Process of debonding in RC beams shear-strengthened with FRP U strips or side strips

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Abstract: Reinforced concrete (RC) beams may be strengthened for shear using externally bonded fibre reinforced polymer (FRP) composites in the form of side bonding, U-jacketing or complete wrapping. The shear failure of almost all RC beams shear-strengthened with side bonded FRP and the majority of those strengthened with FRP U-jackets, is due to debonding of the FRP. The bond behaviour between the externally-bonded FRP reinforcement (referred to as FRP strips for simplicity) and the concrete substrate therefore plays a crucial role in the failure process of these beams. Despite extensive research in the past decade, there is still a lack of understanding of how debonding of FRP strips in such a beam propagates and how the debonding process affects its shear behaviour. This paper presents an analytical study on the progressive debonding of FRP strips in such strengthened beams. The complete debonding process is modelled and the contribution of the FRP strips to the shear capacity of the beam is quantified. The validity of the analytical solution is verified by comparing its predictions with numerical results from a finite element analysis. This analytical treatment represents a significant step forward in understanding how interaction between FRP strips, steel stirrups and concrete affects the shear resistance of RC beams shear-strengthened with FRP strips.

Keywords: FRP; Concrete; Bond; Interface; Interfacial stress; Strengthening; Shear failure; Shear resistance; Stress distribution; Strain distribution; Bond-slip model; Debonding

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1. Introduction

The shear resistance of reinforced concrete (RC) beams can be enhanced using externally bonded fibre-reinforced polymer (FRP) reinforcement. The FRP reinforcement can be bonded around the entire section (complete wrapping), to the two sides as well as the soffit of the beam (U-jacketing), and to the two sides of the beam only (side bonding) (Teng et al. 2002). Without loss of generality, the FRP shear reinforcement is assumed to be in the form of discrete strips in this paper; a continuous sheet can be represented as discrete strips with a zero net gap between strips (Chen and Teng 2003a).

Extensive research has been conducted on the shear strengthening of RC beams with externally bonded FRP in the past decade (e.g. Triantafillou 1998; Khalifa and Nanni 2000, 2002; Taljsten 2003; Chen and Teng 2003a, 2003b; Islam 2005; Pellegrino and Modena 2002, 2006; Bousselham and Chaallal 2004, 2006a, 2006b, 2008; Monti and Liotta 2007; Teng et al. 2009), but several aspects of the behaviour of such strengthened beams are still not well understood (Teng et al. 2002, 2004; Bousselham and Chaallal 2004, 2008; Chen 2010). Examples of these aspects include shear interaction between the different components of the shear resistance (e.g. Mohamed Ali et al. 2005; Teng et al. 2009; Chen et al. 2010a), the effect of beam size (e.g. Leung et al. 2007), and the effect of shear span-to-depth ratio (e.g. Bousselham and Chaallal 2004).

Almost all RC beams shear-strengthened with FRP side strips (side-bonding), and most of those strengthened with U-strips (U-jacketing) fail due to the debonding of FRP strips from the beam sides (Chen and Teng 2003a; Teng and Chen 2009). This failure mode is usually brittle so that the width of the critical shear crack is limited when debonding occurs. As a result, not all steel stirrups intersected by the critical shear crack may reach yielding at the shear failure of the beam (Teng et al. 2002, 2004; Li et al. 2002; Carolin and Taljsten 2005; Mohamed Ali et al. 2006; Pellegrino and Modena 2002, 2006; Monti and Liotta 2007; Bousselham and Chaallal 2008). Consequently, the steel shear reinforcement may not be fully mobilized and may contribute less than what is predicted by existing shear strength models when beam shear failure occurs due to debonding (Chen 2010). Chen et al. (2010a) has recently demonstrated that adverse interaction between external FRP shear reinforcement and internal steel shear reinforcement may significantly affect the effectiveness of FRP shear strengthening for the debonding failure mode, especially when side strips are used. However, the effect of such shear interaction is not appropriately reflected in existing shear strength models for RC beams shear-strengthened with FRP (Chen 2010). To quantitatively assess the effect of shear interaction, it is necessary to understand the development of shear resistance contributed by both the external FRP and the internal steel shear reinforcements as the critical shear crack widens. The development of shear contribution of steel stirrups with the crack width has been numerically investigated by Chen et al. (2010a).

This paper presents a closed-form solution for the entire debonding process of FRP U-strips or side strips used to enhance the shear resistance of RC beams, allowing the different stages of debonding to be clearly explained and understood. A particularly important outcome of the solution is the closed-form expressions that describe the development of shear contribution of externally bonded FRP shear reinforcement to the shear resistance of an RC beam as the critical shear crack widens. While the FRP shear reinforcement is assumed to be in the form of discrete strips, the discrete FRP strips are smeared into an equivalent continuous FRP sheet as a convenient approximation for analytical treatment. As a result, the analytical solution is exact
for FRP shear reinforcement in the form of a continuous FRP sheet or discrete FRP strips with zero net gaps, but is approximate for discrete FRP strips with non-zero gaps.

The solution is developed based on an analytical solution for the full-range behavior of FRP-to-concrete bonded joints, and the assumption of a linear crack shape. The validity of the solution is demonstrated by comparing its predictions with finite element (FE) predictions. An important merit of the closed-form solution is that it can be directly employed in investigating the effect of shear interaction on the shear strength of FRP shear-strengthened RC beams (Chen et al. 2010). It also provides valuable insight into the debonding failure process of RC beams shear-strengthened with FRP.

Monti et al. (2004) [see also Liotta (2006)] were probably the first to present a closed-form solution for the shear contribution of shear-strengthening FRP as a function of the shear crack width. They made the following two assumptions: (1) the crack width varies linearly from the crack tip to the crack end; and (2) the slips at the two sides of the shear crack are symmetrical and equal to half of the crack width. The first assumption is also adopted in the present study because it has been shown that a linear crack width variation is the most critical among the many different crack width shapes examined by Chen et al. (2010b). The slips between FRP and concrete at the two sides of the shear crack can be very different, particularly for FRP U-strips as clearly demonstrated in Chen et al. (2010a). Consequently, the slip symmetry assumption is inaccurate as shown in the Appendix. Therefore, the slips at the two sides of the shear crack are rigorously determined in this study. The present solution also differs from that of Monti et al. (2004) in the following aspects:

1. The present solution is strictly based on an analytical solution for FRP-to-concrete bonded joints with different end anchorage conditions, whilst Monti et al.’s (2004) solution is based on a simplified solution for FRP-to-concrete bonded joints. This leads to substantially different predictions of the FRP shear contribution at the ultimate state especially for FRP side strips as shown in the Appendix.

2. The present solution includes the full-range behaviour for the development of FRP shear contribution with the shear crack width, while Monti et al. (2004) only aimed to obtain the maximum effective stress in FRP strips and hence the FRP contribution to the shear capacity of the strengthened beam at the ultimate state (see Appendix).

3. The present study considers bonded areas of FRP as found in practical beams. Monti et al. (2004) considered only an idealized triangular bonded area of FRP, neglecting the bonded areas above the crack tip and beneath the crack end.

2. Full-range behavior of FRP-to-concrete bonded joints

Only the bond behavior in the fiber direction of the FRP is considered in this study. This means that an FRP strip bonded to the side of a beam can be represented by an FRP strip bonded to a concrete prism (i.e. an FRP-to-concrete bonded joint) as shown in Fig. 1. With this assumption, the FRP-to-concrete bond behaviour can be examined by studying two types of FRP-to-concrete bonded joints: type A joint in which the FRP strip is free at the far end and type B joint in which the FRP strip is fixed at the far end (Figs 1c and 1d). A side-bonded FRP strip can be modelled using two type A joints representing the interfaces above and below the critical shear crack (Figs 1b and 1c). An FRP U-strip is generally wrapped around a rounded corner and extended to the beam soffit. It can be represented by one type A joint for the interface above, and one type B joint for the interface below the critical shear crack, by treating the end of the FRP strip at the beam bottom as fixed (Figs 1b and 1d). It should be noted that
the beam corner may have an effect on the effective FRP rupture strain (Teng et al. 2002) but is unlikely to affect the debonding strain.

