Design optimization of a damped hybrid vibration absorber

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Abstract

In this article, the H_∞ optimization design of a hybrid vibration absorber (HVA), including both passive and active elements, for the minimization of the resonant vibration amplitude of a single degree-of-freedom (SDOF) vibrating structure is derived by using the fixed-points theory. The optimum tuning parameters are the feedback gain, the tuning frequency, damping and mass ratios of the absorber. The effects of these parameters on the vibration reduction of the primary structure are revealed based on the analytical model. Design parameters of both passive and active elements of the HVA are optimized for the minimization of the resonant vibration amplitude of the primary system. One of the inherent limitations of the traditional passive vibration absorber is that its vibration absorption is low if the mass ratio between the absorber mass and the mass of the primary structure is low. The proposed HVA overcomes this limitation and provides very good vibration reduction performance even at a low mass ratio. The proposed optimized HVA is compared to a recently published HVA designed for similar propose and it shows that the present design requires less energy for the active element of the HVA than the compared design.

Keywords: Vibration absorber, tuned mass damper, hybrid control.

1. Introduction

The traditional passive vibration absorber (PVA) is an auxiliary mass-spring-damper system which, when correctly tuned and attached to a vibrating system subject to harmonic excitation, causes to cease the steady-state motion at the point to which it is attached. The first research conducted at the beginning of the twentieth century considered an undamped PVA tuned to the frequency of the disturbing force [1]. Such an absorber is a narrow-band device as it is unable to eliminate structural vibration after a change in the disturbing frequency.

Finding the optimum parameters of a viscous friction PVA in SDOF system drew the attention of many scholars. One of the optimization methods is H_∞ optimization. Ormondroyd and Den Hartog [2] proposed the optimization principle of the damped PVA in terms of minimizing the maximum amplitude response of the primary system, which is called H_∞ optimization of PVA. Following this principle, Hahnkamm [3] derived the expressions for the optimum tuning of PVA used in the SDOF system. Block [4] developed the approximated optimum damping. The optimum design method of the dynamic vibration absorber is called "Fixed-points theory", which was well documented in the textbook by Den Hartog [5]. The exact solution of the H_∞ optimization of a PVA attached to undamped primary system was derived by Nishihara and Asami [6]. However, it was found that the minimum resonant vibration amplitude of the primary system attached with the PVA depends on the mass ratio [7]. When the mass ratio is fixed, the performance of the PVA is also limited.

In order to improve the vibration suppression performance of the PVA, some researchers incorporate an active actuator to a PVA to form a hybrid vibration absorber (HVA). Various methods were proposed to control the active element of the HVA including neural network [8], delayed resonator [9], linear matrix inequalities [10], modal feedback control [11-15] and closed-loop poles by modal feedback [16-17]. However, the control methods of HVA found in literature are very complicated and most of the research reported in literature focused on the improvement of the active controller design rather than the optimization of both the active and passive components of the HVA.

In this article, H_∞ optimal design of a damped hybrid vibration absorber is proposed for the minimization of the resonant vibration amplitude of a SDOF system. Both the active and passive elements are optimized. The proposed optimized tuning of the HVA can also minimize the actuation force of the actuator. Comparisons with the result of H_∞ optimal PD control of HVA by Chatterjee [18] show much better results of our optimum design. Finally, we apply the proposed optimized HVA to a beam structure and compare its vibration suppression performance to that of the optimized PVA [18,19] and also to the optimal PD control of HVA by Chatterjee [18].

2. Theory

A HVA coupled with a primary system is shown as Fig. 1, where x, M and K denote, respectively, displacement, mass and stiffness of the primary system; and x_a, m and k are those of the absorber. c is the damping coefficient of the absorber.

The equations of motion of the primary mass M and the absorber mass m may be written as

$$\begin{cases} M\ddot{x} = -Kx - k(x - x_a) - c(\dot{x} - \dot{x}_a) - f_a + F \\ m\ddot{x}_a = -k(x_a - x) - c(\dot{x}_a - \dot{x}) + f_a \end{cases} \tag{1}$$

where F is a disturbance and $f_a = ax$ is the active force applied by the actuator as illustrated in Fig. 1. Taking Laplace transformation of Eq. (1), the transfer function of the primary mass M may be written as

$$H(p) = \frac{X}{F/K} = \frac{\gamma^2 + p^2 + 2\zeta\gamma p}{\left(1+p^2\right)\left(\gamma^2 + p^2\right) + \left(\mu\gamma^2 + 2\alpha\right)p^2 + 2\zeta\gamma p\left(1 + p^2 + \mu p^2\right)} \tag{2a}$$

where $p = \dfrac{s}{\omega_n}$, $\mu = \dfrac{m}{M}$, $\omega_n = \sqrt{\dfrac{K}{M}}$, $\omega_a = \sqrt{\dfrac{k}{m}}$, $\gamma = \dfrac{\omega_a}{\omega_n}$, $\zeta = \dfrac{c}{2\sqrt{mk}}$ and $\alpha = \dfrac{a}{2K}$.

The transfer function of the absorber mass m may be written as

$$\frac{X_a}{F/K} = \frac{\gamma^2 + \left(2\alpha/\mu\right) + 2\zeta\gamma p}{\left(1+p^2\right)\left(\gamma^2 + p^2\right) + \left(\mu\gamma^2 + 2\alpha\right)p^2 + 2\zeta\gamma p\left(1 + p^2 + \mu p^2\right)}. \tag{2b}$$

Since $f_a = a\,x = 2K\alpha\,x$, the transfer function of the active force in the absorber may be written as

$$\frac{F_a}{F} = 2\alpha\,H(p) = \frac{2\alpha\left(\gamma^2 + p^2 + 2\zeta\gamma p\right)}{\left(1+p^2\right)\left(\gamma^2 + p^2\right) + \left(\mu\gamma^2 + 2\alpha\right)p^2 + 2\zeta\gamma p\left(1 + p^2 + \mu p^2\right)} \tag{2c}$$

where F_a is the Laplace transformation of f_a. According to Eqs. (2a) (2b) and (2c), the characteristic equation of the combined system may be written as

$$p^4 + 2\zeta\gamma\left(1+\mu\right)p^3 + \left(1 + \gamma^2 + \mu\gamma^2 + 2\alpha\right)p^2 + 2\zeta\gamma p + \gamma^2 = 0 \tag{3}$$

and $\forall \zeta, \gamma, \mu, \alpha \in R^+$.

To apply the Routh's stability criterion, the array of coefficient may be written as

$$
\begin{array}{c|ccc}
p^4 & 1 & 1+\gamma^2+\mu\gamma^2+2\alpha & \gamma^2 \\
p^3 & 2\zeta\gamma(1+\mu) & 2\zeta\gamma & 0 \\
p^2 & \dfrac{(1+\mu)^2\gamma^2+\mu+2\alpha(1+\mu)}{1+\mu} & \gamma^2 & \\
p & \dfrac{2\zeta\gamma(\mu+2\alpha+2\alpha\mu)}{(1+\mu)^2\gamma^2+\mu+2\alpha(1+\mu)} & & \\
1 & \gamma^2 & &
\end{array}
\tag{4}
$$

The system is stable if the real parts of all poles are negative. Since all coefficients of the array in Eq. (4) are positive if $\alpha \geq 0$, the control system is stable according to the Routh's stability criterion. That means the proposed HVA control system is applicable in principle if $\alpha \geq 0$.

The frequency response function of mass *M* can be obtained by replacing *p* in Eq. (2a) by jλ where $\lambda = \dfrac{\omega}{\omega_n}$ and $j^2 = -1$. The frequency response function of mass *M* may be written as

$$
H(\lambda) = \frac{\gamma^2 - \lambda^2 + 2\,\mathrm{j}\zeta\gamma\lambda}{(1-\lambda^2)(\gamma^2-\lambda^2) - (\mu\gamma^2 + 2\alpha)\lambda^2 + 2\,\mathrm{j}\zeta\gamma\lambda(1-\lambda^2-\mu\lambda^2)}. \tag{5}
$$

The frequency response amplitude of the primary mass *M*, $|H(\lambda)|$, is calculated according to Eq. (5) with three different damping ratios and the results are plotted in Fig. 2 for illustration. It can be seen in Fig. 2 that the frequency response amplitudes of mass *M* at λ_a and λ_b are independent of the damping ratio ζ and these two points are called 'fixed points'. Considering $H(\lambda)|_{\varsigma=0} = H(\lambda)|_{\varsigma=\infty}$, we may write

$$
\lambda^4 - \frac{2(\gamma^2 + \mu\gamma^2 + 1 + \alpha)}{2+\mu}\lambda^2 + \frac{2\gamma^2}{2+\mu} = 0. \tag{6}
$$

The two roots of Eq. (6) are $\lambda_\mathrm{a}{}^2$ and $\lambda_\mathrm{b}{}^2$ where $0 < \lambda_\mathrm{a} < \lambda_\mathrm{b}$. The amplitudes of the frequency response at λ_a and λ_b may be written as

$$|H(\lambda_a)| = \left|\frac{1}{1-\lambda_a{}^2-\mu\lambda_a{}^2}\right|, \text{ and} \tag{7a}$$

$$|H(\lambda_b)| = \left|\frac{1}{1-\lambda_b{}^2-\mu\lambda_b{}^2}\right| \tag{7b}$$

At any damping ratio, the frequency must pass through these two fixed points. So the optimum condition should obey the following equation:

$$\max(|H(\lambda,\gamma_{H\infty},\zeta_{H\infty})|) = \min_{\gamma,\zeta}\Big(\max(|H(\lambda_a)|,|H(\lambda_b)|)\Big) \tag{8}$$

According to the fixed-points theory [2] originally developed for the design of the passive dynamic vibration absorber, the optimum condition of the dynamic vibration absorber can be achieved by adjusting the frequency ratio γ such that the vibration amplitude responses at λ_a and λ_b are the same, and then finding the damping so that the two fixed points become the maximum points on the response curves. A similar procedure is applied to the optimization of the proposed HVA, i.e. $|H(\lambda_a)| = |H(\lambda_b)|$. Using Eqs. (7a) and (7b) and noting that $H(\lambda_a)$ and $H(\lambda_a)$ are in opposite phases, we may write

$$\frac{1}{1-\lambda_a{}^2-\mu\lambda_a{}^2} = -\frac{1}{1-\lambda_b{}^2-\mu\lambda_b{}^2} \tag{9}$$

Solving Eq. (6) for λ_a and λ_b and substituting them into Eq. (9), the tuning frequency ratio leading to the same response amplitude at the fixed points can be found and written as

$$\gamma_{opt} = \sqrt{\frac{1-\alpha(1+\mu)}{(1+\mu)^2}} \tag{10}$$

From Eq. (10), γ_{opt} exists if $1-\alpha(1+\mu) > 0$, i.e.

$$\alpha < \frac{1}{1+\mu} \tag{11}$$

Substituting Eq. (10) into Eq. (6), the resulting equation may be written as

$$\lambda^4 - \frac{2}{1+\mu}\lambda^2 + \frac{2(1-\alpha-\alpha\mu)}{(2+\mu)(1+\mu)^2} = 0 \tag{12}$$

The roots of Eq. (12) may be written as

$$\lambda_{\mathrm{a}}^{\ 2} = \left(\frac{1}{1+\mu}\right)\left(1-\sqrt{\frac{\mu+2\alpha(1+\mu)}{2+\mu}}\right) \tag{13}$$

$$\text{and} \quad \lambda_{\mathrm{b}}^{\ 2} = \left(\frac{1}{1+\mu}\right)\left(1+\sqrt{\frac{\mu+2\alpha(1+\mu)}{2+\mu}}\right) \tag{14}$$

