SIAM J. OPTIM.
Vol. 15, No. 1, pp. 275–302

GLOBAL MINIMIZATION OF NORMAL QUARTIC POLYNOMIALS BASED ON GLOBAL DESCENT DIRECTIONS*

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Abstract. A normal quartic polynomial is a quartic polynomial whose fourth degree term coefficient tensor is positive definite. Its minimization problem is one of the simplest cases of nonconvex global optimization, and has engineering applications. We call a direction a global descent direction of a function at a point if there is another point with a lower function value along this direction. For a normal quartic polynomial, we present a criterion to find a global descent direction at a noncritical point, a saddle point, or a local maximizer. We give sufficient conditions to judge whether a local minimizer is global and give a method for finding a global descent direction at a local, but not global, minimizer. We also give a formula at a critical point and a method at a noncritical point to find a one-dimensional global minimizer along a global descent direction. Based upon these, we propose a global descent algorithm for finding a global minimizer of a normal quartic polynomial when $n = 2$. For the case $n \geq 3$, we propose an algorithm for finding an ϵ-global minimizer. At each iteration of a second algorithm, a system of constrained nonlinear equations is solved. Numerical tests show that these two algorithms are promising.

Key words. global optimization, normal quartic polynomial, tensor

AMS subject classifications. 90C30, 65K10

DOI. 10.1137/S1052623403420857

1. Introduction.

The multivariate polynomial optimization problem has attracted some attention recently [6, 12, 14, 16, 17, 18, 25]. It has applications in signal processing [2, 5, 18, 22, 23]; merit functions of polynomial equations [6]; $0-1$ integer, linear, and quadratic programs [12]; nonconvex quadratic programs [12]; and bilinear matrix inequalities [12]. It is related to Hilbert's 17th problem on the representation of nonnegative polynomials [14, 20].

In [18], Qi and Teo raised the concept of *normal polynomial*. They used tensors to denote coefficients of a multivariate polynomial. They called an even degree polynomial a normal polynomial if its leading degree term coefficient tensor is positive definite. They showed that the multivariate polynomials resulting from signal processing [2, 5, 22, 23] are normal quartic polynomials. They gave a bound for the norms of all global minimizers of a normal polynomial. They pointed out that normal quartic optimization is one of the simplest nontrivial cases of nonconvex global optimization.

In [17], Qi further studied the extrema structure of a real polynomial, in particular, a normal quartic polynomial. Let f be a real polynomial. Qi [17] called a

*Received by the editors January 8, 2003; accepted for publication (in revised form) March 9, 2004; published electronically December 9, 2004.
http://www.siam.org/journals/siopt/15-1/42085.html

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polynomial factor of $f - c_0$, where c_0 is a constant, an essential factor of f. He showed that essential factors of f play an important role in defining critical and extremum surfaces of f. He proved that a normal polynomial has no odd degree essential factors, and all of its even degree essential factors are normal polynomials, up to a sign change. He also showed that a normal quartic polynomial can have at most one local maximizer.

The results of [18] and [17] indicate that there should be some better methods for finding a global minimizer of a normal quartic polynomial than in the general case.

Let $f : \Re^n \to \Re$ be a nonconvex function. Let $x, y \in \Re^n$, $y \neq 0$. We call y a *global descent direction* of f at x if there is a $t \in \Re$ such that

$$f(x + ty) < f(x).$$

Clearly, x is a global minimizer of f if and only if it has no global descent directions. It is also obvious that a local descent direction of f at x is a global descent direction of f at x. Hence, if x is not a local minimizer of f, it is easy to find a global descent direction of f at x.

Thus, the next questions are as follows: Given x and y, can we judge if y is or is not a global descent direction of f at x? If it is, can we easily find a one-dimensional global minimizer of f along this global descent direction? In general, how can we find such a global descent direction? We seek answers to these questions for the case when f is a normal quartic polynomial.

The paper is organized as follows. In section 2, we review the current knowledge on tensors and normal quartic polynomials.

In section 3, we present a criterion for finding a global descent direction of a normal quartic polynomial f at a saddle point or a local maximizer and for judging whether a given direction is a global descent direction of f at a local minimizer. At a critical point of f, we give a formula for finding a one-dimensional global minimizer along a global descent direction.

