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Analytic reconstruction of Compton scattering tomography

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Compton scattering can be used to determine the electron densities of tissues for medical applications and those of materials for industrial applications. Much work has been devoted in solving the reconstruction problem. Norton proposed an analytic transform method for the reconstruction of Compton scattering tomography [J. Appl. Phys. **76**, 2007 (1994)]. However, it is difficult to relate the response function presented by Norton to the measurement quantity. The aim of this article is to present an improved form of the detector response function which corresponds to the actual measurement and to verify the validation of the transform method for this problem. © *1999 American Institute of Physics.* [S0021-8979(99)02215-X]

I. INTRODUCTION

The effect of Compton scattering is a kind of interactions between the electrons and high energy (over 100 keV to compete against the photoelectric effect) photons. The first introduction of this effect to the electron density measurement and imaging was done by Lale in 1959.[1] From then on, Compton scattering has been used in radiotherapy treatment planning,[2] bone density measurement,[3] lung function estimation,[4] and heart motion detection.[5] Besides the medical applications, Compton scattering has also been used in the nondestructive testing (NDT) for industrial applications.[6]

Comparing the Compton scattering imaging technique with the conventional transmission tomography, the scattering technique has some attractive superiorities. First, we can have more freedom in selecting the relative positions of a radiation source and detectors and can arrange the source and detectors on the same side of the examined sample, which is very important for a superficial testing or for objects embedded under ground. Second, this technique has greater sensitivity in density variations for low density materials[7] and for superficial measurement.[8] The third merit is that we can obtain direct three-dimensional (3D) density images by using this technique.[9] Furthermore, this technique can have larger relative contrast and is more dose efficient for thin objects.[7]

Different methods have been developed corresponding to different experimental configurations for the Compton scattering imaging technique.[10] A representative one of those methods is to reconstruct the electron density image from the scattering energy spectral data, which was first proposed by Farmer and Collins.[11] A crucial requirement of this method is the availability of the detector with both good position and energy sensitivity. Utilizing the energy spectral data, Hussein *et al.* proposed to reconstruct the density image by iteration[12] while Norton proposed an analytic formula for the reconstruction when attenuation is neglected.[13] If the attenuation is considered, an iteration procedure was proposed by Norton beginning with a homogeneous initial guess. However, it seems difficult to relate the response functions presented by Norton to the actual measurement—the detector count rate in an energy increment at a series of energy levels. Hence, the parameters w_1 and w_2 presented there are difficult to be used for the real reconstruction.

As in the paper of Norton, we have studied the analytic reconstruction of the density image for a quasi-two-dimensional thin slice from the Compton scattering energy spectral data. Although the slice can be regarded as a quasi-two-dimensional one, the response function will be obtained through a three-dimensional analysis due to the fact that the Compton scattering is a three-dimensional effect. Through accurate mathematical derivations, the detector response, which corresponds to the actual measurement, is obtained. The derivations begin with a general discussion in the scattering problem. From the final response function, it was discovered that our response function has a similar form as that of Norton and hence the analytic method proposed by Norton can be directly adopted except that the parameter $w(r,\theta;\rho,\phi)$ should be evaluated using our formula. The main contribution of this article is that we have presented an improved form of the response function and validated the transform method proposed by Norton for the Compton scattering image reconstruction. Detailed derivations and discussions are presented in the following sections.

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FIG. 1. Shematic diagram of general scattering.

II. PHYSICAL MODEL

A. General description of scattering problem

When a beam of particles incident on a target that is composed of another kind of particles with number density ρ (as shown in Fig. 1), the detector count rate can be expressed as[14]

$$dN(\alpha)=F\frac{d\sigma}{d\Omega}(\alpha)\rho dVd\Omega, \tag{1}$$

where

(1) $dN(\alpha)$ is the count rate of the detector for the scattered particles at the scattering angle α.
(2) F is the flux of the incident beam.
(3) $d\sigma/d\Omega\,(\alpha)$ is the differential scattering cross section at scattering angle α.
(4) $d\Omega$ is the solid angle subtended by the face of the detector to the scattering point.
(5) dV is volume of the target.