The response amplitude at the fixed points $|H(\lambda_{\mathrm{a}})|$ and $|H(\lambda_{\mathrm{b}})|$ are calculated using Eqs. (7a), (7b), (13) and (14) with $\mu = 0.2$ and $\alpha = 0.5$ and plotted in Fig. 3for illustration. It can be seen that, when the excitation frequency increases, $|H(\lambda_{\mathrm{a}})|$ increases while $|H(\lambda_{\mathrm{b}})|$ decreases. The corresponding frequency ratio γ at the intersection point of the curves in Fig. 3 is the optimum frequency ratio of the HVA such that $|H(\lambda_{\mathrm{a}})| = |H(\lambda_{\mathrm{b}})|$. Substituting Eq. (13) into Eq. (7a) or Eq. (14) into Eq. (7b), the response amplitude at the fixed points may be written as

$$|H(\lambda_{\mathrm{a}})| = |H(\lambda_{\mathrm{b}})| = \sqrt{\frac{2+\mu}{\mu+2\alpha(1+\mu)}} \tag{15}$$

The optimum damping is the damping value which causes the fixed points to become the peaks on the response curve $|H(\lambda)|$ and therefore we may consider

$$\left.\frac{\partial}{\partial\lambda^2}|H(\lambda)|^2\right|_{\lambda=\lambda_{\mathrm{a}}} = \left.\frac{\partial}{\partial\lambda^2}|H(\lambda)|^2\right|_{\lambda=\lambda_{\mathrm{b}}} = 0\,. \tag{16}$$

The damping required leading to maximum vibration amplitude at λ_{a} and λ_{b} may be solved using Eqs. (5), (13) (14) and (16) and written as

$$\zeta_{\mathrm{a,b}} = \sqrt{\frac{\left(3\mu+\alpha\left(\mu^2+7\mu+6\right)\right)\pm\left(\mu-\alpha\left(\mu^2+3\mu+2\right)\right)\sqrt{\frac{\mu+2\alpha(1+\mu)}{2+\mu}}}{8(1+\mu)(1-\alpha-\alpha\mu)}} \tag{17}$$

$|H(\lambda_{\mathrm{a}})|$ would becomes the peak value of the response function $|H(\lambda)|$ if $\zeta=\zeta_{\mathrm{a}}$ and $|H(\lambda_{\mathrm{b}})|$ would becomes the peak value of the response function $|H(\lambda)|$ if $\zeta=\zeta_{\mathrm{b}}$. A convenient approximate value of the optimum damping may be chosen as

$$\zeta_{\mathrm{opt}} = \sqrt{\frac{\zeta_{\mathrm{a}}^2+\zeta_{\mathrm{b}}^2}{2}} = \sqrt{\frac{3\mu+\alpha\left(\mu^2+7\mu+6\right)}{8(1+\mu)(1-\alpha-\alpha\mu)}} \tag{18}$$

Using Eqs. (3), (10) and (18), the root locus of the control system as shown in Fig. 1 is calculated with mass ratio μ = 0.2 and feedback gain α varies from 0 to $\frac{1}{1+\mu}$, and the results are plotted in Fig. 4 for illustration. All four poles of the control system have negative real parts with one pole approaching the origin when α approaches the limiting value $\frac{1}{1+\mu}$.

If the optimum tuning frequency γ_{opt} and the optimum damping ζ_{opt} are applied to the proposed HVA, using Eq. (15), the maximum vibration amplitude of the primary mass M may be written as

$$G = \left|\frac{X}{F/K}\right|_{\max} = \sqrt{\frac{2+\mu}{\mu+2\alpha(1+\mu)}} \tag{19}$$

The frequency response amplitude of the primary mass M, $|H(\lambda)|$, at $\mu=0.05$ with the proposed optimum frequency and damping ratios are calculated according to Eqs. (5), (10) and (18) with α = 0, 0.2 and 0.5 respectively and the results are plotted in Fig. 5 for illustration. All the response curves in Fig. 5 show the typical double peaks in the response spectra of the primary mass *M*. When the feedback gain α = 0, the HVA becomes the traditional passive vibration absorber (PVA) and the resonant vibration amplitude is about 10 times of the static deflection of mass *M*. When the active element is deployed with α = 0.2, the resonant vibration amplitude drops to about 2.2 times of the static deflection of mass *M*. When the feedback gain α increase to 0.5, the resonant vibration amplitude drops further to about 1.4 times of the static deflection of mass *M*. These results show that the proposed HVA is very effective in suppressing the resonant vibration amplitude of the primary vibrating system in comparison to the traditional passive vibration absorption when the mass ratio is low such as the cases of using vibration absorbers to suppress oscillations of tall buildings and bridges.

Since $f_a = ax = 2K\alpha\, x$, the maximum active force required for the HVA may be written using Eq. (19) as

$$\left|\frac{F_a}{F}\right|_{\max} = 2\alpha\, G = \frac{aG}{K} \tag{20}$$

When $\alpha \geq \dfrac{1}{1+\mu}$, it can be shown that the response amplitudes at the fixed point λ_{a} is always higher than that at fixed point λ_{b}, i.e. $|H(\lambda_{\mathrm{a}})| > |H(\lambda_{\mathrm{b}})|$. $|H(\lambda_{\mathrm{a}})|$ and $|H(\lambda_{\mathrm{b}})|$ are calculated using Eqs. (6), (7a) and (7b) with α = 0.5 and μ = 0.2 and they are plotted in Fig. 6 for illustration.

The frequency of the fixed point λ_{a} can be found by solving Eq. (6) and written as

$$\lambda_{\mathrm{a}} = \sqrt{\frac{(1+\mu)\gamma^2 + 1 + \alpha - \sqrt{\gamma^4(1+\mu)^2 - 2\gamma^2\left(2-\alpha(1+\mu)\right) + (1+\alpha)^2}}{2+\mu}}\ . \tag{21}$$

The response amplitude of mass *M* at frequency λ_{a} may be found by substituting Eq. (21) into Eq. (5) and written as

$$\left|H(\lambda_{\mathrm{a}})\right| = \frac{-1+\alpha+\mu\alpha+(1+\mu)^2\gamma^2+(1+\mu)\sqrt{\gamma^4(1+\mu)^2-2\gamma^2(2-\alpha(1+\mu))+(1+\alpha)^2}}{\mu+2\alpha+2\mu\alpha}\,. \tag{22}$$

The optimum damping is the damping value which causes the fixed point λ_{a} to become the peak on the response curve $\left|H(\lambda)\right|$, i.e. $\left.\frac{\partial}{\partial\lambda^2}\left|H(\lambda)\right|^2\right|_{\lambda=\lambda_a}=0$, and it can be derived using Eq. (5) and written as

$$\zeta_{\mathrm{H}} = \sqrt{\frac{S_1\gamma^4+S_2\gamma^2+S_3+S_4\sqrt{\gamma^4(1+\mu)^2-2\gamma^2(2-\alpha(1+\mu))+(1+\alpha)^2}}{8\gamma^2(1+\mu)(2+\mu)}} \tag{23}$$

where $S_1=(2+\mu)(1+\mu)^2$, $S_2=(3\alpha+1)\mu^2+(9\alpha+1)\mu+6\alpha-4$,

$S_3=(\alpha+1)(1+2\alpha+\mu\alpha)$ and $S_4=(2+\mu)(1+\mu)\gamma^2-2+4\alpha+2\mu\alpha$.

The corresponding tuning ratio can be derived using Eq. (23) and written as

$$\gamma_{\mathrm{H}} = \sqrt{\frac{(G-1)(G(\mu+2\alpha+2\alpha\mu)+2+\mu)}{2G(1+\mu)^2}} \tag{24}$$

where *G* is the maximum amplitude response of $\left|H(\lambda)\right|$.

To illustrate the difference of $\left|H(\lambda)\right|$ between the cases of using the low feedback gain

$\alpha < \frac{1}{1+\mu}$ and the high gain $\alpha \geq \frac{1}{1+\mu}$, the frequency amplitude response $|H(\lambda)|$ of both cases are calculated according to Eq. (5) and the corresponding optimum frequency and damping ratios and the result are plotted in Fig. 7. Figure 7 shows double peaks in the frequency spectrum with a low gain with $\alpha < \frac{1}{1+\mu}$ and single peak with a high gain. Since the active force required in the HVA is proportional to the gain α, it is therefore recommend that a low feedback gain with $\alpha < \frac{1}{1+\mu}$ should be used whenever possible.

In practice, the maximum frequency response G is often a design constraint. If $\alpha < \frac{1}{1+\mu}$ is assumed and using Eq. (19), the range of G may be written as

$$\sqrt{\frac{2+\mu}{\mu}} > G > 1 \tag{25}$$

The corresponding feedback gain can be obtained from Eq. (19) and written as

$$\alpha = \frac{2+\mu - G^2\mu}{2G^2(1+\mu)} \tag{26}$$

The optimum tuning frequency and damping ratios of the HVA can then be determined using Eqs. (10) and (18) respectively.

Since the use of multiple feedback signals is common in modern control theory [21], feedback signals from both the primary and absorber masses are considered in the following for the active control of the HVA and compared to the proposed method which uses only the feedback signal from the primary mass. Assuming the active force of the HVA is a function of both the displacements of primary and absorber masses written as $f_a = ax + bx_a$. The active force in the HVA may be rewritten as

$$f_a = k_e(x - x_a) + a'x \tag{27}$$

where $k_e = -b$ and $a' = a + b$.

k_e may be consider to be an added stiffness and a' to be the control gain to the HVA.

Eq. (1) may be rewritten as

$$\begin{cases} M\ddot{x} = -Kx - (k + k_e)(x - x_a) - c(\dot{x} - \dot{x}_a) - ax + F \\ m\ddot{x}_a = -(k + k_e)(x_a - x) - c(\dot{x}_a - \dot{x}) + ax \end{cases} \tag{28}$$

The frequency response function of the mass *M* may be written according to Eq. (5) as

$$H(\lambda) = \frac{X}{F/K} = \frac{\gamma^2 - \lambda^2 + 2\mathrm{j}\zeta\gamma\lambda}{\left((1-\lambda^2)(\gamma^2 - \lambda^2) - (\mu\gamma^2 + 2\alpha)\lambda^2\right) + 2\mathrm{j}\zeta\gamma\lambda\left(1 - \lambda^2 - \mu\lambda^2\right)}, \tag{29}$$

and

and the frequency response of the absorber mass *m* may be written as

$$\frac{X_a}{F/K} = \frac{\gamma^2 + 2(\alpha/\mu) + 2\mathrm{j}\zeta\gamma\lambda}{\left((1-\lambda^2)(\gamma^2 - \lambda^2) - (\mu\gamma^2 + 2\alpha)\lambda^2\right) + 2\mathrm{j}\zeta\gamma\lambda\left(1 - \lambda^2 - \mu\lambda^2\right)} \tag{30}$$

where $\mu = \dfrac{m}{M}$, $\omega_n = \sqrt{\dfrac{K}{M}}$, $\omega_a = \sqrt{\dfrac{k + k_e}{m}}$, $\gamma = \dfrac{\omega_a}{\omega_n}$, $\zeta = \dfrac{c}{2\sqrt{m(k + k_e)}}$, $\lambda = \dfrac{\omega}{\omega_n}$ and $\alpha = \dfrac{a'}{2K}$.

Comparing Eqs. (5) and (29), the two equations are the same except the absorber's frequency ω_a, the damping ratio ζ and the control gain α are different in the two cases. The optimum tuning frequency and damping ratios of the HVA in this case are still expressed as Eqs. (10) and (18) respectively with $\alpha = \frac{a'}{2K}$.

According to Eq. (19), the maximum vibration amplitude of the primary mass M may be written as

$$G = \left|\frac{X}{F/K}\right|_{\max} = \sqrt{\frac{2+\mu}{\mu + 2\alpha(1+\mu)}} \tag{31}$$

where $\alpha = \frac{a'}{2K} < \frac{1}{1+\mu}$.

Comparing Eqs. (19) and (31), the maximum vibration amplitude of the primary mass M cannot be further reduced by using feedback signals from both the primary and absorber masses with the control law of Eq. (27) for the active control of the HVA.

