A Note on Acyclic Domination Number in Graphs of Diameter Two

T.C. Edwin Cheng[a], Yaojun Chen[a,b] and C.T. Ng[a]
[a]Department of Logistics, The Hong Kong Polytechnic University,
Hung Kom, Kowloon, Hong Kong, P.R. CHINA
[b]Department of Mathematics, Nanjing University, Nanjing 210093, P.R. CHINA

Abstract: A subset S of the vertex set of a graph G is called *acyclic* if the subgraph it induces in G contains no cycles. S is called an *acyclic dominating* set of G if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set, denoted by $\gamma_a(G)$, is called the *acyclic domination number* of G. Hedetniemi et al. in [5] introduced the concept of acyclic domination and posed the following open problem: If $\delta(G)$ is the minimum degree of G, is $\gamma_a(G) \leq \delta(G)$ for any graph whose diameter is two? In this paper, we provide a negative answer to this question by showing that for any positive k, there is a graph G with diameter two such that $\gamma_a(G) - \delta(G) \geq k$.

Key words: acyclic domination number, diameter two

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph without loops. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. If $G[S]$ contains no edge, then we call S an independent set. The *distance* of two distinct vertices u and v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v. The *diameter* of G, denoted by $diam(G)$, is defined as:

$$diam(G) = max\{d(u, v) \mid u, v \in V(G)\}.$$

A set $S \subseteq V(G)$ is called a dominating set if every vertex u in $V(G) - S$ is adjacent to at least one vertex v in S. For $X, Y \subseteq V(G)$, we say X dominates Y (or Y is dominated by X) if $N(y) \cap X \neq \emptyset$ for any vertex $y \in Y$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A set $S \subseteq V(G)$ is called an acyclic set if $G[S]$ contains no cycles. A set $S \subseteq V(G)$ is called an *acyclic dominating* set of G if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set in a graph G is called the *acyclic domination number* of G, denoted by $\gamma_a(G)$.

In [4], one can find an appendix listing 75 different types of domination-related parameters that have been studied in the literature (see for instance [1, 2]). The concept of acyclic domination was introduced by Hedetniemi et al. in [5]. This invariant is particularly interesting in that it is a fundamental type of domination and lies between $\gamma(G)$ and $i(G)$, the minimum cardinality of an independent dominating set. In the same paper, they posed some open questions on acyclic domination including the following.

Question 1. Let G be a graph with $diam(G) = 2$. Is $\gamma_a(G) \leq \delta(G)$?

It is shown in [3] that $\gamma_a(G) \leq \delta(G)$ does not hold when $\delta(G) = 3$. In this paper, we show that for any positive integers k and $d \geq 3$, there is a graph G of diameter two with $\delta(G) = d$ such that $\gamma_a(G) - \delta(G) \geq k$.

2. Construction

Let $m \geq 2$ and $n \geq 2$ be integers and $H(m, n)$ be a graph of order mn with vertex set and edge set as follows:

- $V(H(m, n)) = \cup_{1 \leq i \leq m}\{a_{ij} \mid 1 \leq j \leq n\}$;
- $E(H(m, n)) = (\cup_{1 \leq i \leq m}\{a_{ij}a_{ik} \mid j \neq k\}) \cup (\cup_{1 \leq i \leq n}\{a_{ji}a_{ki} \mid j \neq k\})$.

Let F be a complete graph of order $d + 1$ ($d \geq 3$) with $V(F) = \{v_i \mid 0 \leq i \leq d\}$. Take $n = dt$, where t is an integer not less than 2. Let $G(d, n)$ be a graph of order $n^2 + d + 1$ with vertex set and edge set as follows:

- $V(G(d, n)) = V(F) \cup V(H(n, n))$;
- $E(G(d, n)) = E(H(n, n)) \cup E(F) \cup (\cup_{1 \leq k \leq d}\{v_k a_{ij} \mid 1 \leq i \leq n, (k-1)t + 1 \leq j \leq kt\})$.

From the definition of $H(m, n)$, it is easy to see that $H(m, n)$ is the Cartesian product of two complete graphs K_m and K_n, that is, $H(m, n) = K_m \Box K_n$. Thus we can easily obtain the following two lemmas.

Lemma 1. $\gamma(H(m, n)) = \gamma_a(H(m, n)) = \min\{m, n\}$.

Lemma 2. $diam(G(d, n)) = 2$.

Lemma 3. $\gamma_a(G(d, n)) = (d - 2)n/d + 2$.

Proof. Let S be an acyclic dominating set of $G(d, n)$ and $|S \cap N(v_0)| = l$. Obviously, $0 \leq l \leq 2$. If $l = 0$, then in order to dominate $\{v_0\} \cup V(H(n, n))$, we have $|S| \geq n + 1$ by

Lemma 1. If $l \neq 0$, we assume $S \cap N(v_0) = \{v_1, \ldots, v_l\}$ and $G' = G(d,n) - \cup_{1 \leq j \leq l} N[v_j]$. It is easy to see that $G' = H(n,(d-l)t)$. In order to dominate $V(G')$, S must contain at least $(d-l)t$ vertices of $V(H(n,n))$. Thus, we have $|S| \geq (d-l)t + l$. On the other hand, $\{v_1, \ldots, v_l\} \cup \{a_{ii} \mid lt+1 \leq i \leq n\}$ is an acyclic dominating set of order $(d-l)t+l$, and hence we have $\gamma_a(G(d,n)) = \min\{n+1, (d-1)t+1, (d-2)t+2\} = (d-2)t+2 = (d-2)n/d+2$. ∎

Theorem 1. For any positive integers k and $d \geq 3$, there is a graph G of diameter two with $\delta(G) = d$ such that $\gamma_a(G) - \delta(G) \geq k$.