For simplicity and without loss of generality, a linearly softening bond-slip model as shown in Fig. 2 is adopted to obtain the full-range behavior of both type A and type B joints following Chen et al. (2007). The linearly softening bond-slip model as shown in Fig. 2 is characterized by the maximum interfacial shear stress $\tau_{\text{max}}$ and the interfacial slip $\delta_f$ at which the interfacial stress is reduced to zero, giving the interfacial fracture energy $G_f = \tau_{\text{max}} \delta_f / 2$. It should be noted that an accurate bond-slip model should have a nonlinear ascending branch and a nonlinear softening branch and a bilinear model can provide a close approximation (Lu et al. 2005). However, it is much more involved to obtain a closed-form solution using such a more complex bond-slip model. Furthermore, Chen et al. (2007) compared the solutions based on a bilinear bond-slip model and a linearly softening bond-slip model respectively and showed that the latter leads to a simple yet sufficiently accurate solution for FRP-to-concrete bonded joints if the fracture energy is the same in both bond-slip models.

### 2.1 Behaviour of an FRP-to-concrete bonded joint with a free far end (type A joint)

Chen et al. (2007) presented an analytical solution for an FRP-to-concrete bonded joint with both ends of the FRP strip subjected to a tensile force. A type A bonded joint (Fig. 1c) is a special case of this joint with the tensile force equal to zero at one end. Therefore, an analytical solution for the full range load-displacement behaviour of a type A joint can be obtained from Chen et al.’s (2007) solution. A summary of this solution is presented here but further details can be found in Chen (2010).

A typical full range load-displacement response of the bonded joint can be characterized by four key points O, A, B and C when the FRP bond length $L$ is larger than the effective bond length $a_e$ (Fig. 3). Accordingly, the curve features three segments, namely segments $\overline{OA}$, $\overline{AB}$ and $\overline{BC}$, which represent the ascending, plateau, and linearly unloading regions of the load-displacement response respectively. The force $P$, and the displacement (slip between the FRP and the concrete) $\Delta$ at the loaded end of the FRP strip for the three segments are as follows.

For segment $\overline{OA}$:

$$P = \frac{\tau_f b_f}{\lambda} \sin(\lambda a)$$  \hspace{1cm} (1)

$$\Delta = \delta_f \left[1 - \cos(\lambda a)\right]$$  \hspace{1cm} (2)

and hence

$$P = \frac{\tau_f b_f}{\lambda \delta_f} \frac{\sin(\lambda a)}{1 - \cos(\lambda a)} \Delta$$  \hspace{1cm} (3)

where

$$\lambda = \sqrt{\frac{\tau_f}{\delta_f E_f t_f}}$$  \hspace{1cm} (4)

For segment $\overline{AB}$:
P = \frac{\tau_f b_f}{\lambda} \\
\Delta = \delta_f + \delta_f \lambda d 
\tag{5}

\text{where}

0 \leq d \leq L - a_u 
\tag{6}

\frac{a_u}{2\lambda} 
\tag{7}

\text{For segment BC :}

\Delta = \Delta_u + \left(\frac{P_u - P}{P_u}\right) (\Delta_u - \Delta_u) 
\tag{8}

\text{where}

\begin{align*}
P_u & = \begin{cases} 
\frac{\tau_f b_f}{\lambda} \sin(\lambda L) & L \leq a_u \\
\frac{\tau_f b_f}{\lambda} & L > a_u 
\end{cases} \\
\Delta_u & = \begin{cases} 
\delta_f [1 - \cos(\lambda L)] & L \leq a_u \\
\delta_f + \delta_f \lambda (L - a_u) & L > a_u 
\end{cases} 
\tag{9}
\end{align*}

In Eqs (1)-(11), \( P_u \) is the maximum force that can be carried by the FRP strip with a bond length of \( L \), \( \Delta_u \) is the displacement at the loaded end corresponding to \( P_u \), \( b_f, t_f \) and \( E_f \) are the width, thickness, and modulus of elasticity of the FRP strip respectively, \( a \) is the mobilized bond length, beyond which the interfacial shear stress is zero (Chen et al. 2007), \( a_u \) is the effective bond length beyond which an increase of \( L \) does not increase \( P_u \), and \( d \) is the debonded length of the FRP-to-concrete interface starting from the loaded end.

The interfacial shear stress distributions experienced by the FRP-to-concrete interface of a type A joint during the loading process are schematically shown in Figs 4 and 5 for two cases of \( L > a_u \) and \( L \leq a_u \) respectively, corresponding to the different stages of the \( P - \Delta \) curve in Fig. 3. In Figs 4 and 5, the letters “R”, “S” and “D” stand for rigid (intact), softening and debonded parts of the interface respectively.

It should be noted that when \( L \leq a_u \), point \( A \) merges with point \( B \) and segment \( \overline{AB} \) vanishes.

Based on the \( P - \Delta \) response of a type A joint (Fig. 3), the following observations can be made:

1. For the ascending branch of the \( P - \Delta \) curve, there is a unique relationship between \( P \) and \( \Delta \), regardless of the value of the bond length \( L \). This implies that for an FRP side-bonded strip intersected by a critical shear crack (Figs 1a and 1b), the slips on the two sides of the critical shear crack are symmetrical about the critical shear crack before the maximum FRP stress is reached, regardless of the actual FRP bond length above or below the critical shear crack.

2. On the plateau (when the bond length \( L \) is greater than the effective bond length \( a_u \)), the above unique relationship is no longer valid as the displacement is not unique for the same maximum load \( P_u \). This means that the slips on the two sides of the critical shear crack can
be different.

(3) For cases with \( L < a_u \), the load decreases linearly to zero with a linear increase in \( \Delta \) after \( \Delta \) reaches \( \Delta_u \) [Eq. (11)] (segment \( \overline{BC} \) in Fig. 3). Neglecting the resistance of the descending branch results in an underestimation of the force carried by the FRP when \( \Delta \geq \Delta_u \) for \( L < a_u \). However, since the shear-strengthening FRP often covers a large area on the sides of the beam, this underestimation has very little effect on the total force carried by all the FRP since only a very small part of the FRP (with \( L < a_u \)) is affected (as shown later). Furthermore, neglecting the descending branch significantly simplifies the solution presented in this study (Chen 2010). It is thus assumed that for \( L < a_u \), the descending branch of \( P - \Delta \) response is ignored (i.e. \( P \) drops to zero for \( \Delta \geq \Delta_u \)) if not otherwise stated. For \( L \geq a_u \), \( P \) also drops to zero for \( \Delta \geq \Delta_u \) (Fig. 3).

2.2. Behaviour of an FRP-to-concrete bonded joint with a fixed far end (type B joint)

To the best of the authors’ knowledge, no publication to date has been directly concerned with the full-range behaviour of a type B joint (Fig. 1d). However, the solution can be deduced from that for an FRP-to-concrete bonded joint with the FRP strip loaded at both ends (Teng et al. 2006; Chen et al. 2007). As shown in Fig. 6, it is apparent that the behaviour of a type B joint with a bond length equal to \( L \) is identical to that of an FRP-to-concrete bonded joint with a bond length equal to \( 2L \) and loaded at its two ends with the load ratio \( \beta = P_3/P_1 = P_3/P_4 = 1 \).

Based on this observation, the full-range behaviour of a type B joint with a linearly softening bond-slip relationship can be deduced from the solution already given in Chen et al. (2007) as follows.

The full-range behaviour of a type B joint can be characterized by key points \( O, B', D' \) and \( P' \) when the bond length is small (\( L \leq a_u \)) and key points \( O, A, B \) and \( P \) when the bond length is large (\( L > a_u \)) as schematically shown in Fig. 7. Accordingly, the full range load-displacement response can be divided into three segments, namely segments \( \overline{OB'} \), \( \overline{BD'} \), \( \overline{DP'} \) for \( L \leq a_u \) and \( \overline{OA} \), \( \overline{AB} \) and \( \overline{BP} \) for \( L > a_u \).

Similar to Figs 4 and 5, the interfacial shear stress distributions along the FRP-to-concrete interface experienced by a type B joint during the loading process are schematically shown in Fig. 8 for \( L > a_u \) and in Fig. 9 for \( L \leq a_u \), corresponding to the different stages of the \( P - \Delta \) curve (Fig. 7).

The solutions for segments \( \overline{OA} \) and \( \overline{AB} \) for the case of \( L > a_u \) and \( \overline{OB'} \) for the case of \( L \leq a_u \) (Fig. 7) of the load-displacement response are not affected by the boundary condition at the far end so they are the same as those for a type A joint and can be found in Eqs (1)-(11).