In section 4, we give a method at a noncritical point of f to find a one-dimensional global minimizer along a global descent direction.

In section 5, we present a method for finding a global descent direction of f at a local minimizer when $n = 2$. For the case $n \geq 3$, we propose a constrained nonlinear equation approach to find a global descent direction of f at a local minimizer. The latter is valid for general global optimization. Some sufficient conditions for judging whether a local minimizer is global are given in section 6.

Based upon the above analysis, in section 7 we propose a global descent algorithm for finding a global minimizer of f when $n = 2$. When $n \geq 3$, we form another algorithm based upon the constrained nonlinear equation approach described in section 5. This algorithm will find an ϵ-global minimizer of f.

In section 8, we describe an application of the proposed method in signal processing and report numerical testing results for these two algorithms. For the first algorithm, which can find a global minimizer of f when $n = 2$, we solve the example in [23] in eight iterations. We then solve ten problems randomly generated. The results show that the maximum iteration number is 10 and the minimum is 4. The computer time for each example is no more than one second. For the second algorithm which finds an ϵ-global minimizer of f for $n \geq 3$, we solve ten problems randomly generated for $n = 3$ and five problems randomly generated for $n = 4$. The maximum iteration number is 21 and the minimum is 6. The computer time for each example is approximately one second. We then solve four problems for a class of special normal quartic polynomials with $n = 6$. After 21–25 iterations, we obtain the global minimizers of these four problems. Some final remarks are given in section 9.

2. Tensor analysis and normal quartic polynomial. We use A and B to denote fourth order totally symmetric tensors, and use $A_{i_1 i_2 i_3 i_4}, i_l \in \{1, \ldots, n\}$ for $l = 1, 2, 3, 4$ to denote the elements of a fourth order totally symmetric tensor A, i.e.,

$$A_{i_1 i_2 i_3 i_4} = A_{j_1 j_2 j_3 j_4}$$

if $\{i_1, i_2, i_3, i_4\}$ is any reordering of $\{j_1, j_2, j_3, j_4\}$. Let $x \in \Re^n$ define

$$Ax^4 := \sum_{i_1, i_2, i_3, i_4 = 1}^{n} A_{i_1 i_2 i_3 i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

Similarly, we use M and N to denote third order totally symmetric tensors, P and Q to denote second order totally symmetric tensors, p and q to denote first order totally symmetric tensors, and p_0 and q_0 to denote constants. Thus, the elements of M, P, and p are M_{ijk}, P_{ij}, and p_i, respectively, for $i, j, k = 1, \ldots, n$. We also have

$$Mx^3 = \sum_{i,j,k=1}^{n} M_{ijk} x_i x_j x_k,$$

$$Px^2 = \sum_{i,j=1}^{n} P_{ij} x_i x_j,$$

and

$$px = \sum_{i}^{n} p_i x_i.$$

Then we may denote a quartic polynomial $f : \Re^n \to \Re$ as

$$f(x) = Ax^4 + Mx^3 + Px^2 + px + p_0, \tag{1}$$

where A is a fourth order tensor, M is a third order tensor, P is a second order tensor, and p is a first order tensor, while p_0 is a constant.

Actually, a first order tensor p is equivalent to a vector, and a second order tensor P is equivalent to a square matrix. Thus, we have $px = p^T x$ and $Px^2 = x^T Px$. But we will prefer to use the tensor notation px and Px^2 in this paper.

Let $\| \cdot \|$ be a norm in $\Re^n$. Denote

$$S := \{x \in \Re^n : \|x\| = 1\}.$$

We say that a fourth order totally symmetric tensor A is *positive definite* if

$$Ax^4 > 0$$

for all $x \in S$. This definition extends the definition of positive definite matrices. For a fourth order tensor A, it was defined in [18] that

$$[A] := \min\{Ax^4 : x \in S\}.$$

Clearly, A is positive definite if and only if $[A] > 0$. Similarly, for a second order totally symmetric tensor P,

$$[P] := \min\{Px^2 : x \in S\},$$

which is the smallest eigenvalue of the symmetric matrix P when we use the 2-norm. We may also define positive semidefinite totally symmetric tensors similarly.

We also define the norm of A, M, P, a, etc. as

$$\|A\| := \max\{|Ax^4| : x \in S\}, \text{ etc.}$$

A quartic or quadratic polynomial is called a normal quartic or quadratic polynomial, respectively, if its leading coefficient tensor is positive definite.