Since $f_a = ax + bx_a$, the active force required in the HVA may be written as

$$\left|\frac{F_a}{F}\right| = \left(\frac{1}{K}\right)\left|\frac{aX + bX_a}{F/K}\right|. \tag{32}$$

$\left|\frac{F_a}{F}\right| = \left|\frac{a}{K}\right|\left|\frac{X}{F/K}\right|$ if b is zero and therefore the spectrum of $\left|\frac{F_a}{F}\right|$ will have two peaks of equal height similar to the spectrum of $\left|\frac{X}{F/K}\right|$ as shown in Fig. 5. If b is not zero, one peak of the spectrum $\left|\frac{F_a}{F}\right|$ will raise while the other peak will fall as illustrated in Fig.3 for

$\left|\frac{X}{F/K}\right|$. To illustrate the effect of b on the active force amplitude, the dimensionless active force amplitudes $\left|\frac{F_a}{F}\right|$ are calculated using Eqs. (29), (30) and (32) with $\mu = 0.2$ and four different set of $(a/2K, b/2K)$= (0.2, 0), (1.2, -1), (-0.3, 0.5) and (0.1, 0.1) such that $(a+b)/2K = \alpha = 0.2$ in all four cases and the results are plotted in Fig. 8 for illustration. According to Eq. (31), the maximum vibration amplitude of the primary mass *M*, *G* can be found to be 1.8 for all the four cases being considered but the maximum active force required in the first case with b = 0 is smaller than the other three cases with b not equal to zero. This shows that in the first case where the HVA use only the feedback signal ax from the primary mass requires smaller active forces and hence power for optimum performance than the other cases where the HVA use both the feedback signals $ax + bx_a$ from the primary and the absorber masses.

3. Simulation results and discussion

The proposed HVA is compared to a similar design of Chatterjee [18] reported recently, in which the displacement of the absorber mass in a HVA without damping was used as feedback signal. Chatterjee proposed a H_∞ optimum PD control for the minimization of resonant vibration amplitude of a SDOF system with the active force of the HVA being $f_a = ax_a - b\dot{x}_a$ and the frequency response function of the primary and absorber masses may be written respectively as

$$\frac{X}{F/K} = \frac{\mu\gamma^2 - 2\alpha - \mu\lambda^2 + \mathrm{j}\beta\lambda}{(\mu\gamma^2 - \mu\lambda^2 - 2\alpha)(1-\lambda^2) - \mu^2\gamma^2\lambda^2 + 2\,\mathrm{j}\beta\lambda(1-\lambda^2)} \tag{33}$$

$$\frac{X_a}{F/K} = \frac{\mu\gamma^2}{\left(\mu\gamma^2 - \mu\lambda^2 - 2\alpha\right)\left(1-\lambda^2\right) - \mu^2\gamma^2\lambda^2 + 2\,\mathrm{j}\beta\lambda\left(1-\lambda^2\right)} \tag{34}$$

where $\alpha = \dfrac{a}{2K} < 1$ and $\beta = \dfrac{b\omega_n}{2K}$.

The optimum tuning frequency and damping ratios of the absorber can be written respectively as [18]

$$\gamma = \sqrt{\frac{4\alpha + 2\mu}{\mu(2+\mu)}}, \text{ and} \tag{35a}$$

$$\beta = \sqrt{\frac{\beta_{\lambda_a}^2 + \beta_{\lambda_b}^2}{2}} = \sqrt{\frac{3\mu^2(\mu + 2\alpha)}{2+\mu}} \tag{35b}$$

The resonant vibration amplitude of the primary mass may be written as [18]

$$G_{\mathrm{PD}} = \left|\frac{X}{F/K}\right|_{\max} = \sqrt{\frac{2+\mu}{2\alpha + \mu}} \tag{36}$$

The feedback gain α may be written using Eq. (36) as

$$\alpha = \frac{2 + \mu - \mu G_{\mathrm{PD}}{}^2}{2G_{\mathrm{PD}}{}^2} \tag{37}$$

The dimensionless force functions are defined as

$$\left.\frac{F_a}{F}\right|_{PD} = 2(\alpha - \mathrm{j}\beta\lambda)\left(\frac{X_a}{F/K}\right) \tag{38}$$

To compare the proposed HVA to the one by Chatterjee [18], the amplitude response of the proposed HVA, $\left|H\left(\lambda, \gamma_{\mathrm{opt}}, \zeta_{\mathrm{opt}}\right)\right|$ is calculated according to Eqs. (5), (10) and (18) and the amplitude response of the HVA by Chatterjee is calculated according to Eqs. (33), (35a) and (35b) and the results are plotted in Fig. 9. G = 2.1 and μ = 0.2 in both cases. $\gamma = 1.5$ (ω_a in ref. [18]) in the second case. In Fig. 9, both amplitude response curves have similar shape but slightly different resonant frequencies. The active force of the proposed HVA is calculated according to Eqs. (2c), (10) and (18) and that of Chatterjee [18] according to Eqs. (26), (35a), (35b) and (38) and the results are plotted in Fig. 10 for comparison. The maximum actuation force of the *PD* control is 4.6 times of that of the present *P* control with the optimized passive damping of the absorber. As shown in Fig. 10, the proposed control with the proposed optimized parameters can reduce the actuation force while maintaining the vibration suppression performance.

The proposed HVA is tested numerically on a simply-supported beam similar to the one studied by Chatterjee [18], as shown in Fig. 11, with a uniformly distributed force. The mean square displacement of the whole beam is evaluated. The length of the beam is $L = 1\text{m}$ and a HVA is attached at $x = x_0 = 0.5\text{m}$. The dimension of the cross section is $0.025\text{m} \times 0.025\text{m}$. The mass ratio of the HVA is 0.05. The material of the beam is aluminium with $\rho = 2710\text{kgm}^{-3}$ and $E = 6.9\text{GPa}$. The beam is assumed to be an Euler-Bernoulli beam and its equation of motion may be written as

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(t) g(x) + F_h(t) \delta(x - x_0) \tag{39}$$

Here it has been assumed that the externally applied forcing function can be expressed as $p(t)g(x)$, where $p(t)$ is a function of time and $g(x)$ is a deterministic function of *x*. F_h is the force excited by the HVA. The frequency response function of the beam can be derived as shown in the Appendix A and written as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A \omega_n^2} \sum_{p=1}^{\infty} \frac{a_p - \dfrac{\mu L b_p \sum_{q=1}^{\infty} \dfrac{a_q \varphi_q(x_0)}{\gamma_q^2 - \lambda^2}}{-\dfrac{\gamma^2 - \lambda^2 + 2\,\mathrm{j}\zeta\gamma\lambda}{\lambda^2\left(\gamma^2 + \dfrac{2\alpha}{\varepsilon} + 2\,\mathrm{j}\zeta\gamma\lambda\right)} + \mu L \sum_{r=1}^{\infty} \dfrac{b_r \varphi_r(x_0)}{\gamma_r^2 - \lambda^2}}}{\gamma_p^2 - \lambda^2} \varphi_p(x) \quad (40)$$

where φ_p is the p^{th} eigenvector of the beam, a_p and b_p are Fourier coefficients as described in the Appendix A; $\omega_n = \sqrt{\dfrac{EI\beta_1^4}{\rho A}}$, $\omega_r = \sqrt{\dfrac{EI\beta_r^4}{\rho A}}$, $\gamma_r = \dfrac{\omega_r}{\omega_n}$, $\mu = \dfrac{m}{\rho AL}$, $\varepsilon = \mu\varphi_1^2(x_0)$ and $\alpha = \dfrac{a\varphi_1^2(x_0)}{2EIL\beta_1^4}$.

Based on Eq. (40), the mean square motion of the beam may be written as

$$\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx = \left(\frac{1}{\rho A \omega_n^2}\right)^2 \sum_{p=1}^{\infty} \left| \frac{a_p - \dfrac{\mu L b_p \sum_{q=1}^{\infty} \dfrac{a_q \varphi_q(x_0)}{\gamma_q^2 - \lambda^2}}{-\dfrac{\gamma^2 - \lambda^2 + 2\,\mathrm{j}\zeta\gamma\lambda}{\lambda^2\left(\gamma^2 + \dfrac{2\alpha}{\varepsilon} + 2\,\mathrm{j}\zeta\gamma\lambda\right)} + \mu L \sum_{r=1}^{\infty} \dfrac{b_r \varphi_r(x_0)}{\gamma_r^2 - \lambda^2}}}{\gamma_p^2 - \lambda^2} \right|^2 \quad (41)$$

The optimum frequency and damping of the HVA may be rewritten [20] in term of ε, i.e. when $\alpha < \dfrac{1}{1+\varepsilon}$,

$$\gamma_{\text{opt\_HVA}} = \sqrt{\frac{1-\alpha(1+\varepsilon)}{(1+\varepsilon)^2}}, \text{ and} \quad (42a)$$

$$\zeta_{\text{opt\_HVA}} = \sqrt{\frac{\zeta_{\lambda_a}^2 + \zeta_{\lambda_b}^2}{2}} = \sqrt{\frac{3\varepsilon + \alpha(1+\varepsilon)(6+\varepsilon)}{8(1+\varepsilon)[1-\alpha(1+\varepsilon)]}} \qquad (42b)$$

The HVA is tuned for the dimensionless frequency response at $G = 2$. The feedback gain α is determined to be 0.1458 while the optimum tuning ratio and the optimum damping ratios are determined using Eqs. (42a) and (42b) as 0.7569 and 0.4613 respectively.

The proposed optimum HVA is firstly compared to the PVA counterpart with $\alpha = 0$. The optimum tuning frequency and damping ratio may be written respectively as [19]

$$\gamma_{\text{opt\_DVA}} = \frac{1}{(1+\varepsilon)} \text{ , and} \qquad (43a)$$

$$\zeta_{\text{opt\_DVA}} = \sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}} \qquad (43b)$$

Dimensionless mean square displacement of the beam with a passive vibration absorber is calculated with Eq. (41) when $\alpha = 0$, $\gamma = \gamma_{\text{opt\_DVA}}$ and $\zeta = \zeta_{\text{opt\_DVA}}$ using Eqs. (43a) and (43b), respectively, and the result is plotted with the case of the using the proposed HVA in Fig. 12. Fig. 12 shows that the maximum mean square motion of the primary mass using the proposed HVA is 60% lower than the one using the optimized PVA. Suppression of the mean square motion of the primary mass using the proposed HVA at the higher modes is also better than using the PVA.

Secondly the proposed optimum HVA is also compared to the optimized *PD* controlled HVA proposed by Chatterjee [18]. Similar to Eq. (40), the frequency response function of the beam using the optimized PD controlled HVA [17] may be derived and written as

$$
\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A\omega_n^2}\sum_{p=1}^{\infty}\frac{a_p - \dfrac{\mu L b_p \sum_{q=1}^{\infty}\dfrac{a_q\varphi_q(x_0)}{\gamma_q^2-\lambda^2}}{-\dfrac{\gamma^2-\lambda^2-\dfrac{2\alpha}{\varepsilon}+\dfrac{2\mathrm{j}\eta\lambda}{\varepsilon}}{\gamma^2\lambda^2}+\mu L\sum_{r=1}^{\infty}\dfrac{b_r\varphi_r(x_0)}{\gamma_r^2-\lambda^2}}}{\gamma_p^2-\lambda^2}\varphi_p(x) \tag{44}
$$

where $\omega_n = \sqrt{\dfrac{EI\beta_1^4}{\rho A}}$, $\omega_r = \sqrt{\dfrac{EI\beta_r^4}{\rho A}}$, $\gamma_r = \dfrac{\omega_r}{\omega_n}$, $\mu = \dfrac{m}{\rho AL}$, $\varepsilon = \mu\varphi_1^2(x_0)$,

$\alpha = \dfrac{a\varphi_1^2(x_0)}{2EIL\beta_1^4}$ and $\eta = \dfrac{b\omega_n\varphi_1^2(x_0)}{2EIL\beta_1^4}$.