For segments \( \overline{BD'} \) of a type B joint with a short bond length \( (L \leq a_u) \), the solution is

\[
\Delta = \Delta_{\text{max}} + \frac{P}{E_f t_f b_f} - \delta_y \lambda \sin(\lambda L) \left[ \frac{\tan(\lambda L)}{\lambda} \right]
\]  

\[\text{(12)}\]
\[ \Delta_{\text{max}} = \delta_f \left[ 1 - \cos(\lambda L) \right] \]  

At the end of this stage (i.e. at point \( D' \)), the displacement \( \Delta \) and the corresponding force \( P \) at the loaded end are

\[ \Delta = \delta_f \]

\[ P = E_f t_f b_f \lambda \delta_f \left[ \sin(\lambda L) + \frac{\cos(\lambda L)}{\tan(\lambda L)} \right] = \frac{\tau_f b_f}{\lambda} \frac{1}{\sin(\lambda L)} \]  

The solution for segment \( \overline{BP} \) of a type B joint with a long bond length \( (L > a_u) \) is the same as that for segment \( \overline{DP} \) for a short bond length \( (L \leq a_u) \) and is given by

\[ P = \frac{\tau_f b_f}{\lambda} \frac{1}{\sin(\lambda a')} \]  

\[ \Delta = \delta_f + \frac{\delta_f \lambda (L - a')}{\sin(\lambda a')} \]  

where \( a' \) is the length of the softening part of the FRP-to-concrete interface at this stage (Figs 8e and 9d). If the debonded length of the interface is represented by \( d \) \( (0 \leq d \leq L) \), \( a' \) can be related to the debonded length through

\[ a' = L - d \]  

Note that \( a' \) approaches zero (i.e. \( d \) approaches \( L \)) when the FRP-to-concrete interface approaches the state of complete debonding. Accordingly, the slope of the load-displacement curve is

\[ K_{p-\Delta} = \lim_{a \to 0} \frac{P}{\Delta} = \lim_{a \to 0} \frac{\tau_f b_f}{\lambda} \frac{1}{\sin(\lambda a')} \]  

Substituting Eq. (4) into Eq. (19) gives

\[ K_{p-\Delta} = \frac{E_f b_f t_f}{L} \]  

which represents the slope of the asymptote of the load-displacement curve of a type B joint (Fig. 7). For an FRP-to-concrete bonded joint with the FRP strip equally loaded at both ends, there is no complete debonding based on the idealized model as demonstrated by Teng et al. (2006) and Chen et al. (2007) so the upper limit of the load should be controlled by the FRP rupture strength. In Fig. 7, this is schematically marked as \( P \) for \( L \leq a_u \) and \( P' \) for \( L > a_u \).

The maximum force an FRP U-strip can resist is controlled by the bond strength above the critical shear crack which is a type A joint (see Fig. 1). When the bond capacity of the FRP strip above the critical shear crack, \( P = \frac{\tau_f b_f}{\lambda} \), is reached, the FRP slip at the crack due to deformation below the shear crack (a type B joint) \( \Delta \) is greater than \( \delta_f \) and smaller than \( \delta_f + \delta_f \lambda (L - a_u) \) when \( L \geq a_u \) (i.e. it lies between \( A \) and \( B \) in Fig. 7). For \( L < a_u \),

\[ \Delta = \delta_f \left[ 1 - \cos(\lambda L) \right] + \delta_f \left[ 1 - \sin(\lambda L) \right] \tan(\lambda L) \]  

\[ \text{for} \ L < a_u \]
which is marked as $C'$ in Fig. 7.

3. Shear contribution of shear-strengthening FRP

Similar to Chen and Teng (2003a, b), only shear tension failure is considered herein where the shear failure process of an RC beam shear-strengthened with FRP is assumed to be dominated by the development of a single critical shear crack at $\theta$ from the beam longitudinal axis (Fig. 10). In practice, additional shear cracks are likely, but the assumption of a single crack is generally conservative for predicting FRP debonding failure (Teng et al. 2006; Chen et al. 2007). Only the contribution of the FRP strips intersected by the critical shear crack is considered here (Fig. 10). As in Chen and Teng (2003a, b), the upper end (i.e. the crack tip) of the shear crack at failure is assumed to be located at $0.1d$ from the compression face of the beam (Fig. 10), with $d$ being the effective depth of the beam. The vertical distance from the upper end of the shear-strengthening FRP strips to the crack tip is assumed to be $h$. Although the lower end of a shear crack is likely to be located at the tension face of the beam, only the portion of the shear crack between the crack tip and the centre of the steel tension reinforcement is considered in this study. The portion of the crack between the centre of the steel tension reinforcement and the tension face is not considered for reasons given in Chen and Teng (2003a, b). The intersection between the steel tension reinforcement and the critical crack is thus termed the ‘crack end’ (Fig. 10) herein and the shear crack between this crack end and the crack tip is termed the ‘effective shear crack’ (Fig. 10) hereafter. The vertical distance from the tension face of the beam to the crack end is $h_b$, and that from the crack tip to the crack end is $h_{f,e}$ ($h_{f,e} = 0.9d$ when the bonded FRP strips cover the full height of the beam).

Based on vertical equilibrium consideration, the shear contribution of shear-strengthening FRP is given by (Chen and Teng 2003a, b):

$$V_f = 2f_{f,e}t_fw_f \frac{h_{f,e}(\cot \theta + \cot \beta)\sin \beta}{s_f}$$

(22)

where $f_{f,e}$ is the effective (average) stress in the FRP strips intersected by the effective shear crack; $w_f$ is the width of an individual FRP strip perpendicular to the fiber direction (all FRP strips are assumed to have the same $w_f$); $s_f$ is the centre-to-centre spacing of FRP strips measured along the longitudinal axis (the FRP strips are assumed to be evenly distributed); $t_f$ is the thickness of the FRP strips; and $\beta$ is the angle between the fiber direction and the beam longitudinal axis.

It should be noted that for a continuous FRP sheet/plate (or FRP strips with zero net gaps), the following relationship exists between $w_f$ and $s_f$:

$$w_f = s_f \sin \beta$$

(23)

Eq. (22) indicates that to evaluate the shear contribution of shear-strengthening FRP strips, it is essential to obtain the effective stress in the FRP strips $f_{f,e}$. If the development of $f_{f,e}$ with the widening of the shear crack is known, the development of the shear contribution of FRP strips ($V_f$) with the opening-up of a shear crack can be obtained from Eq. (22).
The effect of shear crack shape on FRP shear contribution was explored in Chen et al. (2010b) and it was found that a linear crack width variation is the most critical among various crack shapes studied. Therefore, it is assumed that the shear crack width increases linearly from zero at the crack tip to \( w_e \) at the crack end (termed as “crack end width” hereafter) in this study:

\[
w(z) = \frac{z}{h_{f,e}} w_e
\]  

(24)

where \( h_{f,e} \) is the vertical depth of the effective shear crack; and \( z \) is the vertical downward-coordinate starting from the crack tip (Fig. 10).

Following the definition of Chen et al. (2003a, b), the effective stress in FRP \( f_{f,e} \) can be expressed as

\[
f_{f,e} = \frac{\int_0^{h_{f,e}} \sigma(z)dz}{h_{f,e}}
\]  

(25)

where \( \sigma(z) \) is the stress in the FRP strips intersected by the critical shear crack at a coordinate \( z \). Assuming that discrete FRP strips can be treated as an equivalent FRP continuous sheet/plate, Eq. (25) is applicable to beams strengthened with either FRP discrete strips or FRP continuous sheets/plates.

In the following two sections, the solutions for \( f_{f,e} \) for FRP side strips are given first, followed by those for FRP U-strips. For both strengthening schemes, two different solutions are given for the two cases of \( h_c \cosec \beta \leq a_u \) and \( h_c \cosec \beta > a_u \), respectively as presented below.

During the loading process, the FRP bonded area can be divided into the following three zones (schematically shown in Figs 11, 12, 13 and 14):

1. inactive zone, where the interface between FRP and concrete is not yet stressed;
2. mobilised zone, where the interface is in a softening state; and
3. debonded zone, where complete interfacial debonding has been achieved.

Zones 1 and 2 are separated by the softening front where \( \tau = \tau_m \) and \( \delta = 0 \) (also see Figs 4, 5, 8 and 9). Zones 2 and 3 are separated by the debonding front where \( \tau = 0 \) and \( \delta = \delta_i \) (also see Figs 4, 5, 8 and 9).