Here are some nice properties for a normal quartic polynomial f, expressed by (1) [17, 18].

(a) When $\|x\|$ tends to infinity, the value of f will also tend to infinity.

(b) f always has a global minimizer. If x^* is a global minimizer of f, then

$$\|x^*\| \le L := \max\left\{1, \frac{\|M\| + \|P\| + \|p\|}{[A]}\right\}.$$

For a normal quartic polynomial arising in signal processing, a computational bound for its global minimizers was also given in [18].

(c) If f can be written as

$$f(x) = g(x)h(x) + c_0,$$

where g and h are two nonconstant polynomials and c_0 is a constant, then g and h are normal quadratic polynomials up to a sign change of both g and h. In this case, if at least one of the zero sets of g and h is nonempty, then a global minimizer of f can be found in the interiors of one or two ellipsoids defined by the zero sets of g and h, or at the points defined by the zero sets of g and h. See [17] for details.

(d) f has at most one local maximizer.

According to the Bézout theorem [3, 10], if a quartic polynomial f of n variables has only isolated critical points, then the number of these isolated critical points is less than 3^n. In [4], it was shown that a quartic polynomial of two variables has at most five isolated local extremum points if it has only isolated critical points. See also [21]. But a quartic polynomial or even a normal quartic polynomial may have extremum manifold. So these results have not given a real bound on the number of local extremum points.

Surely, if a function f has a connected extremum manifold C, it has infinitely many local minimizers. Thus, we may not count the number of local extremum points in this case. But we may count the number of extremum levels. It is easy to see that if a function f has a connected critical point manifold C, then f has the same value at this manifold [17]. This motivates us to define the following concept: We call a real number c_0 a *minimum level* (*critical level*) of f if there is a local minimizer (critical point) x of f such that

$$f(x) = c_0.$$

PROPOSITION 1. *The number of minimum (critical) levels of a polynomial is finite.*

Proof. The critical point set of a polynomial is a real algebraic variety [26]. It has at most a finite number of topological components [26]. As stated above, f has the same value on each topological component of this algebraic variety. Hence, the number of critical levels of f is finite. As a minimum level is a critical level, the number of minimum levels of f is also finite. ∎

This proposition has important implications for developing global descent algorithms for finding a global minimizer of a normal quartic polynomial. We say an algorithm is a global descent algorithm if it can always proceed from a local minimizer, when it is not a global minimizer, to another local minimizer with lower function value. By this proposition, such an algorithm will converge in finitely many iterations.

3. Global descent directions. In this section we study global descent directions of f at critical points, where $f : \Re^n \to \Re$ is assumed to be a normal quartic polynomial, i.e., its fourth degree term coefficient tensor is positive definite. We first give a criterion to judge the type of critical points. Then we give a sufficient and necessary condition for a direction y to be a global descent direction of f at a critical point x, and a formula to compute a global minimizer of f along y if y is a global descent direction. We further investigate the way to find a global descent direction if x is a local maximizer or a saddle point of f. If x is a local minimizer, we give a criterion to judge whether it is a global minimizer or not. The issue of finding a global descent direction when x is a local minimizer, but not global, will be treated in section 5.

Let $F : \Re^n \to \Re^n$ be the gradient function of f, i.e.,

$$F = \nabla f.$$

Throughout this section, we assume that x is a critical point of f, i.e.,

$$F(x) = 0.$$

Then we may rewrite f as

$$f(x + ty) = f(x) + t^2 Qy^2 + t^3 Ny^3 + t^4 Ay^4, \tag{2}$$

where $y \in \Re^n, y \neq 0, t \in \Re$, Q is a second order tensor, and N is a third order tensor. It is obvious that

$$Q = \frac{1}{2} f''(x)$$

and

$$N = \frac{1}{6} f'''(x).$$

Hence, it is easy to calculate Q and N.

The next proposition gives a criterion to judge whether a given critical point x of f is a local minimizer, a local maximizer, or a saddle point.

Proposition 2. *Let $f : \Re^n \to \Re$ be a normal quartic polynomial. Assume that x is a critical point of f. Then the following statements hold.*

(i) *If Q is negative definite, then x is the unique local maximizer of f.*

(ii) *If Q is positive definite, or if Q is positive semidefinite and*

$$Ny^3 = 0$$

for all $y \in \{y \in \Re^n : Qy^2 = 0\}$, then x is a local minimizer of f.