From the above equation, we can see that the differential scattering cross section can be regarded as the surface area perpendicular to the incident beam that each target particle presents to the beam particle per-unit solid angle at scattering angle α.[14]

B. Free attenuated detector response for Compton back scattering

Several physical processes can take place between photons and electrons. In the γ-ray energy regime from 0.1 to 1.0 MeV and for materials with low to intermediate atomic numbers, Compton scattering will dominate all the interactions between photons and electrons. Following the method for describing the general scattering process depicted above, the detector response when the attenuation of both the incident and scattered beams is neglected will be presented in this section.

Suppose a beam of γ-ray photons with energy E_0 incident on a target as shown in Fig. 2. Neglecting the attenuation of the incident and scattered beams and considering the single scattering approximation, we can express the photon count rate $dN(r,\theta,x_d)$ of an ideal point detector resulting from the point (r,θ) in the polar system as

$$dN(r,\theta,x_d)=F(\theta)\frac{d\sigma}{d\Omega}[E_0,\alpha(r,\theta,x_d)]\rho_e(r,\theta)dV\Delta\Omega, \tag{2}$$

where

(1) $F(\theta)$ is the photon flux at angle θ.
(2) $\rho_e(r,\theta)$ is the electron number density at the point (r,θ).
(3) dV is a small volume element.
(4) $\alpha(r,\theta,x_d)$ is the scattering angle as shown in Fig. 2.
(5) $d\sigma/d\Omega\,(E_0,\alpha)$ is the Klein–Nishina differential scattering cross section for photons with energy E_0 at scattering angle α. This differential scattering cross section can be given by

$$\frac{d\sigma}{d\Omega}(E_0,\alpha)=\frac{r_e^2}{2}\left[\frac{1}{[1+k(1-\cos\alpha)]^2}\times\left(1+\cos^2\alpha+\frac{k^2(1-\cos\alpha)^2}{1+k(1-\cos\alpha)}\right)\right], \tag{3}$$

where $k=E_0/m_0c^2$, m_0 is the rest mass of the electron, c is the velocity of light in vacuum.
(6) $\Delta\Omega$ is the solid angle subtended by the detector face to the scattering point and can be expressed as

$$\Delta\Omega=\frac{\Delta A\sin\beta}{r_d^2}, \tag{4}$$

where ΔA is the area of one detector element at x_d, r_d is the distance between the detector and the scattering point, β is the angle between the x axis and r_d.

From the geometry in Fig. 2, the volume element dV can be expressed as

$$dV=rd\theta dr\Delta z, \tag{5}$$

where Δz is the thickness of the thin slice in z direction which is determined from the width of the incident beam in this direction (z axis is perpendicular to the paper plane). On the other hand, if the source is confined to the plane of the thin slice and it can be described with a function $S(\theta)$ [$S(\theta)$ stands for the emission rate distribution of the radiation source], we can have the following form for the flux at angle θ

$$F(\theta)=\frac{S(\theta)d\theta}{rd\theta\Delta z}=\frac{S(\theta)}{r\Delta z}. \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (2), the detector count rate then can be written as

