and thus, omitted.

We next show $R^L(d) \le \hat{R}(d) \le R^U(d)$ and then build an upper bound on $\|\hat{R}(d) - R(d)\|_\infty$. By the definition of β_i, $\hat{R}(d)$ can be rewritten as

$$\hat{R}(d) = \beta_i(d - d_i) + r_i, \quad \text{for any } d \in [d_i, d_{i+1}].$$

As $R(d)$ is continuous with Lipschitz constant L_R, the definition of β_i yields $-L_R \le \beta_i \le L_R$, $i = 0, 1, ..., N$. Thus, one can readily prove that $R^L(d) \le \hat{R}(d) \le R^U(d)$. As a result, we have

$$\begin{aligned} \|\hat{R} - R\|_\infty &\le \sup_{d \in [d_0, d_{N+1}]} \max \left\{ \hat{R}(d) - R^L(d), R^U(d) - \hat{R}(d) \right\} \\ &= \max_{i=0,\dots,N} \left\{ \sup_{d \in [d_i, d_{i+1}]} \left\{ \hat{R}(d) - R^L(d) \right\}, \sup_{d \in [d_i, d_{i+1}]} \left\{ R^U(d) - \hat{R}(d) \right\} \right\}. \end{aligned}$$

For any $i = 0, 1, ..., N$ and $d \in [d_i, d_{i+1}]$, $\hat{R}(d)$ is a linear function and $R^L(d)$ is a piecewise linear continuous function with breakpoints $\{d_i, \widehat{d}_i, d_{i+1}\}$. As $\hat{R}(d_i) = R^L(d_i)$ and $\hat{R}(d_{i+1}) = R^L(d_{i+1})$,

$$\begin{aligned} &\sup_{d \in [d_i, d_{i+1}]} \left\{ \hat{R}(d) - R^L(d) \right\} = \hat{R}(\widehat{d}_i) - R^L(\widehat{d}_i) = (\beta_i + L_R)(\widehat{d}_i - d_i) \\ &= (\beta_i + L_R) \left(\frac{r_i - r_{i+1}}{2L_R} + \frac{d_i + d_{i+1}}{2} - d_i \right) = (\beta_i + L_R) \left(-\beta_i \frac{d_{i+1} - d_i}{2L_R} + \frac{d_{i+1} - d_i}{2} \right) \\ &= \frac{d_{i+1} - d_i}{2} (\beta_i + L_R) \left(1 - \frac{\beta_i}{L_R} \right) = \frac{d_{i+1} - d_i}{2} \left(L_R - \frac{\beta_i^2}{L_R} \right) \le \frac{L_R}{2}(d_{i+1} - d_i). \end{aligned}$$

Similarly,

$$\sup_{d \in [d_i, d_{i+1}]} \left\{ R^U(d) - \hat{R}(d) \right\} \le \frac{L_R}{2}(d_{i+1} - d_i).$$

Therefore,

$$\|\hat{R} - R\|_\infty \le \frac{L_R}{2} \max_{i=0,\dots,N} \{d_{i+1} - d_i\}.$$

By Lemma 2, we have

$$\|\bar{R} - R\|_\infty \le \|\bar{R} - \hat{R}\|_\infty + \|\hat{R} - R\|_\infty \le \frac{1}{2} \sum_{j=0}^{N} (\beta_j - \mu_j)(d_{j+1} - d_j) + \frac{L_R}{2} \max_{i=0,\dots,N} \{d_{i+1} - d_i\}.$$

(ii) Now, we consider the case that $R(d)$ is not only Lipschitz continuous with the Lipschitz constant L_R, but also quasi-concave. Define $I = \arg\max\{r_i\}$, which represents the index with the largest values among the realized points. For $i = 0, ..., N$, define

$$\widehat{d_i} = \begin{cases} d_{i+1} + \frac{r_i - r_{i+1}}{L_R} & \text{if } \; i = 0, ..., I-1, \\ d_i + \frac{r_i - r_{i+1}}{L_R} & \text{if } \; i = I, ..., N, \end{cases} \quad \text{and} \quad \widetilde{d_i} = \begin{cases} d_i + \frac{r_{i+1} - r_i}{L_R} & \text{if } \; i = 0, ..., I-1, \\ d_{i+1} + \frac{r_{i+1} - r_i}{L_R} & \text{if } \; i = I, ..., N. \end{cases}$$