Similar to Eq. (41), the mean square motion over the whole domain of the beam using the optimized PD controlled HVA [18] may be derived and written as

$$
\frac{1}{L}\int_0^L\left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx = \left(\frac{1}{\rho A\omega_n^2}\right)^2\sum_{p=1}^{\infty}\left|\frac{a_p - \dfrac{\mu L b_p \sum_{q=1}^{\infty}\dfrac{a_q\varphi_q(x_0)}{\gamma_q^2-\lambda^2}}{-\dfrac{\gamma^2-\lambda^2-\dfrac{2\alpha}{\varepsilon}+\dfrac{2\mathrm{j}\eta\lambda}{\varepsilon}}{\gamma^2\lambda^2}+\mu L\sum_{r=1}^{\infty}\dfrac{b_r\varphi_r(x_0)}{\gamma_r^2-\lambda^2}}}{\gamma_p^2-\lambda^2}\right|^2 \tag{45}
$$

The mean square motion response of the whole beam is calculated according to Eqs. (41) and (45) and the results are plotted in Fig. 13. The corresponding active force spectra are plotted in Fig. 14. Comparing the proposed optimum HVA to that of Chatterjee [18]. The active force required by the proposed optimum HVA is much smaller than that required by the one proposed by Chatterjee [18].

There are some reported design methods of HVA such as the zero-pole placement method [16] which is able to reduce vibration peaks while keeping the absorption dip simultaneously in the frequency response of the closed-loop primary system. However, it is shown in the Appendix B that even though the zero-pole placement method can produce greater vibration reduction of the vibrating structure than the proposed method but the active force required in that method is very much larger than the proposed method. The proposed design method optimizes the damping effect using the passive elements and therefore the active force in the HVA can be very much reduced even though the reduction of vibration amplitude of the primary mass is not as good as the zero-pole placement method. The proposed design method would be a good option if the active force component in the HVA cannot be too large.

8. Conclusion

In this paper, the H_∞ optimization design of a hybrid vibration absorber (HVA) for the minimization of the resonant vibration amplitude of a single degree-of-freedom (SDOF) vibrating structure is derived by using the fixed-points theory. A general design framework is established based on an analytical model. The optimum tuning parameters are the feedback gain, the tuning frequency ratio, the damping ratio and the mass ratio of the absorber. The effects of these parameters on the vibration absorption of the primary structure are systematically revealed. Design parameters of both passive and active elements in the HVA are optimized. The inherent limitation of the traditional passive vibration absorber requiring relatively large mass ratio to achieve a targeted vibration suppression level is bypassed using the proposed design, such facilitating the application of the technique in applications involving large structures such as buildings and bridges. Compared to other existing HVA designs for similar propose, the presently proposed design requires smaller active force and hence less energy for the active element.

Appendix A

Consider the motion of the beam as shown in Fig. 11 excited by a uniformly distributed force located between 0 and *L*. A damped hybrid vibration absorber is attached at x_0. The length of the beam is *L*, and mass per unit length is ρA with bending stiffness *EI*. The added mass and the stiffness of HVA are *m* and *k* respectively. The boundary conditions may be a pinned, clamped or free end. The problem is described by the Bernoulli–Euler equation for small motions of slender beams and the following conditions.

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(t)g(x) + F_h(t)\delta(x - x_0) \qquad \text{(A1a)}$$

$$m\ddot{x}_a = k(x - x_a) + c(\dot{x} - \dot{x}_a) + f_a \qquad \text{(A1b)}$$

$$F_h(t) = -m\ddot{x}_a \qquad \text{(A1c)}$$

Here it has been assumed that the externally applied forcing function is $p(t)g(x)$, where $p(t)$ is a function of time and $g(x)$ is a deterministic function of *x*. $F_h(t)$ is the force applied to the beam from the HVA. f_a is the active force from the HVA. The solution to this problem may be expanded in a Fourier series written as [21],

$$w(x,t) = \sum_{p=1}^{\infty} q_p(t)\varphi_p(x) \qquad \text{(A2)}$$

$$\text{where } \int_0^L \varphi_i^2(x)dx = L \text{, where } i \in N \qquad \text{(A3)}$$

Similarly, the spatial part of the forcing function can be expanded as

$$g(x) = \sum_{p=1}^{\infty} a_p \varphi_p(x) \qquad \text{(A4)}$$

And the derivative of Dirac delta functions can also be expanded as

$$\delta(x-x_0)=\sum_{p=1}^{\infty} b_p \varphi_p(x) \tag{A5}$$

where the Fourier coefficients a_i and b_i are respectively

$$a_i = \frac{\int_0^L g(x)\varphi_i(x)dx}{L} \quad \text{and} \quad b_i = \frac{\varphi_i(x_0)}{L} \tag{A6}$$

Here a_i depend only on the spatial distribution of the forcing function $g(x)$. If the Eqs. (A2), (A3), (A4), (A5) and (A6) are substituted into Eq. (A1a) and the Laplace transform is taken with respect to time, the result is a set of algebraic equations

$$\rho A s^2 Q_i(s) + EI\beta_i^4 Q_i(s) = a_i P(s) + b_i F_h(s), \text{ where } \; i \in N \tag{A7}$$

If this is solved for the generalized co–ordinates $Q_i(s)$ the result is

$$Q_i(s) = \frac{a_i P(s) + b_i F_h(s)}{\rho A s^2 + EI\beta_i^4} \tag{A8}$$

Then if $P(s)$ and $F_h(s)$ were known then the *s*–domain motion of any point on the beam could be given as

$$W(x,s) = \sum_{p=1}^{\infty} \frac{a_p P(s) + b_p F_h(s)}{\rho A s^2 + EI\beta_p^4} \varphi_p(x) \tag{A9}$$

where $W(x,s)$ is the Laplace transform of $w(x,t)$ with respect to time. Let the active force be $f_a = ax$. By Eqs. (A1b) and (A1c), the relations between the motion of the point of attachment and the force transmitted to the beam at the point of attachment is

$$F_h(s) = -W(x_0,s)\frac{ms^2(cs+k+a)}{ms^2+cs+k} \tag{A10}$$

$W(x,s)$ can be obtained by Eqs. (A1d), (A9) and (A10), i.e.

$$W(x,s)=\sum_{p=1}^{\infty}\frac{a_pF(s)-b_pW(x_0,s)\dfrac{ms^2(cs+k+a)}{ms^2+cs+k}}{\rho As^2+EI\beta_p^4}\varphi_p(x) \tag{A11}$$

By Eq. (A11), $W(x_0,s)$ can be obtained when $x=x_0$, i.e.,

$$W(x_0,s)=\frac{\displaystyle\sum_{p=1}^{\infty}\frac{a_p\varphi_p(x_0)F(s)}{\rho As^2+EI\beta_p^4}}{1+\displaystyle\sum_{p=1}^{\infty}\frac{b_p\varphi_p(x_0)\dfrac{ms^2(cs+k+a)}{ms^2+cs+k}}{\rho As^2+EI\beta_p^4}} \tag{A12}$$

Substitute Eq. (A12) into (A11), the transfer function of the beam is

$$\frac{W(x,s)}{F(s)}=\sum_{p=1}^{\infty}\frac{a_p-b_p\dfrac{\displaystyle\sum_{p=1}^{\infty}\frac{a_p\varphi_p(x_0)}{\rho As^2+EI\beta_p^4}}{\dfrac{ms^2+cs+k}{ms^2(cs+k+a)}+\displaystyle\sum_{p=1}^{\infty}\frac{b_p\varphi_p(x_0)}{\rho As^2+EI\beta_p^4}}}{\rho As^2+EI\beta_p^4}\varphi_p(x) \tag{A13}$$

Replacing the complex variable s in Eq. (A13) by $\mathrm{j}\omega$, the frequency response function of the beam may be written in a dimensionless forma as

$$\frac{W(x,\lambda)}{F(\lambda)}=\frac{1}{\rho A\omega_n^2}\sum_{p=1}^{\infty}\frac{a_p-\dfrac{\mu Lb_p\displaystyle\sum_{q=1}^{\infty}\frac{a_q\varphi_q(x_0)}{\gamma_q^2-\lambda^2}}{-\dfrac{\gamma^2-\lambda^2+2\mathrm{j}\zeta\gamma\lambda}{\lambda^2\left(\gamma^2+\dfrac{2\alpha}{\varepsilon}+2\mathrm{j}\zeta\gamma\lambda\right)}+\mu L\displaystyle\sum_{r=1}^{\infty}\frac{b_r\varphi_r(x_0)}{\gamma_r^2-\lambda^2}}}{\gamma_p^2-\lambda^2}\varphi_p(x) \tag{A14}$$

where $\omega_n=\sqrt{\dfrac{EI\beta_1^4}{\rho A}}$, $\omega_r=\sqrt{\dfrac{EI\beta_r^4}{\rho A}}$, $\gamma_r=\dfrac{\omega_r}{\omega_n}$, $\mu=\dfrac{m}{\rho AL}$, $\varepsilon=\mu\varphi_1^2(x_0)$ and

$$\alpha = \frac{a\varphi_1^2(x_0)}{2EIL\beta_1^4}$$

The eigenfunctions of the beam obey the orthogonality relations and the orthogonality relations can be written as,

$$\int_0^L \varphi_i(x)\varphi_j(x)dx = 0 \text{, if } \; i \neq j \tag{A15a}$$

$$\int_0^L \varphi_i(x)\varphi_j(x)dx = L \text{, if } \; i = j \tag{A15b}$$

Consider the orthogonality relations and the equation $\left|\frac{W(x,s)}{F(s)}\right|^2 = \frac{W(x,s)}{F(s)} \times \overline{\left(\frac{W(x,s)}{F(s)}\right)}$, the mean square motion over the whole domain of the beam can be written as,

$$\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx = \left(\frac{1}{\rho A\omega_n^2}\right)^2 \sum_{p=1}^{\infty} \left| \frac{a_p - \dfrac{\mu L b_p \displaystyle\sum_{q=1}^{\infty} \frac{a_q \varphi_q(x_0)}{\gamma_q^2 - \lambda^2}}{-\dfrac{\gamma^2 - \lambda^2 + 2\,\mathrm{j}\zeta\gamma\lambda}{\lambda^2\left(\gamma^2 + \dfrac{2\alpha}{\varepsilon} + 2\,\mathrm{j}\zeta\gamma\lambda\right)} + \mu L \displaystyle\sum_{r=1}^{\infty} \frac{b_r \varphi_r(x_0)}{\gamma_r^2 - \lambda^2}}}{\gamma_p^2 - \lambda^2} \right|^2$$

(A16)

Appendix B

Consider $c = 0$ in Fig. 1, the equations of motion may be written using Eq. (1) as

$$\begin{cases} M\ddot{x} = -Kx - k(x - x_a) - f_a + F \\ m\ddot{x}_a = -k(x_a - x) + f_a \end{cases} \tag{B1}$$

where F is a disturbance and f_a an actuation force. Laplace transformation is taken with respect to time, the result is a set of algebraic equation written as

$$\begin{cases} Ms^2 X = -KX - k(X - X_a) - F_a + F \\ ms^2 X_a = -k(X_a - X) + F_a \end{cases} \tag{B2}$$

Following the approach of [16], the active force may be written as

$$F_a = \left(\frac{a_{-2}}{s^2} + \frac{a_{-1}}{s} + a_0 + sa_1 \right) X \tag{B3}$$

where a_{-2}, a_{-1}, a_0 and a_1 are the feedback gains. The transfer function of the primary mass may be solved using Eq. (B2) and written as

$$\frac{X}{F} = \frac{ms^2 + k}{mMs^4 + m(K + k + kM + a_1)s^2 + m(a_{-1} + a_0)s + ma_{-2} + kK} \tag{B4}$$

Replacing s by $j\omega$ in Eq. (B4), the frequency response function of the primary mass may be rewritten in a dimensionless form as

$$\frac{X}{F/K} = \frac{\gamma^2 - \lambda^2}{\lambda^4 - 2j\alpha_1\lambda^3 - (1 + 2\alpha_0 + \gamma^2 + \mu\gamma^2)\lambda^2 + 2j\alpha_{-1}\lambda + 2\alpha_{-2} + \gamma^2} \tag{B5}$$

where $\omega_n = \sqrt{\dfrac{K}{M}}$, $\omega_a = \sqrt{\dfrac{k}{m}}$, $\lambda = \dfrac{\omega}{\omega_n}$, $\gamma = \dfrac{\omega_a}{\omega_n}$, $\mu = \dfrac{m}{M}$, $\alpha_1 = \dfrac{a_1 \omega_n}{2K}$, $\alpha_0 = \dfrac{a_0}{2K}$,

$\alpha_{-1} = \dfrac{a_{-1}}{2K\omega_n}$, $\alpha_{-2} = \dfrac{a_{-2}}{2K\omega_n^2}$ and $j = \sqrt{-1}$.