4. Development of FRP shear contribution with crack width: FRP side strips

The solutions are different depending on the thickness of concrete cover \( h_c \) relative to the effective bond length of the FRP \( a_u \) (see Figs 11 and 12).

4.1. Thin concrete cover with \( h_c \cosec \beta \leq a_u \)

When \( h_c \cosec \beta \leq a_u \), the debonding process of FRP side strips can be divided into three stages:

a) softening stage at which no debonding has taken place. This stage is characterised by \( 0 \leq L_m < h_c \cosec \beta \) (Fig. 11a). Here \( L_m \) is the maximum mobilized bond length in the fibre direction from the critical shear crack to the softening front (Fig. 11);

b) one-way debonding stage at which complete debonding occurs only below the critical shear crack. This stage is characterised by \( h_{f,e} \cosec \beta \leq L_m < a_u \) (Fig. 11b); and
c) two-way debonding stage at which debonding occurs both below (completely) and above (partially) the critical shear crack. This stage is characterised by $L_m \geq a_u$ (Fig. 11c).

For each of the above three stages, the effective FRP stress $f_{e,x}$ and the corresponding crack end width $w_e$ can be deduced as follows.

(1) Softening stage with $0 \leq L_m < h_b \csc \beta$ (Fig. 11a)

During this stage, all the FRP strips are still bonded to the beam sides. The maximum mobilized length $L_m$ is located at the crack end for this stage. The slip and the mobilized bond length $L(z)$ for a given FRP strip are the same on the two sides of the critical shear crack during this stage. Based on this consideration together with Eq. (2), the slips of the FRP strips in the fibre direction at the critical shear crack, $\Delta(z)_{\beta}$, can be obtained as

$$\Delta(z)_{\beta} = \delta_t \left[1 - \cos(\lambda L(z))\right]$$

where $z$ is the vertical coordinate starting from the crack tip (Fig. 11), and $L(z)$ is the length of the mobilized zone in the fiber direction on one side of the critical shear crack (Fig. 11).

Due to compatibility, the slip of an FRP strip at either side of the crack must be equal to half the crack width in the fibre direction:

$$\Delta(z)_{\beta} = \frac{w(z)}{2} \sin(\theta + \beta)$$

Substituting Eqs (26) and (27) into Eq. (24) gives the following relationship between $L(z)$ and $L_m$:

$$\left[1 - \cos(\lambda L(z))\right] = \frac{Z}{h_{f,e}} \left[1 - \cos(\lambda L_m)\right]$$

(28)

Since $0 \leq L_m < a_u$, according to Eq. (1), the stress in FRP at $z$ can be expressed as

$$\sigma(z) = \frac{P(z)}{b_f t_f} = \frac{\tau_f}{\lambda t_f} \sin\left[\lambda L(z)\right] = \sqrt{\frac{2G_f E_{t}}{t_f}} \sin\left[\lambda L(z)\right]$$

(29)

Substituting Eq. (29) into Eq. (25) and applying Eq. (28) give

$$f_{e,x} = \frac{1}{h_{f,e}} \sqrt{\frac{2G_f E_{t}}{t_f}} \sin\left[\lambda L(z)\right] dz = 2 \sqrt{\frac{G_f E_{t}}{t_f}} D_{fp}$$

(30)

$$D_{fp} = \frac{1}{2} \frac{a_u - \sin(\lambda L_m) \cos(\lambda L_m)}{2 - 2 \cos(\lambda L_m)}$$

(31)

where $D_{fp}$ is the stress distribution factor as defined in Chen and Teng (2003b).

The corresponding shear crack width at the crack end can be expressed as
(2) One-way debonding stage with \( h_b \csc \beta \leq L_m < a_u \) (Fig. 11b)

During this stage, some of the FRP strips near the crack end have debonded completely below the critical shear crack (Fig. 11b). Let the rightmost debonded fibre intersect the critical shear crack at a height \( h_{db} \) above the steel tension reinforcement so \( L_m = (h_{db} + h_b) \csc \beta \) (Fig. 11b).

Following the same procedure as in deriving Eqs (28) and (30), the relationship between \( L(z) \) and \( L_m \), and the effective stress in FRP strips are obtained as

\[
[f_{f,e}] = \frac{2G_fE_f}{t_f} D_{fp}
\]

where

\[
D_{fp} = \frac{\lambda L_m - \sin(\lambda L_m)\cos(\lambda L_m)}{2[1 - \cos(\lambda L_m)]} \left(1 - \frac{h_{db}}{h_{f,e}}\right)
\]

The corresponding crack end width can be expressed as

\[
w_e = 2\delta_f \frac{1 - \cos(\lambda L_m)}{\sin(\theta + \beta)} \frac{h_{f,e}}{h_{f,e} - h_{db}}
\]

(3) Two-way debonding stage with \( L_m \geq a_u \) (Fig. 11c)

During this stage, the height of the debonded FRP area \( h_{db} \) continues to increase as \( L_m \) increases following the relationship of \( L_m = (h_{db} + h_b) \csc \beta \) and Eq. (26) can be rewritten as

\[
\Delta(z) = \frac{z}{h_{f,e}} - \frac{w_e}{2} \sin(\theta + \beta) = \frac{z}{h_{f,e} - h_{db}} \delta_f [1 + \lambda(L_m - a_u)]
\]

The point of intersection between the debonding front and the critical shear crack, where \( z = h_{df} \), \( L(z) = a_u \), and thus \( \Delta(z) = \delta_f \), \( w(z) = 2\delta_f \cos\csc(\theta + \beta) \) [see Fig. 11(c)], can be obtained from the following condition:

\[
\Delta(z) \bigg|_{z = h_{df}} = \delta_f
\]

Substituting Eq. (38) into Eq. (37) gives

\[
h_{df} = \frac{h_{f,e} - h_{db}}{1 + \lambda (L_m - a_u)}
\]

The stress in the FRP strips is constant with \( \sigma(z) = \frac{2G_fE_f}{t_f} \sqrt{\frac{1}{1 + \lambda (L_m - a_u)}} \) within the debonded zone with \( h_{df} \leq z \leq h_{f,e} - h_{db} \) (Fig. 11c). The effective (average) stress in the FRP strips intersected by the
critical shear crack can be found as

\[
f_{j,x} = h_{j,x} \sqrt{\frac{2G_f E_f}{t_f}} \int_{h_{j,x}}^{b_x} \frac{\lambda L(z)}{h_{j,x}} dz + \int_{b_x}^{h_{j,x}} \frac{2G_f E_f}{t_f} dz = \frac{2G_f E_f}{t_f} D_{fp}
\]  

(40)

In which

\[
D_{fp} = \frac{h_{j,x}}{h_{j,x}} \frac{\pi}{4} + \left( 1 - \frac{h_{db} + h_{uf}}{h_{j,x}} \right)
\]  

(41)

The corresponding crack end width can be expressed as

\[
w_e = 2\delta_f \frac{1 + \lambda (L_m - a_u)}{\sin(\theta + \beta)} \frac{h_{j,x}}{h_{j,x} - h_{db}}
\]  

(42)

4.2. Thick concrete cover with \( h_y \cosec \beta > a_u \)

For the case of \( h_y \cosec \beta > a_u \), the debonding process of FRP side strips can also be divided into three stages, including the softening stage \((0 \leq L_m < a_u)\), the two-way partial debonding stage \((a_u \leq L_m < h_y \cosec \beta)\) and the two-way debonding stage (with one side being partial debonded and the other side being completely debonded) \((L_m \geq h_y \cosec \beta)\), as shown in Figs 12a, 12b and 12c respectively. The last stage features a debonded FRP height of \( h_{db} \) measured from the crack end (Fig. 12c). Eqs (26)-(42) can be used for these three stages either directly or with slight modifications as follows.

(1) Softening stage with \( 0 \leq L_m < a_u \) (Fig. 12a)

For this stage, Eqs (30)-(32) apply directly.