Proof. Take $G = G(d,n)$. By Lemmas 2 and 3, we have $diam(G) = 2$ and $\gamma_a(G) = (d-2)n/d + 2$. Obviously, $\delta(G) = d$. Thus, $\gamma_a(G) - \delta(G) = (d-2)n/d + 2 - d$. Since $(d-2)n/d + 2 - d \to \infty$ as $n \to \infty$ for a fixed d, it is not difficult to see that the conclusion holds. ∎

As for the domination number of $G(d,n)$, we have the following result.

Theorem 2. $\gamma(G(d,n)) = d$.

Proof. Let S be a minimum dominating set of $G(d,n)$. Since $N(v_0)$ is a dominating set, we have $|S| \leq d$. We now show that $|S| = d$. Suppose to the contrary that $|S| < d$. If $S \cap N(v_0) = \emptyset$, then in order to dominate $\{v_0\} \cup V(H(n,n))$, we have $|S| \geq n+1 \geq 2d+1$ by Lemma 1, a contradiction. Hence we may assume without loss of generality that $S \cap N(v_0) = \{v_{l+1}, v_{l+2}, \ldots, v_d\}$. If $l \geq 1$, then we must have $S - N(v_0) \neq \emptyset$ and $S - N(v_0) \subseteq V(H(n,n))$. Let $G' = G(d,n) - \cup_{l+1 \leq j \leq d} N[v_j]$, then $G' = H(n, lt)$. In order to dominate $V(G')$, S must contain at least lt vertices of $V(H(n,n))$. Thus, we have $|S| \geq (d-l) + lt > d$, a contradiction. Therefore, we have $l = 0$, that is, $S \cap N(v_0) = N(v_0)$. Noting that $N(v_0)$ is a dominating set, we have $\gamma(G(d,n)) = d$. ∎

Corollary 1. $N(v_0)$ is the unique minimum dominating set of $G(d,n)$.

3. Final Remark

Let G be a graph. If $diam(G) = k$ and $diam(G-e) > k$ for any edge $e \in E(G)$, then we call G *k-diameter-critical*. It is easy to see that $G(d,n)$ is not 2-diameter-critical since the graph G_0 obtained from $G(d,n)$ by deleting all the edges v_iv_j $(1 \leq i < j \leq d)$ has diameter two. Since $diam(G_0) = 2$ and G_0 is a subgraph of $G(d,n)$, we have $\gamma(G_0) = d$ by Theorem 2. Obviously, $\gamma_a(G_0) = \gamma(G_0) = \delta(G_0) = d$ if n is large enough. Let $G(l,s,t)$ be a graph as shown in Figure 1, where $l \geq 1$, $s \geq 2$ and $t \geq 3$.

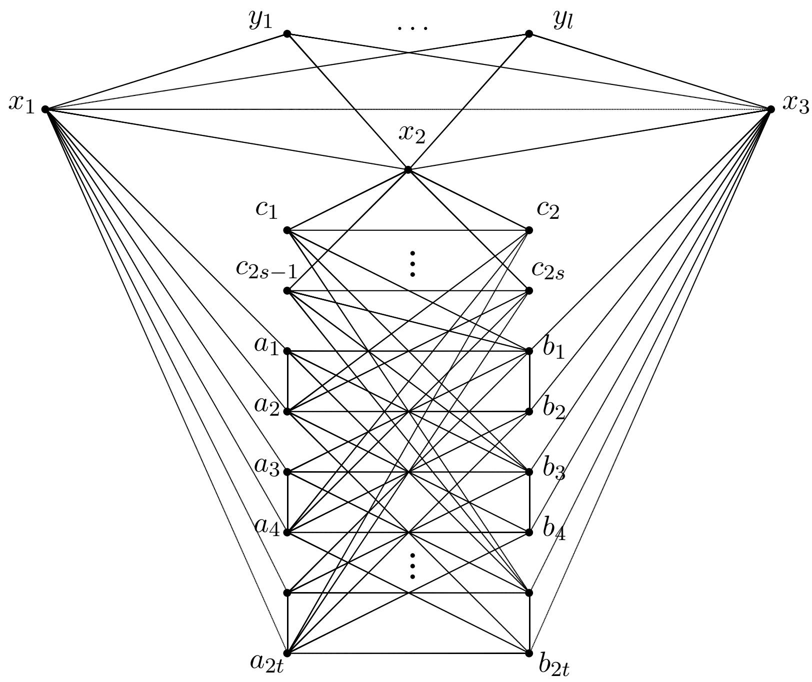

Figure 1

It has been shown in [3] that $\gamma(G(l,s,t)) = 3$ and $\gamma_a(G(l,s,t)) > \delta(G(l,s,t)) = 3$. It is worth noting that $G(l,s,t)$ is not 2-diameter-critical either. In fact, $G(l,s,t) - x_1x_3$ is a 2-diameter-critical graph and $\gamma(G(l,s,t) - x_1x_3) = \delta(G(l,s,t) - x_1x_3) = 3$. A natural problem is the following.

Question 2. Let G be a 2-diameter-critical graph. Is $\gamma_a(G) \leq \delta(G)$?

If the answer to the question above is "YES", then the upper bound for $\gamma_a(G)$ is the best possible in the sense that "$\leq$" cannot be replaced by "$<$" as can be seen by the graphs G_0 and $G(l,s,t) - x_1x_3$.

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