Similarly, the frequency response function of the active force of the HVA may be rewritten in a dimensionless form as

$$\frac{F_a}{F} = \left(-\frac{\alpha_{-2}}{\lambda^2} - \frac{\mathrm{j}\alpha_{-1}}{\lambda} + \alpha_0 + \mathrm{j}\alpha_1 \lambda \right) \left(\frac{X}{F/K} \right) \tag{B6}$$

A numerical example of the HVA design using zero-pole assignment method [16] is presented in the following. Assume $\gamma = 1$ and $\zeta = 1$ in Eq. (B5), the frequency response of the primary structure with damped HVA using the zero-pole assignment method may be written as

$$\frac{X}{F/K} = \frac{1-\lambda^2}{\lambda^4 - 4j\lambda^3 - 6\lambda^2 + 4j\lambda + 1} \tag{B7}$$

where $\begin{cases} 2\alpha_1 = 4 \Rightarrow \alpha_1 = 2 \\ 2.2 + 2\alpha_0 = 6 \Rightarrow \alpha_0 = 1.9 \\ 2\alpha_{-1} = 4 \Rightarrow \alpha_{-1} = 2 \\ 2\alpha_{-2} + 1 = 1 \Rightarrow \alpha_{-2} = 0 \end{cases}$

Eq. (B5) is plotted together with frequency response function of the primary mass using the proposed design method of the HVA in Fig. B1 for comparison. As shown in Fig. B1, the vibration amplitude of the primary mass using the zero-pole assignment method is smaller than using the proposed method. The zero-pole assignment method [16] is able to reduce vibration peaks while keeping the absorption dip simultaneously in the frequency response of the closed-loop primary system. However, as shown in Fig. B2 the plots of Eq. (B6) together with the frequency response function of the active force using the proposed design method of the HVA, the highest frequency response of the active force of HVA using the zero-pole assignment method is forty two times higher than using the proposed design method. Comparing the areas under the frequency response curves of the active force of HVA using the two different design methods as shown in Fig. B2, the area using zero-pole assignment method is found to be one hundred thirty five times higher than using the proposed design method.

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Figure Captions

Fig. 1. Schematic diagram of the proposed hybrid vibration absorber (*m*-*k*-*c*-f_a system) attached to the primary (*M*-*K*) system.

Fig. 2. The frequency response of the primary mass *M* with HVA at $\mu = 0.2$ and $\alpha = 0.1$. ——— $\zeta = 0$, – – – – $\zeta = 0.2$, -·-·-·- $\zeta = 1$.

Fig. 3. The amplitude response at the fixed points versus tuning ratio γ at $\mu = 0.2$ and $\alpha = 0.5$. ——— $|H(\lambda_{\mathrm{a}})|$, – – – – $|H(\lambda_{\mathrm{b}})|$.

Fig. 4. Root locus of the SDOF primary system with the proposed HVA in Fig. 1 with $\mu = 0.2$ and $\alpha \in \left(0, \frac{1}{1+\mu}\right)$.

——— Root 1, – – – – root 2, ·········· root 3, -·-·-·- root 4 of Eq. (3).

Fig. 5. The frequency response of the primary mass *M* with HVA at $\mu = 0.02$.

·········· $\alpha = 0.5$, - - - - - - $\alpha = 0.2$, ——— $\alpha = 0$ (PVA).

Fig. 6. The amplitude response at the fixed points versus tuning ratio γ at $\mu = 0.2$ and $\alpha = 1$. ——— $|H(\lambda_{\mathrm{a}})|$, – – – – $|H(\lambda_{\mathrm{b}})|$.

Fig. 7. The frequency response of the primary mass *M* with HVA tuned to the optimum tuning at $\mu = 0.2$ and $G = 1.5$.

——— $|H(\lambda, \gamma_{\mathrm{opt}}, \zeta_{\mathrm{opt}})|$, $\alpha < \frac{1}{1+\mu}$;

– – – – $|H(\lambda, \gamma_{\mathrm{H}}, \zeta_{\mathrm{H}})|$, $\alpha \geq \frac{1}{1+\mu}$.

Fig. 8. Dimensionless active force $\left|\frac{F_a}{F}\right|$ of the HVA in Fig. 1 where $|F_a| = |aX + bX_a|$ with $\mu = 0.2$ and $G = 2.1$. ——— *a* = 0.2*K*, *b* = 0; ·········· *a* = 1.2*K*, *b* = - *K*; –·–·– *a* = -0.3 *K*, *b* = 0.5 *K*; – – – – *a* = 0.1 *K*, *b* = 0.1 *K*.

Fig. 9. The frequency response at $\mu = 0.2$ and $G = 2.1$.

——— present theory; – – – optimum control by Chatterjee [18].

Fig. 10. Dimensionless active force $\left|\frac{F_a}{F}\right|$ of the HVA with $\mu = 0.2$ and $G = 2.1$.

——— present theory; – – – - optimum control by Chatterjee [18].

Fig. 11. Schematics of a simply supported beam with a hybrid vibration absorber excited by a uniform disturbed force.

Fig. 12. The mean square motion response $\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx$ of the beam as shown in Fig. 11 with $G=2$.

——— present theory using Eq. (41);

.... optimum PVA [19].

Fig. 13. The mean square motion response $\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx$ of the beam as shown in Fig. 11 with $G=2$.

——— present theory using Eq. (41);

-·-·-·- optimum control by Chatterjee [18].

Fig. 14. Active force spectra of the HVA in Fig. 10 with $G=2$.

——— present theory, $2\alpha\left|\frac{W(x_o,\lambda)}{P(\lambda)}\right|$ using Eq. (40);

.... optimum control by Chatterjee [18].

Fig. B1. Fig. B1 The frequency response of the primary mass M in Fig. 1 with $\mu=0.2$ and $G=1.5$

——— present theory using Eq. (B6);

.... zero-pole assignment method [16].

Fig. B2 Active force spectra of the HVA in Fig. 1 with $\mu=0.2$ and $G=1.5$

——— present theory using Eq. (B6);

.... zero-pole assignment method [16].

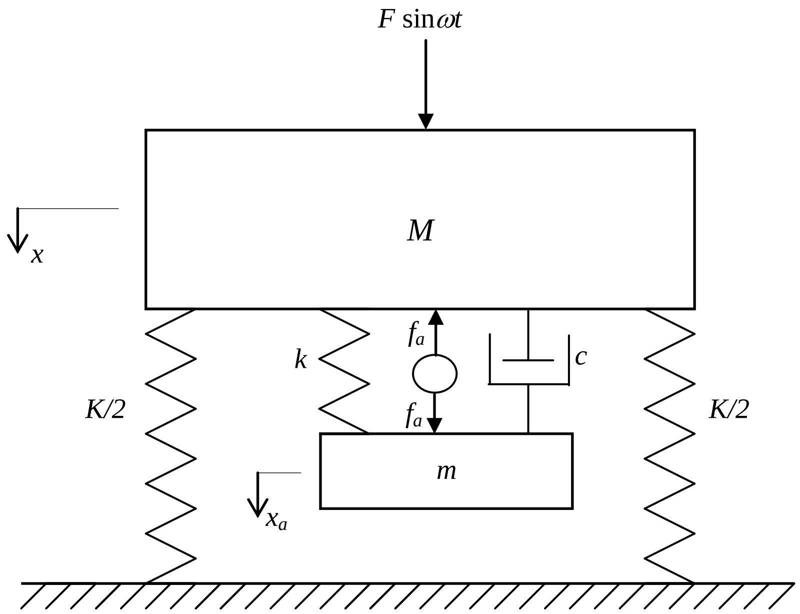

Fig. 1. Schematic diagram of the proposed hybrid vibration absorber (m-k-c-f_a system) attached to the primary (M-K) system.

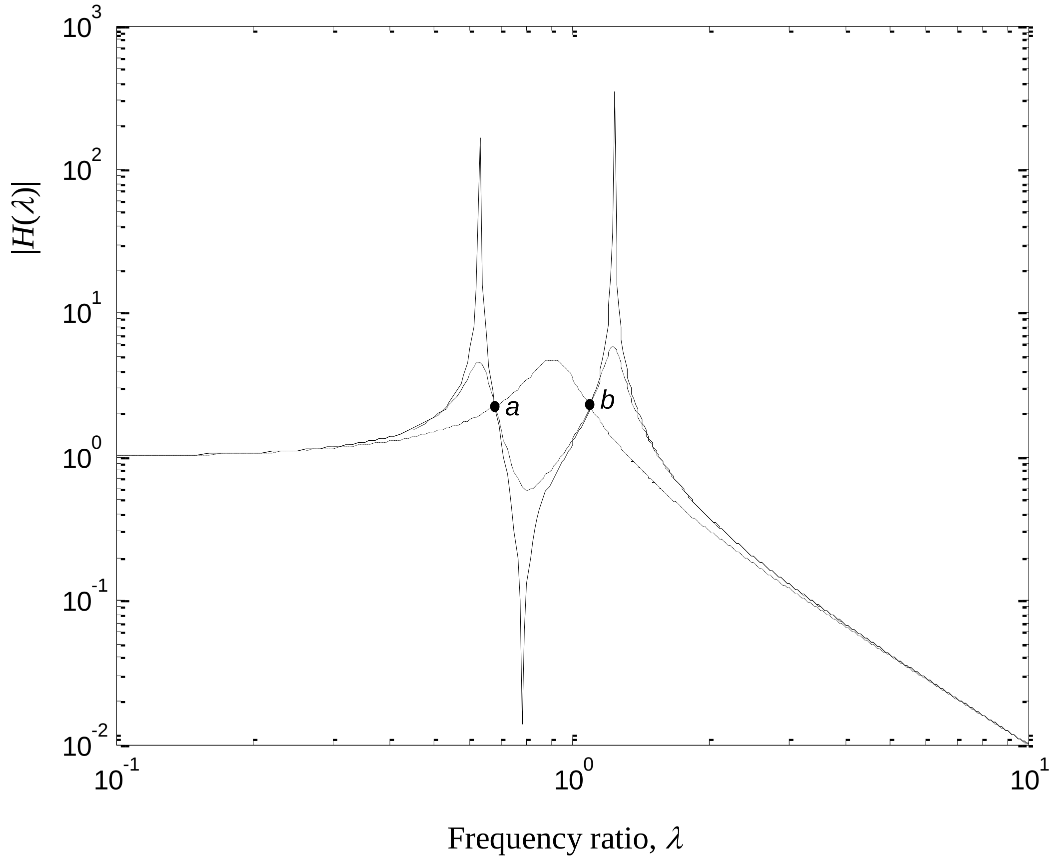

Fig. 2. The frequency response of the primary mass M with HVA at $\mu = 0.2$ and $\alpha = 0.1$. ——— $\zeta = 0$, – – – – $\zeta = 0.2$, -·-·-·-· $\zeta = 1$.