(2) Two-way partial debonding stage with \( a_u \leq L_m < h_y \cosec \beta \) (Fig. 12b)

For this stage, the solution is given by Eqs (39)-(42) with slight modifications. Since the FRP strips have not completely debonded yet, the crack end width can be expressed as [based on Eq. (11)]

\[
w_e = 2\delta_f \frac{1 + \lambda (L_m - a_u)}{\sin(\theta + \beta)}
\]  

(43)

From Eqs (24) and (27), the slip in the fibre direction on each side of the crack is

\[
\Delta(z)_{\beta} = \frac{z}{h_{j,x}} \frac{w_e}{2} \sin(\theta + \beta) = \frac{z}{h_{j,x}} \delta_f \left[ 1 + \lambda (L_m - a_u) \right]
\]  

(44)

Substituting Eq. (44) into Eq. (38) gives

\[
h_{df} = \frac{h_{j,x}}{1 + \lambda (L_m - a_u)}
\]  

(45)

The effective stress in the FRP is
\[ f_{f,s} = h_f \int_0^{b_f} \frac{2G_f E_f}{t_f} \sin[\lambda L(z)]dz + h_f \int_{b_f}^{h_f} \frac{2G_f E_f}{t_f} dz = \frac{2G_f E_f}{t_f} D_{fp} \]  

(46)

where

\[ D_{fp} = \frac{h_{df}}{h_{f,s}} + \left(1 - \frac{h_{df}}{h_{f,s}} \right) \]  

(47)

(3) Two-way debonding stage with \( L_m \geq h_b \csc \beta \) (Fig. 12c)

The debonded FRP height is \( h_{db} \) and \( L_m = (h_{db} + h_b) \csc \beta \) for this stage. The solution is the same as Eqs (39)-(42).

5. Development of FRP shear contribution with crack width: FRP U-strips

As discussed earlier, the behaviour of an FRP U-strip in a shear-strengthened RC beam can be represented by a type A joint representing the interface above the effective shear crack, connected to a type B joint representing the interface below the effective shear crack (Fig. 1). Because the load-displacement relationship of a type A joint (Fig. 3) is very different from that of a type B joint (Fig. 7), the debonding process of FRP U-strips is more complicated than that of FRP side strips. Normally, the slips between FRP and concrete at the two sides of the critical shear crack are different for an FRP U-strip (see the Appendix for more details). To simplify the analysis without compromising the accuracy of the solution, a number of additional assumptions are introduced during the derivation process as detailed below. Similar to FRP side strips, the solutions are different depending on the concrete cover thickness relative to the effective bond length.

5.1. Thin concrete cover with \( h_b \csc \beta \leq a_u \)

For a thin concrete cover with \( h_b \csc \beta \leq a_u \), the debonding process of FRP U-strips can be divided into three stages which include a softening stage \((0 \leq L_m < a_u)\) (Figs 13a and 13b), a partial debonding stage \([(a_u \leq L_m \leq (h_{f,s} + h_f) \csc \beta)]\) (Figs 13c and 13d), a descending stage \((h_{db} \geq 0)\) (Fig. 13e) as explained next.

(1) Softening stage with \( 0 \leq L_m < a_u \) (Figs 13a and 13b)

During the softening stage (Figs 13a and 13b), the crack end width \( w_e \) is very small and the stress in FRP increases with \( w_e \). The length of the mobilized zone \( L(z) \) is also very small.

This stage ends when the maximum mobilized length \( L_m \), which is located at the crack end, reaches \( a_u \). If the concrete cover thickness is small relative to the effective bond length so that \( h_b \csc \beta \leq L_m < a_u \) is satisfied (Fig. 13b), the lengths of the softening zones of the FRP-to-concrete interface, and thus the slips of the interface, on the two sides of the critical shear crack are not the same. However, this asymmetry is assumed to be insignificant and thus neglected in this initial stage of loading in this study, i.e., the slips below the critical shear crack are...
assumed to be the same as those above it and the latter are used in the analysis. This assumption greatly simplifies the solution for the case of \( h \csc \beta \leq L_m < a_u \), but results in very small errors in the prediction of the effective FRP stress \( f_{f,e} \) according to a more accurate analysis presented in Chen (2010) mainly because the affected area is very small compared with the total bonded area of FRP. The errors due to this simplification are further explored later in this paper by comparing the current closed-from solution with FE predictions where this assumption is not used.

Based on the above simplification, Eqs (30)-(32) can be used directly for this stage.

(2) Partial debonding stage with \( a_u \leq L_m \leq (h_{f,e} + h_t)\csc \beta \) (Figs 13c and 13d)

The partial debonding stage (Figs 13c and 13d) starts when the leftmost fibre intersected by the critical shear crack starts to debond so that the softening length \( L_m \) is there, corresponding to \( h_{df} = h_{f,e} \). This stage ends when the leftmost fibre debonds completely from the beam sides [i.e. \( L_m = (h_{f,e} + h_t)\csc \beta \)] (Fig. 13d). The maximum FRP contribution to shear capacity is usually reached at this time (Chen et al. 2010a), with \( h_{df} = h_{df,u} \) and \( w_e = w_{e,u} \).

When the leftmost fibre debonds completely, i.e. \( L_m = (h_{f,e} + h_t)\csc \beta \), the crack end width is equal to the slip of the FRP strip below the shear crack plus that above it. These two parts can be determined from Eqs (11) and (21) respectively, giving

\[
w_{e,u} = \frac{\delta_t + \delta_t \lambda [(h_{f,e} + h_t)\csc \beta - a_u] + \delta_t [1 - \cos(\lambda h_t \csc \beta)] + \delta_t [1 - \sin(\lambda h_t \csc \beta)]\tan(\lambda h_t \csc \beta)}{\sin(\theta + \beta)}
\]

(48)

Noting that \( \Delta(z)_{\beta} \big|_{z = h_{df,u}} = \delta_t \) at \( z = h_{df,u} \) where the FRP-to-concrete interface just reaches the full bond strength (and where the debonding front intersects the critical shear crack), the crack width at this position can be found as

\[
w(z)_{z = h_{df,u}} = \frac{2\delta_t}{\sin(\theta + \beta)}
\]

(49)

From Eq. (24), the crack width at \( z = h_{df,u} \) can also be expressed as

\[
w(z)_{z = h_{df,u}} = \frac{h_{df,u}}{h_{f,e}} w_{e,u}
\]

(50)

Substituting Eq. (50) into Eq. (49) gives

\[
h_{df,u} = \frac{2h_{f,e} \delta_t}{w_{e,u} \sin(\theta + \beta)}
\]

(51)

The effective FRP stress \( f_{f,e} \) in this stage can be obtained as:

\[
f_{f,e} = \int_0^{h_e} \frac{2G_f E_f}{h_{f,e}} \frac{1}{t_f} \frac{\sin[\lambda L(z)]}{t_f} dz + \int_{h_e}^{h_t} \frac{2G_f E_f}{h_{f,e}} \frac{1}{t_f} \frac{dz}{t_f} = \frac{2G_f E_f}{t_f} D_{hy}
\]

(52)
From Eqs (24) and (38), the corresponding crack end width during this stage can be expressed as:

\[ w_e = \frac{2h_{f,e} \delta_f}{h_{df} \sin(\theta + \beta)} \]  

(54)

(3) Descending stage with \( h_{db} \geq 0 \) (or \( w_e \geq w_{e,u} \)) (Fig. 13e)

The descending stage starts when the leftmost fibre intersected by the critical shear crack has debonded completely from the beam side with \( w_e = w_{e,u} \) and \( h_{df} = h_{df,u} \). This stage is characterized by an FRP debonded height \( h_{db} \) (Fig. 13e) and usually corresponds to the descending branch of the \( V_f - w_e \) response (Chen et al. 2010a). The effective FRP stress \( f_{f,e} \) can be obtained with reference to the “two-way debonding stage” of FRP side-strips [see Fig. 11c and Eqs (37)-(42)], as explained below.