Moreover, we define

$$R^L(d) = \begin{cases} r_i & \text{if } \; d \in [d_i, \widehat{d_i}], \quad i = 0, ..., I-1, \\ L_R(d - d_{i+1}) + r_{i+1} & \text{if } \; d \in [\widehat{d_i}, d_{i+1}], \quad i = 0, ..., I-1, \\ -L_R(d - d_i) + r_i & \text{if } \; d \in [d_i, \widehat{d_i}], \quad i = I, ..., N, \\ r_{i+1} & \text{if } \; d \in [\widehat{d_i}, d_{i+1}], \quad i = I, ..., N. \end{cases}$$

and

$$R^U(d) = \begin{cases} L_R(d - d_i) + r_i & \text{if } \; d \in [d_i, \widetilde{d_i}], \quad i = 0, ..., I-1, \\ r_{i+1} & \text{if } \; d \in [\widetilde{d_i}, d_{i+1}], \quad i = 0, ..., I-1, \\ r_i & \text{if } \; d \in [d_i, \widetilde{d_i}], \quad i = I, ..., N, \\ -L_R(d - d_{i+1}) + r_{i+1} & \text{if } \; d \in [\widetilde{d_i}, d_{i+1}], \quad i = I, ..., N. \end{cases}$$

Next, we prove that $R^L(d)$ and $R^U(d)$ are lower and upper bounds on the real revenue function, i.e., $R^L(d) \leq R(d) \leq R^U(d)$. We first show that $R^L(d)$ is a lower bound of $R(d)$ by contradiction. Suppose there exists $\nu \in [d_i, d_{i+1}]$ such that $R(\nu) < R^L(\nu)$. If $\nu \in [d_i, \widehat{d_i}]$, where $i \leq I-1$, then $R(\nu) - R(r_i) < R^L(\nu) - R(r_i) = 0$, which contradicts to the fact that $R(d)$ is increasing over $[d_0, d_{I-1}]$. If $\nu \in [\widehat{d_i}, d_{i+1}]$, where $i \leq I-1$, then, we have

$$\frac{R(d_{i+1}) - R(\nu)}{d_{i+1} - \nu} = \frac{r_{i+1} - R(\nu)}{d_{i+1} - \nu} > \frac{r_{i+1} - R^L(\nu)}{d_{i+1} - \nu} = L_R,$$

which contradicts to the fact that $R(d)$ is Lipschitz continuous with Lipschitz constant L_R. Similarly, one can prove the case with $\nu \in [d_i, d_{i+1}]$, where $i \geq I$, and prove that $R^U(d) \geq R(d)$.

With the same logic, one can prove $R^L(d) \leq \hat{R}(d) \leq R^U(d)$. Similar to part (i), we obtain

$$\begin{aligned} \|\hat{R} - R\|_\infty &\leq \max_{i=0,...,N} \left\{ \sup_{d \in [d_i, d_{i+1}]} \left\{ \hat{R}(d) - R^L(d) \right\}, \sup_{d \in [d_i, d_{i+1}]} \left\{ R^U(d) - \hat{R}(d) \right\} \right\} \\ &= \max_{i=0,...,N} \left\{ \hat{R}(\widehat{d_i}) - R^L(\widehat{d_i}), R^U(\widetilde{d_i}) - \hat{R}(\widetilde{d_i}) \right\}. \end{aligned}$$

For any $i = 0, ..., I-1$,

$$\hat{R}(\widehat{d_i}) - R^L(\widehat{d_i}) = \beta_i(\widehat{d_i} - d_i) = \beta_i \left(d_{i+1} + \frac{r_i - r_{i+1}}{L_R} - d_i \right)$$

$$= \beta_i(d_{i+1} - d_i) \left(1 - \frac{\beta_i}{L_R} \right) = \frac{d_{i+1} - d_i}{L_R} \left(- \left(\beta_i - \frac{L_R}{2} \right)^2 + \frac{L_R^2}{4} \right) \leq \frac{L_R}{4}(d_{i+1} - d_i).$$