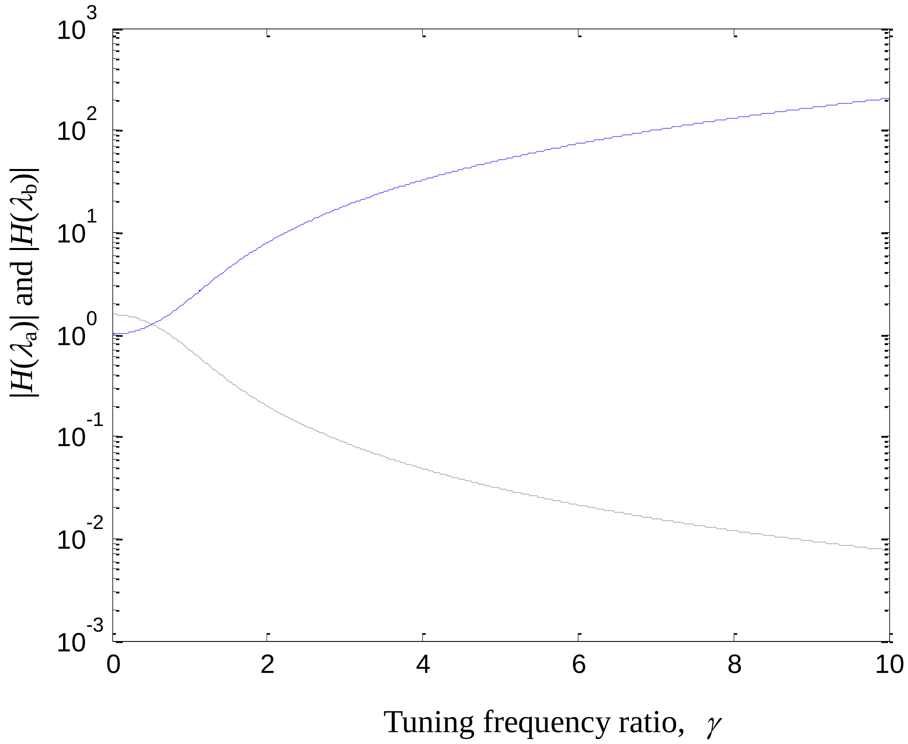

Fig. 3. The amplitude response at the fixed points versus tuning ratio γ at $\mu = 0.2$ and $\alpha = 0.5$. ——— $|H(\lambda_{\mathrm{a}})|$, - - - - $|H(\lambda_{\mathrm{b}})|$.

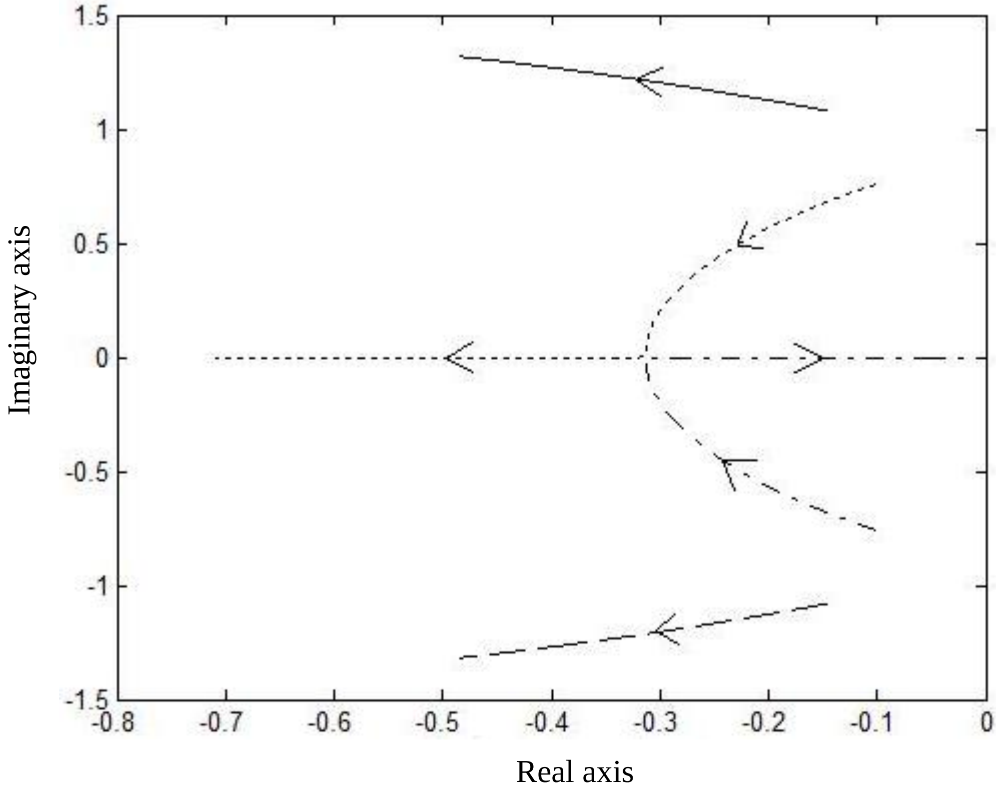

Fig. 4. Root locus of the SDOF primary system with the proposed HVA in Fig. 1 with $\mu = 0.2$ and $\alpha \in \left(0, \frac{1}{1+\mu}\right)$.

——— Root 1, – – – – root 2, ·············· root 3, -·-·-·- root 4 of Eq. (3).

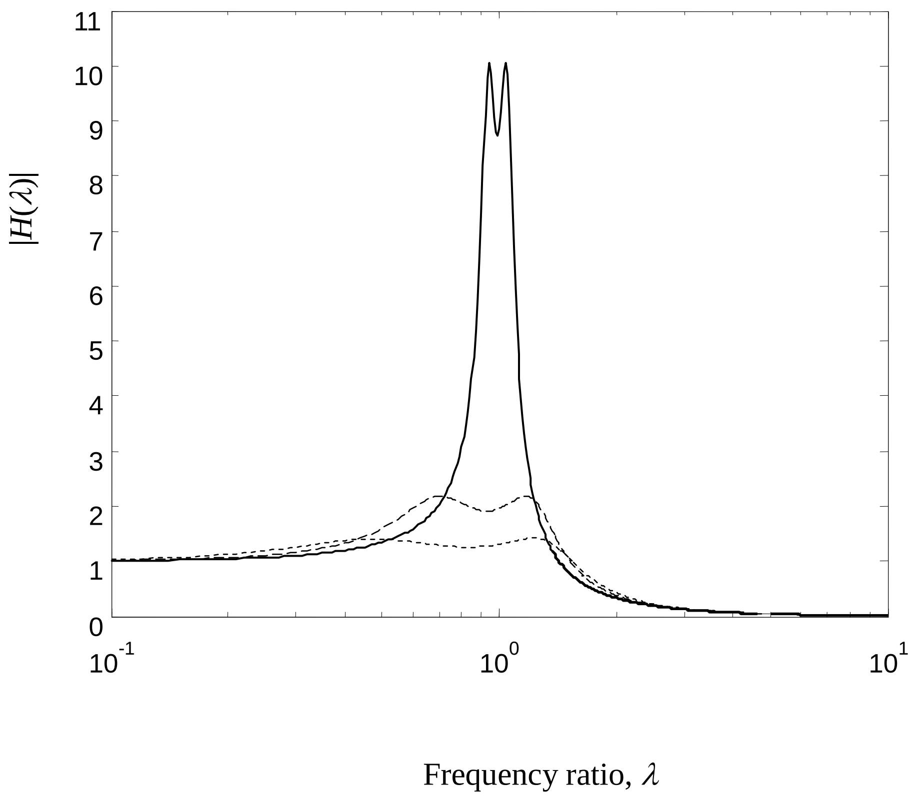

Fig. 5. The frequency response of the primary mass M with HVA at $\mu = 0.02$.

········· $\alpha = 0.5$, - - - - - - $\alpha = 0.2$, ——— $\alpha = 0$ (PVA).

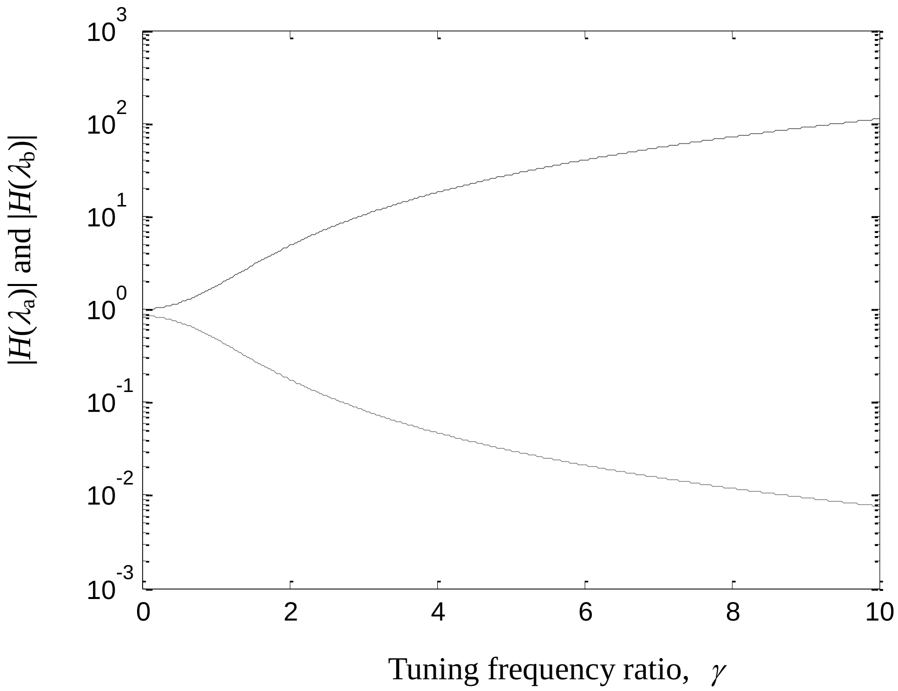

Fig. 6. The amplitude response at the fixed points versus tuning ratio γ at $\mu = 0.2$ and $\alpha = 1$. ——— $|H(\lambda_{\mathrm{a}})|$, - - - - $|H(\lambda_{\mathrm{b}})|$.

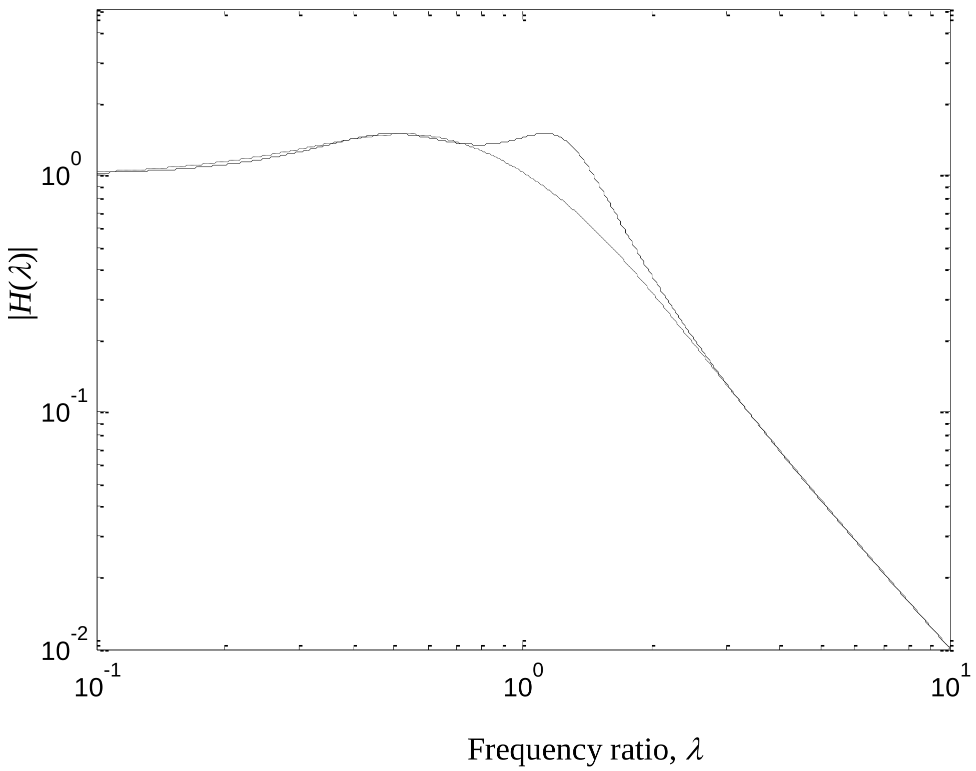

Fig. 7. The frequency response of the primary mass M with HVA tuned to the optimum tuning at $\mu = 0.2$ and $G = 1.5$.