(iii) *Otherwise, x is a saddle point of f.*

Proof. If Q is positive definite, negative definite, or has both positive and negative eigenvalues, the conclusions are clear, as the sign of $f(x+ty)-f(x)$ is dominated by t^2Qy^2 when y is fixed and t is small.

If Q is positive semidefinite and for all

$$y \in \{y \in \Re^n : Qy^2 = 0\},$$

$$Ny^3 = 0,$$

then when t is sufficiently small, for

$$y \notin \{y \in \Re^n : Qy^2 = 0\},$$

the sign of $f(x+ty)-f(x)$ is dominated by t^2Qy^2, which is always positive, while for

$$y \in \{y \in \Re^n : Qy^2 = 0\},$$

we have

$$f(x+ty) - f(x) = t^4Ay^4 \geq 0,$$

as A is positive definite. Hence, x is a local minimizer in this case.

If Q is positive or negative semidefinite and for some fixed

$$y \in \{y \in \Re^n : Qy^2 = 0\},$$

$$Ny^3 \neq 0,$$

then when t is sufficiently small, the sign of $f(x+ty)-f(x)$ is dominated by t^3Ny^3, which changes sign when t changes sign. Hence, x is a saddle point in this case.

If Q is negative semidefinite and for all

$$y \in \{y \in \Re^n : Qy^2 = 0\},$$

$$Ny^3 = 0,$$

then for

$$y \notin \{y \in \Re^n : Qy^2 = 0\},$$

we have $f(x+ty)-f(x) < 0$ when t is sufficiently small, while for

$$y \in \{y \in \Re^n : Qy^2 = 0\},$$

we have

$$f(x+ty) - f(x) = t^4Ay^4 \geq 0,$$

as A is positive definite. Hence, x is a saddle point in this case.

This exhausted all the cases. ∎

We still assume that x is a critical point of f. The next issue is to determine whether a given direction $y \in \Re^n$ is a global descent direction of f at x.

Denote

$$a \equiv a(y) = Ay^4,$$

$$b \equiv b(y) = -Ny^3,$$

and

$$c \equiv c(y) = Qy^2.$$

Since A is positive definite, we have $a > 0$. Define $\phi : \Re \to \Re$ by

$$\phi(t) := f(x + ty) - f(x) \equiv at^4 - bt^3 + ct^2.$$

Clearly, y is a global descent direction of f at x if and only if there is a $t \in \Re$ such that $\phi(t) < 0$. If t^* is a global minimizer of ϕ, then $x + t^*y$ is the best candidate for the next iterate of a global descent algorithm for finding a global minimizer of f if we regard x as the current iterate.

Hence, ϕ plays a fundamental role in our discussion of the global descent direction. We call ϕ the *fundamental polynomial* of f at x along the direction y.

We also denote

$$\Delta \equiv \Delta(y) = b^2 - 4ac.$$

The next theorem provides a sufficient and necessary condition for y to be a global descent direction of f at x and provides a formula to compute exactly a global minimizer of the fundamental polynomial.

THEOREM 3 (fundamental polynomial test). *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a critical point of f. Then $y \in \Re^n$ is a global descent direction of f at x if and only if*

$$\Delta > 0.$$

Furthermore, if $\Delta > 0$, a global minimizer of ϕ is

$$t^* = \begin{cases} \frac{3b+\sqrt{9b^2-32ac}}{8a} & \text{if} \quad b \geq 0, \\ \frac{3b-\sqrt{9b^2-32ac}}{8a} & \text{otherwise} \end{cases} \tag{3}$$

and we have

$$\phi(t^*) = -\frac{c^2}{4a} + \frac{9b^2c}{32a^2} - \frac{27b^4}{512a^3} - \frac{\sqrt{b^2(9b^2 - 32ac)^3}}{512a^3} < 0 = \phi(0). \tag{4}$$

Proof. By calculus, if and only if $\Delta \leq 0$, we have $\phi(t) \geq 0 = \phi(0)$, i.e.,

$$f(x + ty) \geq f(x)$$

for all $t \in \Re$. This proves the first conclusion.