Using a similar argument, we can also show that

$$\hat{R}(\widehat{d}_i) - R^L(\widehat{d}_i) \leq \frac{L_R}{4}(d_{i+1} - d_i) \text{ for any } i = I, ..., N,$$
$$R^U(\widetilde{d}_i) - \hat{R}(\widetilde{d}_i) \leq \frac{L_R}{4}(d_{i+1} - d_i) \text{ for any } i = 0, ..., N.$$

Consequently, we have

$$\|\hat{R} - R\|_\infty \leq \frac{L_R}{4} \max_{i=0,\ldots,N}\{d_{i+1} - d_i\}$$

and hence

$$\|\bar{R} - R\|_\infty \leq \|\bar{R} - \hat{R}\|_\infty + \|\hat{R} - R\|_\infty \leq \frac{1}{2}\sum_{j=0}^{N}(\beta_j - \mu_j)(d_{j+1} - d_j) + \frac{L_R}{4} \max_{i=0,\ldots,N}\{d_{i+1} - d_i\}.$$

(iii) When $R(d)$ is concave, β_i is decreasing in i. For $i = 0, 1, ..., N$, define $R^L(d) = \beta_i(d - d_i) + r_i$, $d \in [d_i, d_{i+1}]$ and

$$\widetilde{d}_i = \frac{r_i - r_{i-1} + \beta_{i-2}d_{i-1} - \beta_i d_i}{\beta_{i-2} - \beta_i}.$$

Moreover, define

$$R^U(d) = \begin{cases} \beta_1(d - d_1) + r_1 & \text{if } d \in [d_0, d_1], \\ \beta_{i-2}(d - d_{i-1}) + r_{i-1} & \text{if } d \in [d_{i-1}, \widetilde{d}_i], \quad i = 2, ..., N, \\ \beta_i(d - d_i) + r_i & \text{if } d \in [\widetilde{d}_i, d_i], \quad i = 1, 2, ..., N, \\ \beta_{N-1}(d - d_N) + r_N & \text{if } d \in [d_N, d_{N+1}]. \end{cases}$$

Similar to parts (i) and (ii), we next prove that $R^L(d)$ and $R^U(d)$ are lower and upper bounds on the real revenue function, respectively, i.e., $R^L(d) \leq R(d) \leq R^U(d)$. We first prove that $R^L(d)$ is a lower bound of $R(d)$ by contradiction. Suppose there exists $\nu \in [d_i, d_{i+1}]$ such that $R(\nu) < R^L(\nu)$. Then, we have

$$\frac{R(\nu) - R(d_i)}{\nu - d_i} = \frac{R(\nu) - r_i}{\nu - d_i} < \frac{R^L(\nu) - r_i}{\nu - d_i} = \beta_i,$$

and

$$\frac{R(d_{i+1}) - R(\nu)}{d_{i+1} - \nu} = \frac{r_{i+1} - R(\nu)}{d_{i+1} - \nu} > \frac{r_{i+1} - R^L(\nu)}{d_{i+1} - \nu} = \beta_i.$$

Hence, $\frac{R(\nu)-R(d_i)}{\nu-d_i} < \frac{R(d_{i+1})-R(\nu)}{d_{i+1}-\nu}$ which contradicts to the concavity of $R(d)$.

Now, we prove that $R^U(d)$ is an upper bound again by contradiction. Suppose there exists $\nu \in [d_0, d_{N+1}]$ such that $R(\nu) > R^U(\nu)$. If $\nu \in [d_0, d_1]$, then one can readily prove that

$$\frac{R(d_1) - R(\nu)}{d_1 - \nu} = \frac{r_1 - R(\nu)}{d_1 - \nu} < \frac{r_1 - R^U(\nu)}{d_1 - \nu} = \beta_1,$$

which contradicts to the concavity of $R(d)$. The proofs for the cases $\nu \in [d_i, d_{i+1}]$, $i = 1, ..., N$, are similar and thus omitted.