——— $\left|H\left(\lambda, \gamma_{\text{opt}}, \zeta_{\text{opt}}\right)\right|$, $\alpha < \dfrac{1}{1+\mu}$;

– – – – $\left|H(\lambda, \gamma_{\text{H}}, \zeta_{\text{H}})\right|$, $\alpha \geq \dfrac{1}{1+\mu}$.

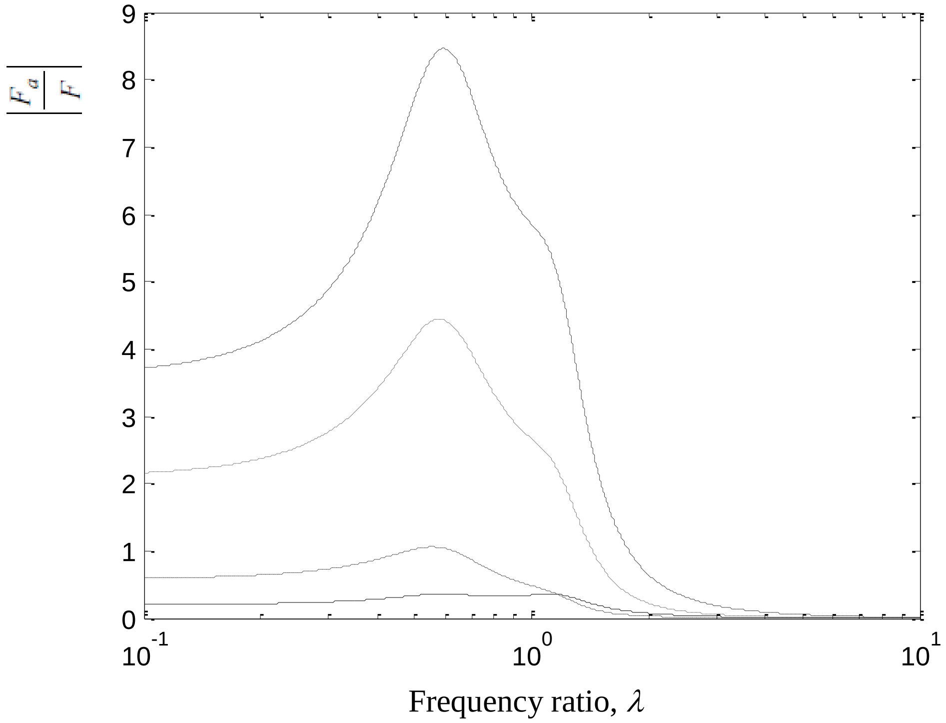

Fig. 8. Dimensionless active force $\left|\frac{F_a}{F}\right|$ of the HVA in Fig. 1 where $|F_a| = |aX + bX_a|$ with $\mu = 0.2$ and $G = 2.1$. ——— a = 0.2K, b = 0; ·········· a = 1.2K, b = -K; –·–·– a = -0.3K, b = 0.5K; – – – – a = 0.1K, b = 0.1K.

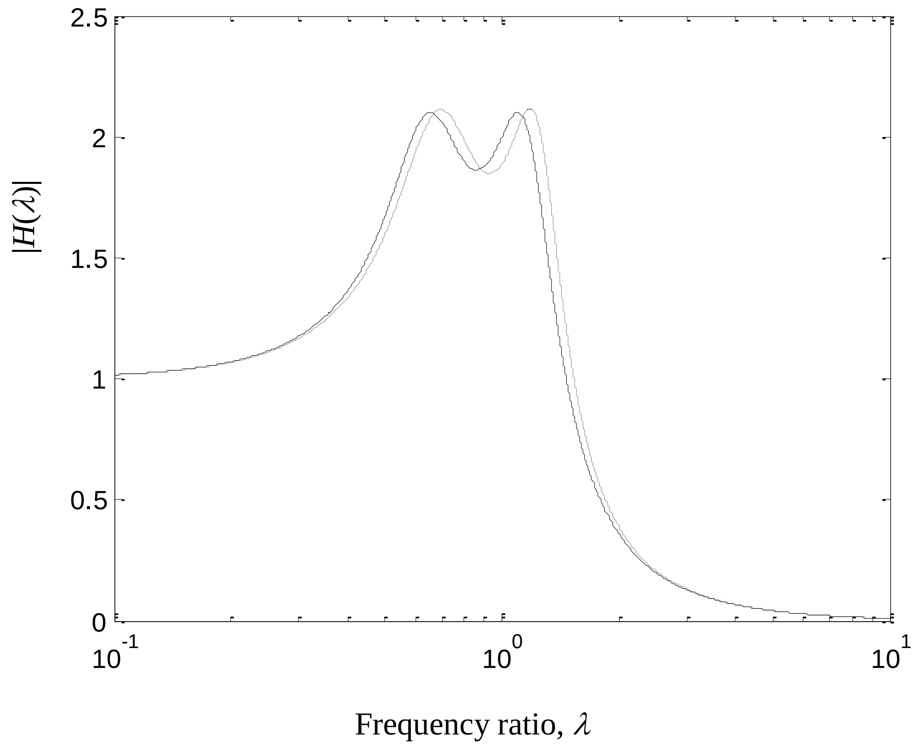

Fig. 9. The frequency response of the primary mass M with $\mu = 0.2$ and $G = 2.1$.

——— present theory; – – – optimum control by Chatterjee [18].

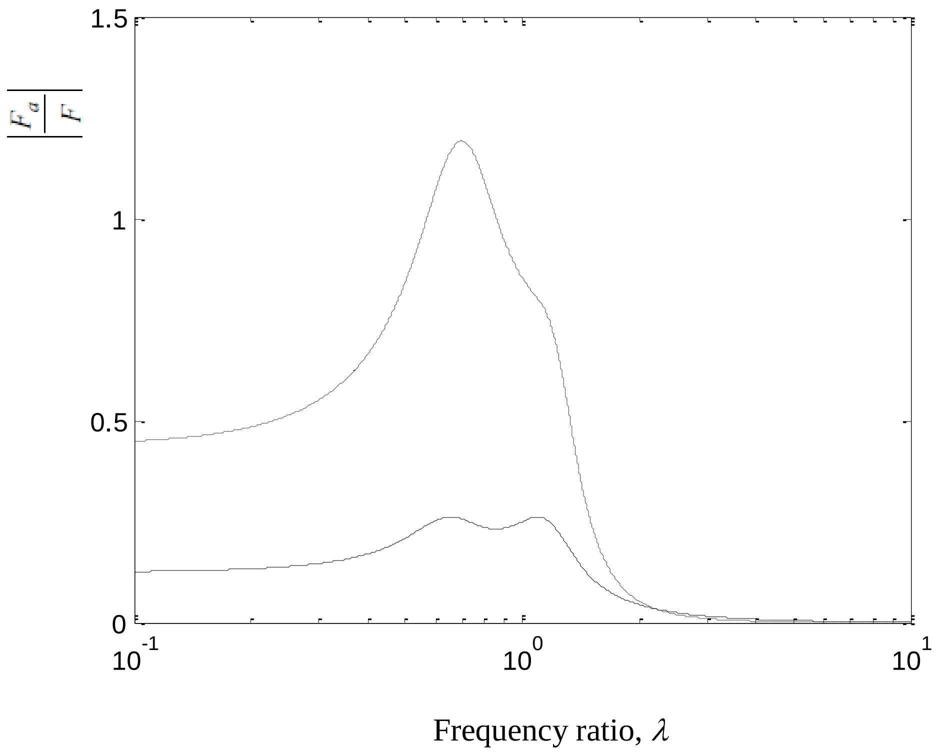

Fig. 10. Dimensionless active force $\left|\dfrac{F_a}{F}\right|$ of the HVA with $\mu = 0.2$ and $G = 2.1$.

——— present theory; – – – - optimum control by Chatterjee [18].

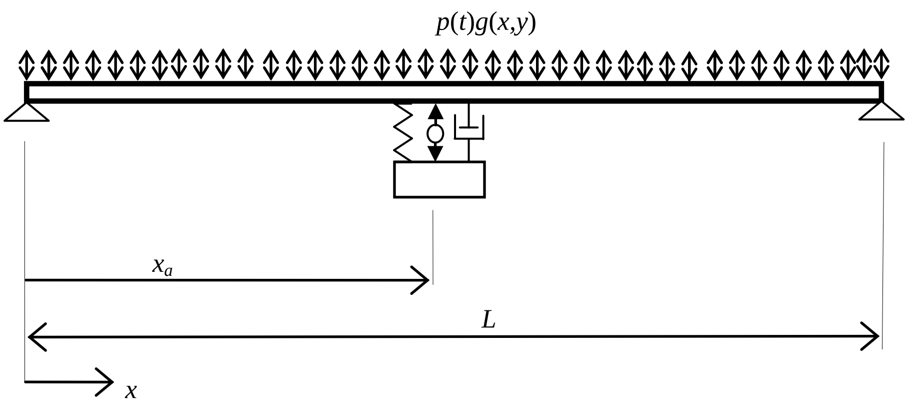

Fig. 11. Schematics of a simply supported beam with a hybrid vibration absorber excited by a uniform disturbed force.

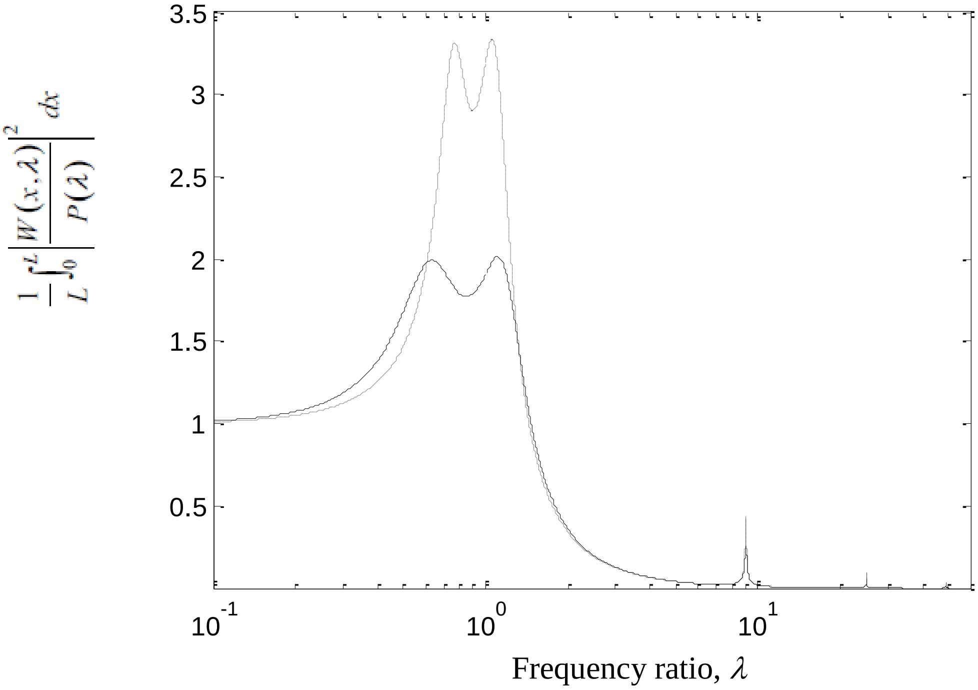

Fig. 12. The mean square motion response $\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx$ of the beam as shown in Fig. 11 with $G = 2$.