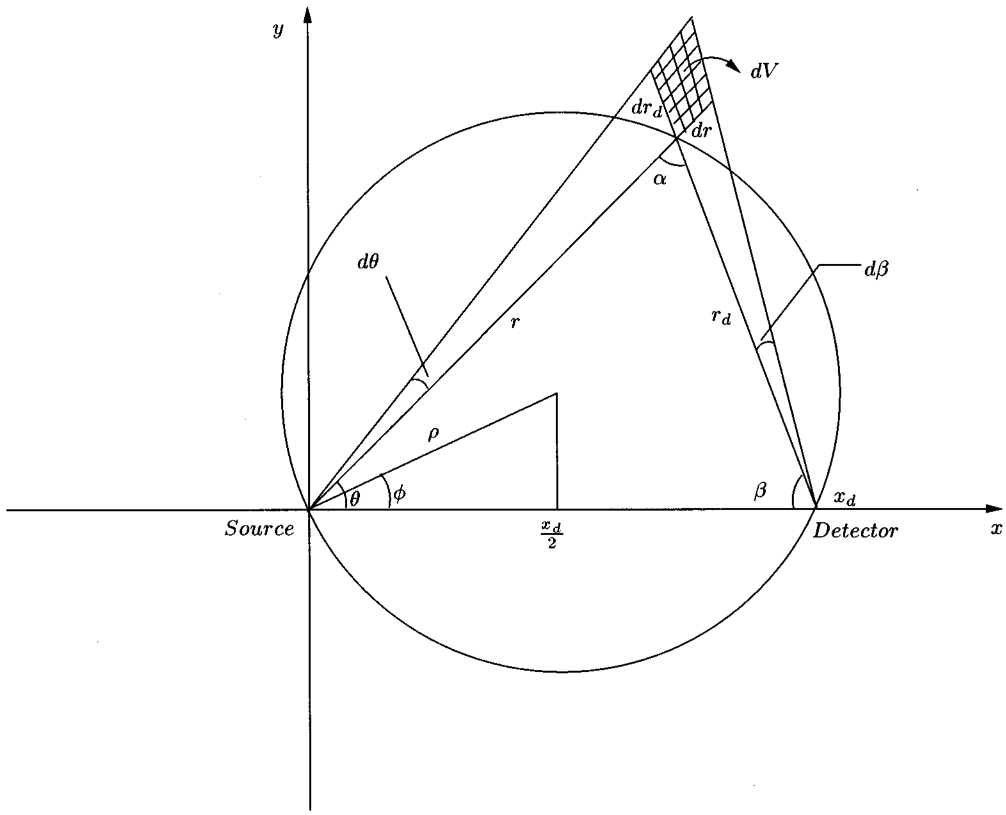

FIG. 2. Shematic diagram of Compton scattering.

$$dN(r,\theta,x_d) = S(\theta)d\theta\frac{d\sigma}{d\Omega}[E_0,\alpha(r,\theta,x_d)]\rho_e(r,\theta)dr\Delta\Omega. \tag{7}$$

If we choose (α,θ,x_d) as variables, Eq. (7) becomes

$$dN(\alpha,\theta,x_d) = S(\theta)d\theta\frac{d\sigma}{d\Omega}(E_0,\alpha)d\alpha\rho_e[r(\alpha,\theta,x_d),\theta]\Delta\Omega|J|, \tag{8}$$

where J is the *Jacobian* determinant which can be expressed as

$$J = \frac{\partial(r,\theta,x_d)}{\partial(\alpha,\theta,x_d)} = \begin{vmatrix} \frac{\partial r}{\partial\alpha} & \frac{\partial r}{\partial\theta} & \frac{\partial r}{\partial x_d} \\ \frac{\partial\theta}{\partial\alpha} & \frac{\partial\theta}{\partial\theta} & \frac{\partial\theta}{\partial x_d} \\ \frac{\partial x_d}{\partial\alpha} & \frac{\partial x_d}{\partial\theta} & \frac{\partial x_d}{\partial x_d} \end{vmatrix}. \tag{9}$$