During this stage, all the FRP strips left to the position with \( z = h_{f,e} - h_{db} \) (i.e. \( h_{f,e} - h_{db} \leq z \leq h_{f,e} \)) have completely debonded above the critical crack (Fig. 13e). Following the same procedure of deriving Eq. (48), the shear crack width at \( z = h_{f,e} - h_{db} \) can be found as

\[
\begin{align*}
    w_{db} &= w_{db}(z) \bigg|_{z=h_{f,e}-h_{db}} \\
    &= \begin{cases} \\
    \frac{\delta_f + \delta_f \lambda \left( (h_f + h_{f,e} - h_{db}) \csc \beta - a_s \right)}{1 \csc \beta} & \text{if } \lambda \left( h_f + h_{f,e} \right) \csc \beta \leq a_s \\
    \frac{2 \delta_f + \delta_f \lambda \left( (h_{f,e} + h_f + h_{db}) \csc \beta - 2a_s \right)}{1 \csc \beta} & \text{if } \lambda \left( h_f + h_{f,e} \right) \csc \beta > a_s 
    \end{cases}
\end{align*}
\]

(55)

From Eq. (24), the crack end width is

\[ w_e = \frac{h_{f,e}}{h_{f,e} - h_{db}} w_{db} \]  

(56)

Following the same procedure of deriving Eq. (51), the location where the FRP-to-concrete interface just reaches the full bond strength [and where the debonding front intersects the critical shear crack, with \( w(z) = 2 \delta_f \csc (\theta + \beta) \), \( \Delta(z) = \delta_f \)] can be found as

\[ h_{df} = \frac{2h_{f,e} \delta_f}{w_e \sin(\theta + \beta)} \]  

(57)

The effective FRP stress \( f_{f,e} \) can be obtained as

\[
    f_{f,e} = \frac{h_{df}}{t_f} \sqrt{\int_{h_{f,e}}^{h_{f,e} - h_{db}} \frac{2G_E f}{h_{f,e}} \sin \lambda L(z) dz} + \int_{h_{f,e}}^{h_{f,e} - h_{db}} \frac{2G_E f}{h_{f,e}} \frac{dz}{t_f} \sqrt{\int_{h_{f,e}}^{h_{f,e} - h_{db}} \frac{2G_E f}{h_{f,e}} \frac{dz}{t_f} D_{frp}}
\]  

(58)
where
\[ D_{frp} = \frac{h_{df}}{h_{f,e}} \cdot \pi \cdot 4 \left( 1 - \frac{h_{db} + h_{df}}{h_{f,e}} \right) \]  
(59)

5.2. **Thick concrete cover with** \( h_b \text{cosec} \beta > a_u \)

When the concrete cover is thick relative to the effective bond length so \( h_b \text{cosec} \beta > a_u \), the debonding process of FRP U-strips can be divided into four stages which include a softening stage \((0 \leq L_m < a_u)\) (Fig. 14a), stage 1 of two-way partial debonding with \( a_u \leq L_m < h_b \text{cosec} \beta \) (Fig. 14b), stage 2 of two-way partial debonding with \( h_b \text{cosec} \beta \leq L_m \leq (h_{f,e} + h) \text{cosec} \beta \) (Figs 14c and 14d) and a descending stage \((h_{db} \geq 0)\) (Fig. 14e). Equations (30)-(32), (43)-(47), (48)-(54) and (55)-(59) can be used for these four stages directly or with slight modifications as detailed below.

(1) **Softening stage with** \( 0 \leq L_m < a_u \) (Fig. 14a)

The solution for this stage is exactly the same as that for the corresponding stage of FRP side strips in the case of a thick concrete cover with \( 0 \leq L_m < a_u \) (Fig. 12a) and consequently, Eqs (30)-(32) apply directly.

(2) **Stage 1 of two-way partial debonding with** \( a_u \leq L_m < h_b \text{cosec} \beta \) (Fig. 14b)

During this partial debonding stage, the fixed end condition of U-strips does not affect the solution, i.e., the lower softening front is above the fixed lower boundary. This stage is the same as the corresponding stage for FRP-side strips, so the solution for this stage is exactly the same as that for the two-way partial debonding stage with \( a_u \leq L_m < h_{f,e} \text{cosec} \beta \) for FRP side strips (Fig. 12b) and consequently, Eqs (43)-(47) apply directly. The stage ends when the softening front below the critical shear crack reaches the beam bottom (i.e. \( L_m = h_b \text{cosec} \beta \)) where FRP is assumed be fixed (Fig. 14b).

(3) **Stage 2 of two-way partial debonding with** \( h_b \text{cosec} \beta \leq L_m \leq (h_{f,e} + h) \text{cosec} \beta \) (Figs 14c and 14d)

During this second partial debonding stage, the softening front intersects with the fixed bottom boundary. The position where the debonding front intersects the critical shear crack \((z = h_{df})\) at the start of this stage can be determined from the condition that the softening front just reaches the fixed bottom boundary by substituting \( L_m = h_b \text{cosec} \beta \) into Eq. (45):

\[ h_{df} = \frac{h_{f,e}}{1 + \lambda \left( h_b \text{cosec} \beta - a_u \right)} \]  
(60)

The crack end width \( w_e \) at end of this stage \([L_m = (h_{f,e} + h) \text{cosec} \beta]\) is obtained from Eqs (11) and (21) following the same principle as in deriving Eq. (48) as:

\[ w_{e,u} = 2 \delta + \delta h \left( h_{f,e} + h_b \right) \text{cosec} \beta - 2a_u \]  
(61)
The solution for this stage is exactly the same as that for the partial debonding stage of FRP U-strips \( (a_c \leq L_m \leq (h_{f,e} + h_1) \cosec \beta) \) for the case of a thin concrete cover as presented earlier. Consequently, Eqs (49)-(54) apply directly here but the expression of \( w_{e,u} \) (at which this stages ends) should be determined from Eq. (61) instead of Eq. (48).

(4) Descending stage with \( h_{db} \geq 0 \) (or \( w_e \geq w_{e,u} \)) (Fig. 14e)

The solution for this stage is exactly the same as for the descending stage of FRP U-strips \( (w_e \geq w_{e,u}) \) in the case of a thin concrete cover \( (h_0 \cosec \beta \leq a_c) \). Accordingly, Eqs (55)-(59) apply directly here.

### 6. Verification of the closed-form solution

To validate the closed-form solution presented above, its predictions are compared with FE predictions in this section. The same FE model as presented in Chen et al. (2010b) was adopted where the continuous FRP sheet is represented by 20 discrete FRP strips. The following parameters were used in both the analytical and FE analyses unless otherwise stated: concrete cylinder compressive strength \( f_c = 30 \text{ MPa} \) [with an equivalent cube strength of 37 MPa according to CEB-FIP (1993)]; elastic modulus of FRP \( E_f = 2.3 \times 10^5 \text{ MPa} \); and tensile strength of FRP \( f_f = 3900 \text{ MPa} \). In the analytical solution, the maximum interfacial shear stress \( \tau_f \) and interfacial fracture energy \( G_f \) were calculated according to Lu et al.’s (2005) bond-slip model. The maximum interfacial slip \( \delta_f \) was calculated from \( \delta_f = 2G_f / \tau_f \) (see Fig. 2). In the FE analyses, both the linearly softening bond-slip model (Fig. 2) and the accurate nonlinear bond-slip model of Lu et al. (2005) were used. The results of the linearly softening bond-slip model (FEM1) are used to verify the accuracy of the analytical solution, and those of the nonlinear bond-slip model (FEM2) are used mainly to demonstrate the effect of approximation introduced by the linearly softening bond-slip model. The shear crack end was assumed to be at 50 mm from the beam soffit (i.e. \( h_b = 50 \text{ mm} \)) which is in the practical range as explained in Chen et al. (2010a, b) (Figs 10-14). It was further assumed that the beam sides are fully covered with FRP (and thus \( h_t = h_{f,e}/9 \), see Figs 10-14) with all fibres oriented vertically \( (\beta = 90^\circ) \) and the angle of the shear crack is \( \theta = 45^\circ \).

Analyses were conducted for a series of cases in which the beam height varies in the range of \( h_{f,e} = 300 – 600 \text{ mm} \) and the FRP thickness varies in the range of \( t_f = 0.11 - 0.88 \text{ mm} \). Close agreement was observed between the closed-form solution and the FE predictions for all the cases examined. Figs 15a-15c show the comparisons for three typical cases: (a) \( h_{f,e} = 600 \text{ mm}, t_f = 0.11 \text{ mm} \); (b) \( h_{f,e} = 300 \text{ mm}, t_f = 0.11 \text{ mm} \); and (c) \( h_{f,e} = 300 \text{ mm}, t_f = 0.88 \text{ mm} \). It can be seen that the ascending branch of the \( V_f - w_e \) curve predicted by the closed-form solution is nearly identical to the FE prediction for all three cases regardless of the FRP configuration (i.e. FRP side strips and U-strips), beam height and FRP thickness. After the peak, the FE analysis predicts stepwise drops, with each drop representing the complete debonding of an individual FRP strip [see Chen et al. (2010a) for a more detailed explanation]. The closed-form solution, being the solution for an equivalent continuous FRP sheet, generally passes through the mid-
point of each stepwise drop on the FE curve. It is only in the last stage that the closed-form solution deviates significantly from the FE prediction (point $F$ in Fig. 15). This is because, after about one half of the FRP strips have debonded, the remaining FRP strips debond in a much more abrupt manner mainly as the critical bond length (for the bonded area above the shear crack) now varies in a manner similar to that of the crack width. As a result, the assumption adopted in the closed-form solution that debonding proceeds sequentially from the crack end to the crack tip is not consistent with the FE prediction for this final stage of debonding failure. A more detailed discussion about this phenomenon can be found in Chen et al. (2010a). This discrepancy should affect neither the applicability nor the accuracy of the closed-form solution in any significant way as it occurs only in the very last stage of FRP debonding failure as can be seen from Fig. 15.