If $\Delta > 0$, then ϕ has three critical points:

$$t_0 = 0, \quad t_1 = \frac{3b + \sqrt{9b^2 - 32ac}}{8a}, \quad t_2 = \frac{3b - \sqrt{9b^2 - 32ac}}{8a}.$$

In the case $b \geq 0$, we have $\phi(0) = 0$, $\phi(t_1) = \phi(t^*)$ is given by (4), and

$$\phi(t_2) = -\frac{c^2}{4a} + \frac{9b^2c}{32a^2} - \frac{27b^4}{512a^3} + \frac{\sqrt{b^2(9b^2 - 32ac)^3}}{512a^3}.$$

It is not difficult to see that

$$\phi(0) = 0 > \phi(t_1)$$

and

$$\phi(t_2) \geq \phi(t_1).$$

This shows that $t^* = t_1$ is a global minimizer of ϕ.

In the case $b < 0$, the proof is similar. This completes our proof. ∎

With this theorem, we may analyze global descent directions at a critical point case by case.

The next theorem shows that eigenvectors corresponding to any negative eigenvalue of Q are global descent directions.

THEOREM 4 (negative eigenvalue of Q). *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a critical point of f.*

If $y \in \Re^n$ is an eigenvector of Q corresponding to a negative eigenvalue of Q, then y is a global descent direction of f at x, and $x + t^*y$ is a one-dimensional global minimizer of f from x along y, with the function value

$$f(x + t^*y) = f(x) + \phi(t^*) < f(x),$$

where t^* and $\phi(t^*)$ are given by (3) *and* (4), *respectively.*

Proof. Since $a > 0$ and $c = Qy^2 < 0$, we have $\Delta > 0$. The conclusions of the theorem follow from Theorem 3. ∎

When Q has a zero eigenvalue but no negative eigenvalue, by the next theorem, we may also determine whether an eigenvector corresponding to a zero eigenvalue of Q is a global descent direction or not.

THEOREM 5 (zero eigenvalue of Q). *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a critical point of f.*

If $y \in \Re^n$ is an eigenvector of Q corresponding to a zero eigenvalue of Q, then y is a global descent direction of f at x if and only if $b \neq 0$.

Furthermore, if $b \neq 0$, then $x + t^*y$ is a one-dimensional global minimizer of f from x along y, with the function value

$$f(x + t^*y) = f(x) + \phi(t^*) < f(x),$$

where

$$t^* = \frac{3b}{4a}$$

and

$$\phi(t^*) = -\frac{27b^4}{256a^3}.$$

Proof. We have $c = Qy^2 = 0$. We now have $\Delta > 0$ if and only if $b \neq 0$. Again, the conclusions of the theorem follow from Theorem 3. ∎

Note that the formulas about t^* and $\phi(t^*)$ in the above theorem are special cases of (3) and (4) when $c = 0$.

Theorems 4 and 5 actually address the issue of how to compute a global descent direction when x is a local maximizer or a saddle point of f. When x is a local minimizer but not global, it is not an easy task to compute a global descent direction of f. In section 5 we will deal with this case. In the following we give a criterion for judging whether a local minimizer is global.

THEOREM 6 (global minimizers). *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a local minimizer of f. Then x is a global minimizer of f if and only if for all $y \in \Re^n, y \neq 0$ with $c \neq 0$, we have $\Delta \leq 0$.*

If for all $y \in \Re^n, y \neq 0$ with $c \neq 0$, we have $\Delta < 0$, then x is the unique global minimizer of f.

If for all $y \in \Re^n, y \neq 0$ with $c \neq 0$, we have $\Delta \leq 0$, and for some $y \in \Re^n, y \neq 0$ with $c \neq 0$, we have $\Delta = 0$, then x is a global minimizer of f but there are other global minimizers of f.

Proof. Since x is a local minimizer, we have $c \geq 0$, and $c = 0$ implies $b = 0$ by Proposition 2(ii).

If x is a global minimizer, then there does not exist a global descent direction. Thus by Theorem 3, $\Delta(y) \leq 0$ for all $y \neq 0$. This proves the "only if" part of the first assertion.