Considering the geometry in Fig. 2, there exists the following relation,

$$r = x_d\frac{\sin(\theta+\alpha)}{\sin\alpha}. \tag{10}$$

Using the above relation, the *Jacobian* determinant can be given as

$$J = -x_d\frac{\sin\theta}{\sin^2\alpha}. \tag{11}$$

In order to obtain the photon count rate in an energy increment ΔE for different scattered energies E (that is the actual measurement), we should use (E,θ,x_d) instead of (α,θ,x_d). After the variable transformation, we have the following expression for $dN(E,\theta,x_d)$,

$$dN(E,\theta,x_d) = S(\theta)d\theta\frac{d\sigma}{d\Omega}[E_0,\alpha(E,\theta,x_d)]dE\rho_e \times[r(E,\theta,x_d),\theta]\Delta\Omega|J||J'|, \tag{12}$$

where J' is also a *Jacobian* determinant which has a similar form as in Eq. (9). For the evaluation of the above determinant, the Compton equation is adopted. The Compton equation, which relates the energy E of the scattered photons and the scattered angle, has the following form:

$$E = \frac{E_0}{1+k(1-\cos\alpha)}. \tag{13}$$

From the Compton equation, the *Jacobian* determinant J' can be given by

$$J' = -\frac{[1+k(1-\cos\alpha)]^2}{E_0 k\sin\alpha}. \tag{14}$$

Combining Eqs. (11), (12), and (14), we can obtain the final form for $dN(E,\theta,x_d)$

$$dN(E,\theta,x_d) = S(\theta)d\theta \frac{d\sigma}{d\Omega}[E_0,\alpha(E,\theta,x_d)]dE\rho_e[r(E,\theta,x_d),\theta]\Delta\Omega \times x_d \frac{\sin\theta[1+k(1-\cos\alpha)]^2}{E_0 k\sin^3\alpha}. \quad (15)$$

The photon count rate $I(E,x_d)$ of the detector at energy E with an energy increment ΔE can then be obtained by integrating $dN(E,\theta,x_d)$ with respect to θ,

$$I(E,x_d) = \int_0^\pi dN(E,\theta,x_d)|_E = \int_0^\pi d\theta S(\theta) \frac{d\sigma}{d\Omega}[E_0,\alpha(E,\theta,x_d)]\Delta E\rho_e \times [r(E,\theta,x_d),\theta]\Delta\Omega \times x_d \frac{\sin\theta[1+k(1-\cos\alpha)]^2}{E_0 k\sin^3\alpha}. \quad (16)$$

On the other hand, the photons scattered from the locations on each circle passing through both the source and the detector will have the same scattering angle and hence have the same scattering energy. Such a scattering circle can be given in polar form by

$$r = 2\rho\cos(\theta-\phi), \quad (17)$$

where ρ is the radius of the circle

$$\rho = \frac{x_d}{2\cos\phi} \quad (18)$$

and ϕ is the angle between the x axis and the line from the source (origin) to the center of the circle. It can be shown that $\phi=\pi/2-\alpha$.[13] Using the above relations, we can transform the one-dimensional integration into a two-dimensional one by introducing a Dirac δ function in the integral function,

$$I(E,x_d) = \int_0^\pi d\theta \int_0^\infty dr S(\theta)\frac{d\sigma}{d\Omega} \times [E_0,\alpha(E,\theta,x_d)]\Delta E\rho_e(r,\theta)\Delta\Omega \times x_d \frac{\sin\theta[1+k(1-\cos\alpha)]^2}{E_0 k\sin^3\alpha} \times \delta[r-2\rho\cos(\theta-\phi)]. \quad (19)$$

Substituting Eq. (4) into the above equation and noticing the fact that $r_d = x_d\ (\sin\theta/\sin\alpha)$ and $\sin\beta = r(\sin\alpha/x_d)$, we have

$$I(E,x_d) = \int_0^\pi d\theta \int_0^\infty dr S(\theta) \frac{d\sigma}{d\Omega}[E_0,\alpha(E,\theta,x_d)]\Delta E\rho_e(r,\theta) \times \frac{\Delta A r[1+k(1-\cos\alpha)]^2}{x_d^2\sin\theta E_0 k}\delta[r-2\rho\cos(\theta-\phi)]. \quad (20)$$