The location of point $F$ where the closed-form solution fails to capture the abrupt debonding of several FRP strips may be estimated from the condition where FRP debonding along the critical shear crack from the left to the right reaches a point (featured by a debonded FRP height $h_{db,f}$) where the bond length above the critical shear crack equals to that below it (see Figs 10-14), i.e.

$$h_{db,f} + h_b = h_{f,e} + h_t - h_{db,f}$$  \quad (62)

Rearranging Eq. (62) gives

$$h_{db,f} = \frac{h_{f,e} + h_t - h_b}{2}$$ \quad (63)

Substituting $L_m = \left(h_{db,f} + h_b\right) \csc \beta$ (the maximum mobilized bond length in the fibre direction) into Eq. (42) gives the crack end width $w_e$ at point $F$ [the same expression can be obtained from Eqs (55)-(56)]:

$$w_{e,f} = \frac{2h_{f,e} \cdot \left[2\delta_f + \delta_f \lambda \left(h_{f,e} + h_b + h_t\right) \cos \csc \beta - 2a_u\right]}{\left(h_{f,e} + h_b - h_t\right) \sin(\theta + \beta)}$$ \quad (64)

It should be noted that when the FRP membrane stiffness is very large (e.g. when $E_{df} > 2 \times 10^5$ N/mm), the closed-form solution may deviate slightly from the mid-heights of the FE predicted stepwise drops before point $F$ (Fig. 15c). Further analyses showed that this is because some FRP strips close to the crack tip may debond before $h_{db,f}$ is reached, but this is not duly considered in the analytical solution for simplicity as the effect of FRP debonding near the crack tip is insignificant when $h_{f,e}/a_u$ is in its practical range (see Appendix for details).

Figure 15 also shows that the closed-form solution slightly overestimates $V_f$ when compared with the second FE model (FEM2) based on Lu et al.’s (2005) accurate nonlinear bond-slip model, but this overestimation is very limited. For example, the differences are 4.68 % and 4.69 % for FRP side strips and FRP U-strips respectively when $t_f = 0.88$ mm and $h_{f,e} = 300$ mm. All methods predict very similar slopes of the descending branch of the $V_f - w_e$ curve, but $w_e$ at point $F$ as determined from Eq. (64) is generally larger than that predicted by FEM 2 at which $V_f$ experiences an abrupt drop. For example, when $t_f = 0.11$ mm and $h_{f,e} = 300$ mm, $w_e$
determined from Eq. (64) is 2.85 mm which is about 14% larger than that predicted by FEM2 (about 2.5 mm).

7. Concluding remarks

This paper has presented a closed-form analytical solution for the entire debonding process of FRP U-strips or side strips used to enhance the shear resistance of RC beams, allowing the different stages of debonding to be clearly explained and understood. The solution is based on a closed-form analytical solution for the full-range behavior of FRP-to-concrete bonded joints based on a linearly softening bond-slip model and the assumption of a linear shear crack shape. A particularly important outcome of the solution is the closed-form expressions that describe the development of shear contribution of externally bonded FRP shear reinforcement to the shear resistance of an RC beam as the critical shear crack widens. The closed-form solution has been validated by comparing its predictions with finite element predictions.

A major benefit among many others of the closed-form solution is that it can be used directly to evaluate the effect of shear interaction between external FRP strips and internal steel stirrups on the shear strength of RC beams shear-strengthened with FRP. Such exploitation of the present analytical solution will be discussed in a forthcoming paper (Chen et al. 2011) for the development of an advanced shear strength model for RC beams shear-strengthened with FRP. The present closed-formed solution also provides valuable insight into the debonding failure process of RC beams shear-strengthened with FRP.

References


Teng, J.G., Yuan, H., Chen, J.F., 2006. FRP-to-concrete interfaces between two adjacent cracks:


Fig. 1. RC beam shear-strengthened with FRP strips: (a) elevation; (b) cross-section; (c) type A joint; (d) type B joint.
Fig. 2. Linearly softening bond-slip model.

Fig. 3. Full-range behavior of FRP-to-concrete bonded joint with a free far end (type A joint).
Fig. 4. Interfacial shear stress distributions at various stages for a type A joint with $L > a_u$; (a) development of softening zone (OA in Fig. 3); (b) initiation of debonding at loaded end ($\Delta = \delta_f$, point A in Fig. 3); (c) propagation of debonding (AB in Fig. 3); (d) softening front at the free end (point B in Fig. 3); (e) linear unloading (BC in Fig. 3).
Fig. 5. Interfacial shear stress distributions at various stages for a type A joint with $L \leq a_s$: (a) development of softening zone (OA’ in Fig. 3); (b) softening front at the free end (point B’ in Fig. 3); (c) linear unloading (B’C in Fig. 3).
Fig. 6. Relationship between a type B joint and an FRP-to-concrete bonded joint loaded at both ends (Teng et al. 2006; Chen et al. 2007).

Fig. 7. Full-range behavior of an FRP-to-concrete bonded joint with a fixed far end (type B joint).
Fig. 8. Interfacial shear stress distribution at various stages for a type B joint with $L > a_u$: (a) development of softening zone; (b) initiation of debonding at the loaded end (point A in Fig. 7); (c) propagation of debonding (AB in Fig. 7); (d) softening front at the fixed end (point B in Fig. 7); (e) final propagation of debonding (BP in Fig. 7).
Fig. 9. Interfacial shear stress distributions at various stages for a type B joint with $L \leq a_u$: (a) development of softening zone (OB’ in Fig. 7); (b) softening front at the fixed end (point B’ in Fig. 7); (c) initiation of debonding at the loaded end (point D’ in Fig. 7); (d) final propagation of debonding (D’P’ in Fig. 7).
Fig. 10. Notation for a general shear strengthening scheme.
Fig. 11. Debonding process of an FRP side strips for a thin concrete cover \((h_b \cos ec \beta \leq a_u)\): (a) \(0 \leq L_m < h_b \cos ec \beta\); (b) \(h_b \cos ec \beta \leq L_m < a_u\); (c) \(L_m \geq a_u\).
Fig. 12. Debonding process of an FRP side strips for a thick concrete cover ($h_b \cos ec \beta > a_u$): (a) $0 \leq L_m < a_u$; (b) $a_u \leq L_m < h_b \cos ec \beta$; (c) $L_m \geq h_b \cos ec \beta$. 

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Fig. 13. Debonding process of an FRP U-strips for a thin concrete cover \( h_0 \cos \beta \leq a_y \): 
(a) \( 0 \leq L_m < h \cos \beta \); (b) \( h \cos \beta \leq L_m < a_y \); (c) \( a_y \leq L_m < \left( h_{f,c} + h \right) \cos \beta \); 
(d) \( L_m = \left( h_{f,c} + h \right) \cos \beta \) (i.e. \( w_e = w_{f,c} \)); (e) \( h_{db} \geq 0 \).
Fig. 14. Debonding process of an FRP U-strips for a thick concrete cover ($h_b \cos \beta > a_u$):

(a) $0 \leq L_m < a_u$; (b) $a_u \leq L_m < h_b \csc \beta$; (c) $h_b \csc \beta \leq L_m < (h_{f,e} + h_t) \csc \beta$;

(d) $L_m = (h_{f,e} + h_t) \csc \beta$ (i.e. $w_e = w_{e,m}$); (e) $h_{db} \geq 0$. 