——— present theory using Eq. (41);

·················· optimum PVA [19].

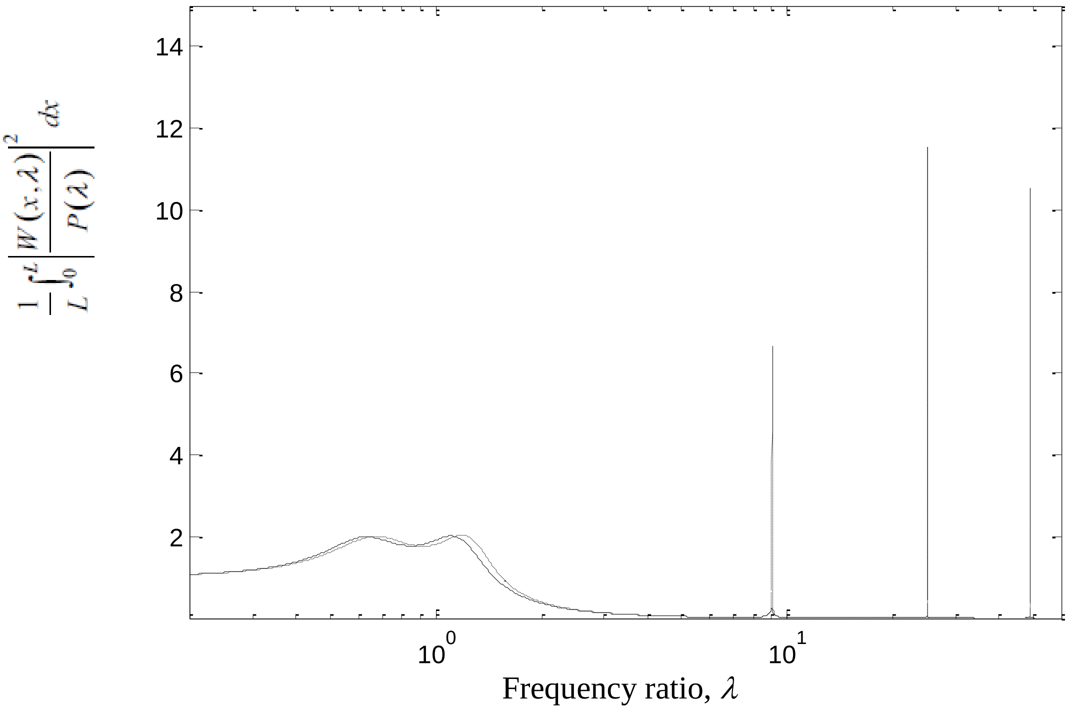

Fig. 13. The mean square motion response $\frac{1}{L}\int_0^L \left|\frac{W(x,\lambda)}{P(\lambda)}\right|^2 dx$ of the beam as shown in Fig. 11 with $G = 2$.

——— present theory using Eq. (41);

-·-·-·-· optimum control by Chatterjee [18].

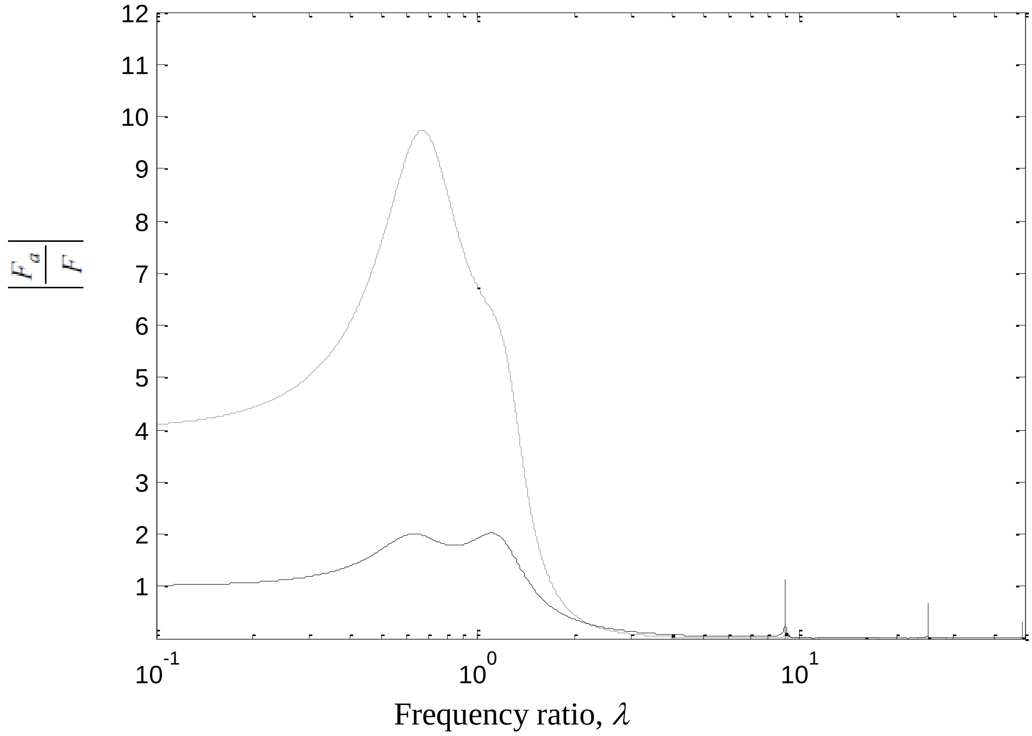

Fig. 14. Active force spectra of the HVA in Fig. 10 with $G = 2$.

——— present theory, $2\alpha\left|\frac{W(x_o,\lambda)}{P(\lambda)}\right|$ using Eq. (40);

.... optimum control by Chatterjee [18].

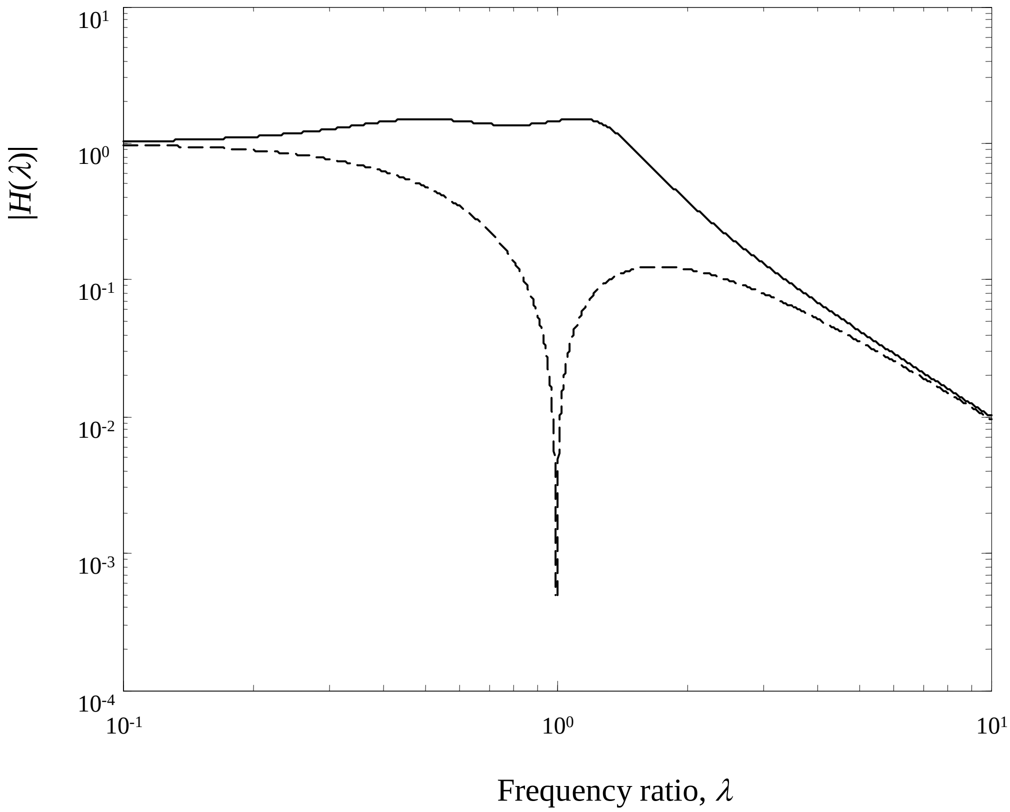

Fig. B1 The frequency response of the primary mass M in Fig. 1 with $\mu = 0.2$ and $G = 1.5$

——— present theory using Eq. (B6);

··········· zero-pole assignment method [16].

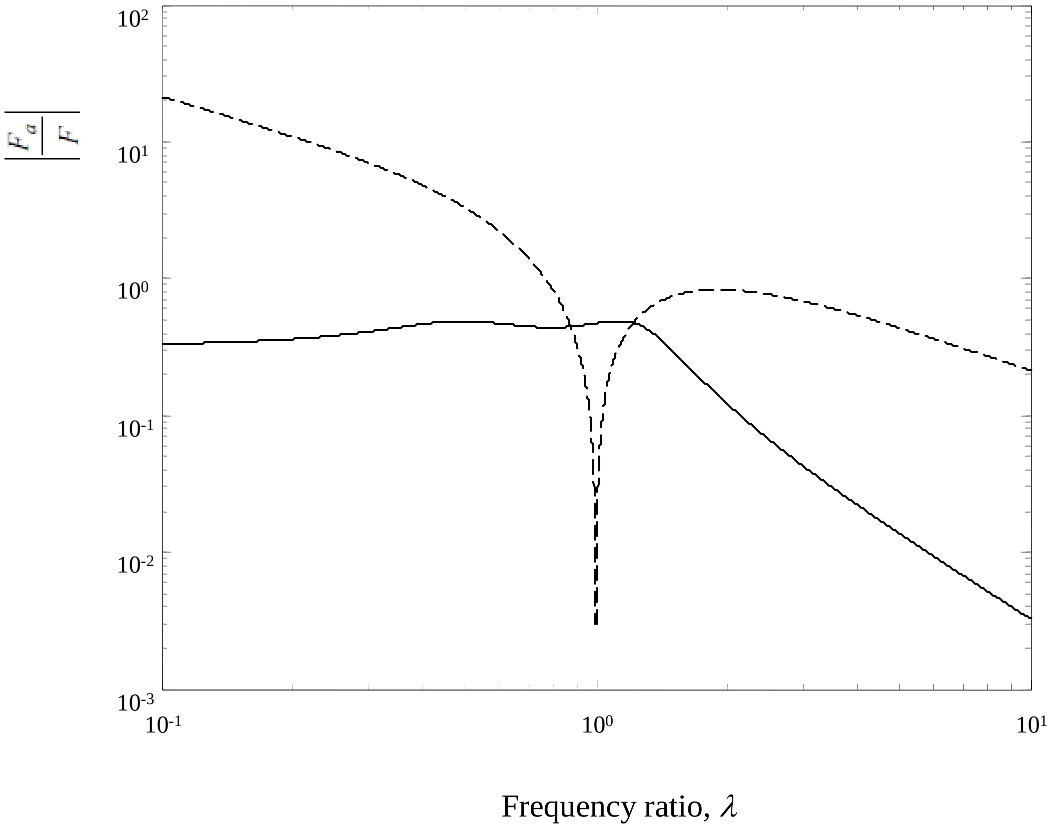

Fig. B2 Active force spectra of the HVA in Fig. 1 with $\mu = 0.2$ and $G = 1.5$

——— present theory using Eq. (B6);

– – – – zero-pole assignment method [16].