The variable pairs (x_d,E) and (ρ,ϕ) are inter-related through Eqs. (13) and (18). Thereafter, we can also use (ρ,ϕ) instead of (x_d,E) and have the following expression:

$$I(\rho,\phi) = \int_0^\pi d\theta \int_0^\infty dr w(r,\theta;\rho,\phi)\rho_e(r,\theta) \times \delta[r-2\rho\cos(\theta-\phi)], \quad (21)$$

where

$$w(r,\theta;\rho,\phi) = \frac{d\sigma}{d\Omega}(E_0,\phi)\frac{\Delta A S(\theta) r[1+k(1-\sin\phi)]^2\Delta E}{E_0 k\sin\theta(2\rho\cos\phi)^2}. \quad (22)$$

The integral nucleus $w(r,\theta;\rho,\phi)$ can be decomposed into two functions, one of (r,θ) and the other of (ρ,ϕ). That is, $w(r,\theta;\rho,\phi)=w_1(r,\theta)w_2(\rho,\phi)$, where

$$w_1(r,\theta) = \frac{\Delta A S(\theta) r}{\sin\theta}, \qquad w_2(\rho,\phi) = \frac{d\sigma}{d\Omega}(E_0,\phi)\frac{[1+k(1-\sin\phi)]^2}{E_0 k(2\rho\cos\phi)^2}\Delta E. \quad (23)$$

Hence, we can express the detector count rate function as

$$I(\rho,\phi) = w_2(\rho,\phi)\int_0^\pi d\theta \int_0^\infty dr w_1(r,\theta)\rho_e(r,\theta) \times \delta[r-2\rho\cos(\theta-\phi)]. \quad (24)$$

We have finally obtained the detector response $I(\rho,\phi)$ of the Compton scattering. This response stands for the photon count rate of an ideal point detector positioned at $x_d(\rho,\phi)$ and measured at energy $E(\phi)$ within a energy increment ΔE, where ΔE is the energy discrimination of the point detector. Different from that quantity obtained by Norton in Ref. 13 to which it is difficult to relate any physical meaning, our response function corresponds to the actual measurement of the detector.

C. Attenuated detector response of Compton back scattering

The detector response function obtained above is applicable only when the attenuation of both the incident and the scattered beams is neglected. However, in the practical situation, the attenuation is an important process and the measurements are seriously affected by such attenuation. In general, the attenuation factor can be described as

$$f(E_0,E) = \exp\left[-\sigma(E_0)\int_{L_S}\rho_e(r,\theta)dl_S - \sigma(E) \times \int_{L_D}\rho_e(r,\theta)dl_D\right], \quad (25)$$

where $\sigma(E_0)$ and $\sigma(E)$ are the total scattering cross sections at energy E_0 and E, respectively. These total scattering cross sections can be obtained from the integrations of the Klein–Nishina formula with respect to the solid angle at those energies.

In the case when the attenuation of both the incident and the scattered photons is accounted, the photon count rate of the ideal point detector can be obtained from Eq. (21) except that $w(r,\theta;\rho,\phi)$ should be multiplied by $f(E_0,E)$. However, owing to the introduction of the attenuation factor, there are two new problems with $w(r,\theta;\rho,\phi)$. First, it can no longer be decomposed into two functions of (r,θ) and (ρ,ϕ). Second, w is now a function of $\rho_e(r,\theta)$ and thus the detector response depends nonlinearly on the electron density. It has been shown by Norton that these problems prevent us from using an accurate transform method for the reconstruction.

III. TRANSFORMATION RECONSTRUCTION OF COMPTON SCATTERING TOMOGRAPHY

In the above section, we have studied the detector response resulting from Compton scattering, which is called the forward problem. The inverse problem is to reconstruct the electron density image from the measured detector response. Norton has proposed an analytic transform method for such a reconstruction. However, his method is based on the response function to which we cannot relate a physical meaning. In this section, we will study the analytic transform method based on our response function. It has been pointed out that the crucial point for validating the analytic reconstruction is that the integral nucleus $w(r,\theta;\rho,\phi)$ can be decomposed into two functions, one of (r,θ) and the other of (ρ,ϕ). It has been shown in the previous section that our response function can be decomposed when the attenuation for both the incident and scattered beams is neglected. However, we cannot decompose it if the attenuation is considered. This circumstance is similar to that of Norton. Therefore, the transform reconstruction method developed in Ref. 13 can still be used for Compton scattering image reconstruction. The detailed discussion of this transform method can be found in the work of Norton. We brief some results here.