(e)
Fig. 15. Closed-form solution versus FE predictions: (a) $h_{f,x} = 600 \text{ mm}, t_f = 0.11 \text{ mm}$; (b) $h_{f,x} = 300 \text{ mm}, t_f = 0.11 \text{ mm}$; (c) $h_{f,x} = 300 \text{ mm}, t_f = 0.88 \text{ mm}$. 
Appendix  Comparison of the present solution with Monti et al.’s (2004) solution

The present solution is compared with Monti et al.’s (2004) solution in this Appendix in terms of the relationship between the FRP shear resistance and the crack end width (i.e. the $V_f - w_c$ curve) and the corresponding debonding sequence of FRP strips. The same material properties (strength of concrete $f'_c$, FRP elastic modulus $E_f$ and FRP tensile strength $f_f$) as those for the example beams in the main body of the paper are adopted. In the present solution, the maximum interfacial stress $\tau_f$ and the interfacial fracture energy $G_f$ are determined according to Lu et al.’s (2006) bond-slip model. In Monti et al.’s (2004) solution, the bond parameters including the effective bond length $L_e$, the interfacial fracture energy $G_f$ and the bond strength of FRP-to-concrete joints $f_{fdd}$ are determined following Liotta (2006) which provides a more detailed presentation of the solution. It is assumed that the beam sides are fully covered with FRP with all fibers oriented vertically ($\beta = 90^\circ$) and the angle of the shear crack $\theta = 45^\circ$. The comparison here is made for two beam heights: $h_{f,e} =$ 300 mm and 600 mm.

For both cases it is assumed that $h_f = h_b = 0$ (see Figs 11-14) which is also assumed in Monti et al.’s (2004) solution and $t_f = 0.11$ mm.

Fig. A.1 compares the $V_f - w_c$ curves from the two solutions. The FE predictions for a linearly softening bond-slip model are also shown for reference. It may be noted that Monti et al.’s (2004) solution only gives part of the $V_f - w_c$ curve because it is subjected to several limiting conditions [e.g. $l_{b}(z) \geq L_e$ for both FRP side strips and U-strips, where $l_{b}(z)$ is the governing FRP bond length at the location $z$, defined as the FRP bond length above the critical shear crack for FRP U-strips and the smaller FRP bond lengths above and below the critical shear crack for FRP side strips].

For FRP U-strips, the maximum FRP shear contribution $V_{f,\text{max}}$ predicted by Monti et al.’s (2004) solution is very close to that from the present solution and both are close to the FE prediction. When $h_{f,e} = 300$ mm and $t_f = 0.11$ mm, the former predicts $V_{f,\text{max}} = 33.2$ kN which is only about 3% higher than that predicted by the latter, 32.4 kN; both are close to the FE prediction of 31.5 kN. However, the crack end width corresponding to $V_{f,\text{max}}$ (i.e. $w_{e,u}$) predicted by the former is significantly larger than the value predicted by the present solution. These values are respectively 1.86 mm and 1.26 mm for the case of $h_{f,e} = 300$ mm and $t_f = 0.11$ mm, and 3.72 mm and 2.63 mm for the case of $h_{f,e} = 600$ mm and $t_f = 0.11$ mm, with the former being respectively 47.6% and 41.4% higher than the latter for the two cases. The present predictions are again close to the FE predictions of 1.13 mm and 2.63 mm for the two cases. The significant difference in $w_{e,u}$ between the two closed-form solutions is mainly caused by the assumption adopted by Monti et al. (2004) that the slips at both sides of the shear crack are symmetrical, as further discussed below. The difference in the predicted $w_c$ value is not important if interaction between internal steel reinforcement, external FRP shear reinforcement and concrete is not considered, but would have a significant consequence if this interaction is considered.
For FRP side strips, the $V_{f,\text{max}}$ value predicted by Monti et al.’s (2004) solution is generally slightly smaller than that predicted by the present solution. The difference between the two solutions decreases with an increase of $h_{f,e}$. For example, this difference is 8.3% for $h_{f,e} = 300 \text{ mm}$ and reduces to 4.1% for $h_{f,e} = 600 \text{ mm}$. The corresponding $w_{e,u}$ predicted by Monti et al.’s (2004) solution is generally larger than that from the present solution. For example, the two predictions are respectively 0.87 mm and 0.62 mm for the case of $h_{f,e} = 300 \text{ mm}$ and $t_f = 0.11 \text{ mm}$, with the former being 40% larger than the latter. The difference reduces to 33.3% for the case of $h_{f,e} = 600 \text{ mm}$ and $t_f = 0.11 \text{ mm}$. Further analyses showed that these differences are mainly caused by the different FRP-to-concrete bond-slip laws adopted in the two solutions. Monti et al. (2004) assumed that the slip at the loaded end of the FRP-to-concrete joint at the onset of deboning $u_i = 0.33 \text{ mm}$ when $l_h(z) = L_e$ for the case of FRP sheet [and $u_t(l_h) = (l_h/L_e)u_i$ for $l_h(z) < L_e$], which is considerably larger than the value of about 0.2 mm calculated using Lu et al.’s (2005) bond-slip model.

Fig. A.2 shows the distribution of the maximum crack width at coordinate $z$, $w_{ab}(z)$, beyond which the FRP strip at this location is fully debonded, along the shear crack (in terms of coordinate $z$) for FRP U-strips. In Monti et al.’s (2004) solution, $w_{ab}(z)$ is determined by the governing bond length $l_h(z)$. In the present solution, $w_{ab}(z)$ can be calculated respectively from Eqs (32) and (43) (by setting $L_m = z/\sin \beta$) for $0 \leq z/\sin \beta \leq h_{f,e}/2$ and from Eq. (55) (by setting $h_l = h_b = 0$) for $h_{f,e}/2 < z/\sin \beta \leq h_{f,e}$. Also shown in Fig. A.2 is the assumed linear crack width distribution determined by the following equation (Monti et al. 2004) which is an alternative expression of Eq. (24):

$$w(z) = \alpha z / \sin \theta$$

(A1)

where $\alpha$ is the crack opening angle of the shear crack (in radians).

In Fig. A.2, the sequence of debonding of FRP strips can be determined by comparing $w_{ab}(z)$ and $w(z)$. When $\alpha$ increases continuously from zero, debonding will occur at locations where $w(z) > w_{ab}(z)$. From Fig. A.2 it is clear that, if symmetrical slips at both sides of the shear crack are assumed as in Monti et al. (2004), debonding will start from a location near the crack tip and propagate towards the crack end as the shear crack widens ($\alpha$ increases); $w(z)$ at the crack end (i.e. $w_e$) corresponding to the initiation of FRP debonding is 1.86 mm and 3.72 mm respectively for $h_{f,e} = 300 \text{ mm}$ and $h_{f,e} = 600 \text{ mm}$, with the corresponding $\alpha$ being 0.00439 for both cases. The present study (in which the above assumption is removed) predicts that debonding starts from the crack end and propagates towards the crack tip as $\alpha$ increases. The $w_e$ values corresponding to the initiation of FRP debonding are 1.26 mm ($\alpha=0.00297$) and 2.63 mm ($\alpha=0.00311$) respectively for $h_{f,e} = 300 \text{ mm}$ and $h_{f,e} = 600 \text{ mm}$ respectively. It should be noted that a small portion of FRP near the crack tip is predicted to debond nearly at the same time as the initiation of FRP debonding at the crack end especially for the case of $h_{f,e} = 300 \text{ mm}$ but this is due to the simplifying assumption of a brittle post-peak response of the $P-\Delta$ curve of the type A joint as explained in the main body of the paper. If the descending branch is considered, then debonding near the crack tip does not occur when $w(z)<0.4 \text{ mm}$ for any bond length (Fig A.2). In the present analytical solution, the consequence of the simplification is counter-balanced by the assumption that debonding is a sequential process from the crack end to the crack tip, so early debonding near the crack tip is not considered. This observation may
explain why the present solution predicts a slightly larger $V_{f,max}$ than the FE prediction (Fig. A.1) in which all debonding events are captured. For the same reason, the post-peak branch of the $V_f - w_{max}$ curve predicted by the present solution is slightly higher than the mid-points of the stepwise drops predicted by the FE model (also see Fig. A.1). Further analyses have shown that early debonding near the crack tip generally does not occur for practical cases with $h_f > 0$ (see Figs 11-14), and the effects of these two simplifying assumptions are further reduced especially when $h_{f,c}$ is large (Figs A.1a and A.1b).

![Graph A.1a](image1.png)

**Fig. A.1a.** The present solution versus Monti et al.’s solution in terms of $V_f - w_e$ curves: (a) $h_{f,c} = 300$ mm, $t_f = 0.11$ mm. (b) $h_{f,c} = 600$ mm, $t_f = 0.11$ mm.
Fig. A.2. Comparison of debonding sequence between Monti et al.’s (2004) solution and the present solution for FRP U-strips: (a) $h_{f,e} = 300$ mm, $t_f = 0.11$ mm. (b) $h_{f,e} = 600$ mm, $t_f = 0.11$ mm.