Note that in this case, $\bar{R}(d) = \hat{R}(d) = R^L(d)$. Then, we have

$$\|\bar{R} - R\|_\infty \leq \|R^U - R^L\|_\infty \leq \max_{i=1,\dots,N} \{(\beta_{i-1} - \beta_i)(d_{i+1} - d_i)\}.$$

Hence, the result holds. $\square$

Proof of Theorem 2. Besides the result in Theorem 2, we also show in this proof that $\|V_t - W_t\|_\infty \leq \sum_{i=0}^{T-t} \alpha^i K$ for any $t \in \{1, ..., T\}$. Note that $V_{T+1}(x_{T+1}) = W_{T+1}(x_{T+1}) = \bar{V}_{T+1}(x_{T+1})$ and hence both results hold when $t = T+1$. Suppose that $\|V_{t+1} - W_{t+1}\|_\infty \leq \sum_{i=0}^{T-t-1} \alpha^i K$ and $\bar{V}_{t+1}(x_{t+1}) \geq V_{t+1}(x_{t+1}) - 2K \sum_{i=0}^{T-t-1} (i+1)\alpha^i$ for some $t \in \{1, ..., T\}$.

Consider any $x_t \in \mathbb{R}$. Define

$$(\bar{y}_t(x_t), \bar{d}_t(x_t)) \in \arg \max_{\substack{y_t \geq x_t, \\ d_t \in [\underline{d}_t, \bar{d}_t]}} \Big\{ \bar{R}(d_t) - c_t(y_t - x_t) + E\big[-H_t(y_t - a_t d_t - b_t) + \alpha W_{t+1}(y_t - a_t d_t - b_t)\Big\}.$$

The induction assumption yields

$$\begin{aligned}
\bar{V}_t(x_t) &= R(\bar{d}_t(x_t)) - c_t(\bar{y}_t(x_t) - x_t) \\
&\qquad + E\big[-H_t(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t) + \alpha \bar{V}_{t+1}(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t)\big] \\
&\geq R(\bar{d}_t(x_t)) - c_t(\bar{y}_t(x_t) - x_t) - 2K \sum_{i=0}^{T-t-1} (i+1)\alpha^{i+1} \\
&\qquad + E\big[-H_t(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t) + \alpha V_{t+1}(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t)\big].
\end{aligned} \tag{26}$$

For all $y_t \geq x_t$ and $d_t \in [\underline{d}_t, \bar{d}_t]$,

$$\begin{aligned}
&\Big| \Big\{ R(d_t) - c_t(y_t - x_t) + E\big[-H_t(y_t - a_t d_t - b_t) + \alpha V_{t+1}(y_t - a_t d_t - b_t)\big] \Big\} \\
&\qquad - \Big\{ \bar{R}(d_t) - c_t(y_t - x_t) + E\big[-H_t(y_t - a_t d_t - b_t) + \alpha W_{t+1}(y_t - a_t d_t - b_t)\big] \Big\} \Big| \\
&\leq \Big| R(d_t) - \bar{R}(d_t) \Big| + \Big| E\big[\alpha V_{t+1}(y_t - a_t d_t - b_t)\big] - E\big[\alpha W_{t+1}(y_t - a_t d_t - b_t)\big] \Big| \\
&\leq \Big| R(d_t) - \bar{R}(d_t) \Big| + \alpha E\Big[\big| V_{t+1}(y_t - a_t d_t - b_t) - W_{t+1}(y_t - a_t d_t - b_t) \big| \Big] \\
&= K + \alpha \sum_{i=0}^{T-t-1} \alpha^i K = \sum_{i=0}^{T-t} \alpha^i K.
\end{aligned} \tag{27}$$

Applying Lemma 1 (a), we have $|V_t(x_t) - W_t(x_t)| \leq \sum_{i=0}^{T-t} \alpha^i K$, i.e., $\|V_t - W_t\|_\infty \leq \sum_{i=0}^{T-t} \alpha^i K$. According to Lemma 1 (b), (27) yields

$$\begin{aligned}
V_t(x_t) - 2 \sum_{i=0}^{T-t} \alpha^i K &\leq R(\bar{d}_t(x_t)) - c_t(\bar{y}_t(x_t) - x_t) \\
&\qquad + E\big[-H_t(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t) + \alpha V_{t+1}(\bar{y}_t(x_t) - a_t \bar{d}_t(x_t) - b_t)\big].
\end{aligned}$$

Combining with (26), we obtain

$$\bar{V}_t(x_t) \geq V_t(x_t) - 2\sum_{i=0}^{T-t} \alpha^i K - 2K \sum_{i=0}^{T-t-1} (i+1)\alpha^{i+1} = V_t(x_t) - 2K \sum_{i=0}^{T-t} (i+1)\alpha^i. \qquad \square$$