If the attenuation is neglected, the integral nucleus $w(r,\theta;\rho,\phi)$ can be decomposed into functions, one of (r,θ) and the other of (ρ,ϕ). In this case the electron density image can be analytically reconstructed from the detector response through the following formula

$$\rho_e(r,\theta)=\frac{1}{\pi^2}\int_0^{2\pi}d\phi\int_0^{\infty}\rho d\rho\,\frac{I(\rho,\phi)}{w(r,\theta;\rho,\phi)}\times h[r-2\rho\cos(\theta-\phi)], \tag{26}$$

where

$$h(x)=\int_{-\infty}^{\infty}e^{-i\xi x}|\xi|d\xi. \tag{27}$$

In the case when the attenuation cannot be neglected, the integral nucleus $w(r,\theta;\rho,\phi)$ can no longer be decomposed and the reconstruction of Eq. (26) cannot be obtained. If uniform attenuation is considered, the detector response is still a linear function of the electron density. An approximate reconstruction formula can then be developed

$$\rho_e(r,\theta)=\frac{1}{\pi^2}\int_0^{2\pi}d\phi\int_0^{\rho_{\max}}\rho d\rho\,\frac{I(\rho,\phi)}{W(\theta;\rho,\phi)}\times h[r-2\rho\cos(\theta-\phi)], \tag{28}$$

where

$$W(\theta;\rho,\phi)=w[2\rho\cos(\theta-\phi),\theta;\rho,\phi]. \tag{29}$$

In the above integration, the infinitive upper limit has been changed to $\rho_{\max}$. This means that we impose an upper limit on the radii of the scattering circles over which data are collected. If nonuniform attenuation is considered, the detector response is a nonlinear function of the electron density. It was proposed by Norton that the reconstruction can be accomplished by iteration. The whole process starts with a constant density distribution and the image is reconstructed using Eq. (28) in which $w(r,\theta;\rho,\phi)$ is evaluated from Eqs. (22) and (25). The reconstructed values of $\rho_e(r,\theta)$ are used to determine $w(r,\theta;\rho,\phi)$ for the next iteration.

IV. DISCUSSION

We have presented a detector response function for the Compton backscattering in this article. The validation of the transform method for Compton scattering image reconstruction has been done based on the response function we obtained. Different from Norton's procedure, we start with an initial analysis of the process of Compton scattering and obtain the detector response function by accurate mathematical derivations. The target studied here is a thin slice and the variations in the direction perpendicular to this slice are neglected, hence the problem can be regarded as a quasi-two-dimensional problem. However, since the Compton scattering takes place in three dimensions, the process can only be understood in three dimensions. Our response function directly corresponds to the actual measurement—the photon count rate in an energy gap determined by the detector discrimination at different energy levels. From the response function presented in Eq. (21), we can see that the quantity $I(\rho,\phi)$ has a dimension of $[T]^{-1}$ which is in accordance with the definition of the count rate (the count per unit time).

There are two problems in the response function presented by Norton. First, the response function $I(\rho,\phi)$ obtained there has a dimension of $[L]^{-1}[T]^{-1}$ which is not in accordance with the definition. Second, it is difficult to assign any physical meaning to $I(\rho,\phi)$ and it does not correspond to the actual measurement.

In conclusion, the transform method proposed by Norton[13] is suitable for the image reconstruction from the Compton scattering energy spectral data, although it was based on a problematic detector response function. We solve the problems of Norton's response function based on the physical and mathematical analysis and derivations. The reconstruction formulas presented by Norton can be directly used here except that the integral nucleus w should be evaluated with the formulas derived by us.

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