For the "if" part of the first assertion, we proceed by contradiction. Assume the contrary, i.e., x is not a global minimizer. Then there exists a global descent direction $y \neq 0$. By Theorem 3 we have $\Delta(y) > 0$. By the "if" condition, we have $c = 0$. This implies $b = 0$ by the argument at the beginning of this proof. Hence $\phi(t) = at^4 > 0$, which contradicts the assumption that y is a global descent direction.

We now prove the two other assertions.

Suppose that $\Delta < 0$ for all $y \in \Re^n, y \neq 0$ with $c \neq 0$. Let $y \neq 0$. Again, by the argument at the beginning of this proof, if $c = 0$, then $b = 0$ and thus

$$\phi(t) = at^4 > 0$$

for all $t \neq 0$. If $c \neq 0$, then since $a > 0$ and $\Delta < 0$, we also have

$$\phi(t) > 0$$

for all $t \neq 0$. Hence x is the unique global minimizer of f.

Similarly, if $\Delta \leq 0$ for all $y \in \Re^n, y \neq 0$ with $c \neq 0$, then x is a global minimizer of f. But if $\Delta = 0$ for some $y \in \Re^n, y \neq 0$ with $c \neq 0$, then $x + t^* y$ will be another global minimizer of f, where $t^* = \frac{b}{2a}$. In this case, x is a global minimizer but not a unique global minimizer of f. ∎

4. One-dimensional normal quartic minimization at a noncritical point. In this section we study the global minimizers of the normal quartic polynomial f at a noncritical point along a global descent direction, which can be reformulated as a one-dimensional global minimization. To address this issue, we propose an approach based on the one-dimensional Newton method.

Assume that x is not a critical point of f, i.e.,

$$q := F(x) \neq 0.$$

Let $y \in \Re^n$ such that

$$qy < 0. \tag{5}$$

Then y is a descent direction, and hence a global descent direction of f at x.

Our aim is to find a one-dimensional global minimizer of f on the line $\{x + ty : t \in \Re\}$, which is equivalent to finding a global minimizer of

$$\phi(t) := f(x + ty) - f(x).$$

We may rewrite

$$\phi(t) = at^4 - bt^3 + ct^2 - dt, \tag{6}$$

where the meanings of a, b, and c are defined in the last section, while $d = -qy$.

We now have $a > 0$ and $d > 0$. Denote

$$\psi(t) := \phi'(t) = 4at^3 - 3bt^2 + 2ct - d$$

and

$$\eta(t) := \psi'(t) = 12at^2 - 6bt + 2c.$$

Let

$$\Delta_1 = 9b^2 - 24ac. \tag{7}$$

The next proposition states a useful property of ψ.

PROPOSITION 7. *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a noncritical point of f. Suppose that y is a descent direction satisfying* (5). *If $\Delta_1 > 0$, then ψ has a local maximizer*

$$\bar{t}_1 = \frac{3b - \sqrt{\Delta_1}}{12a} \tag{8}$$

and a local minimizer

$$\bar{t}_2 = \frac{3b + \sqrt{\Delta_1}}{12a} \tag{9}$$

with $\bar{t}_1 < \bar{t}_2$ and $\psi(\bar{t}_1) > \psi(\bar{t}_2)$.

Proof. If $\Delta_1 > 0$, then $\bar{t}_1$ and $\bar{t}_2$ are two zeros of the quadratic polynomial η, and hence two critical points of the cubic polynomial ψ. We also have

$$\eta'(\bar{t}_1) = -2\sqrt{\Delta_1} < 0$$

and

$$\eta'(\bar{t}_2) = 2\sqrt{\Delta_1} > 0,$$

i.e., $\bar{t}_1$ is a local maximizer of ψ and $\bar{t}_2$ is a local minimizer of ψ. Since $a > 0$, we have

$$\bar{t}_2 - \bar{t}_1 = \frac{\sqrt{\Delta_1}}{6a} > 0.$$

Since ψ is a cubic polynomial, $\bar{t}_1$ is a local maximizer and $\bar{t}_2$ is a local minimizer, we have $\psi(\bar{t}_1) > \psi(\bar{t}_2)$. $\square$

Denote

$$\bar{t} = \frac{b}{4a}, \tag{10}$$

$$\bar{t}_0 = \max\left\{1, \frac{3b_+ + 2(-c)_+ + d}{4a}\right\}, \tag{11}$$

and

$$\hat{t}_0 = \min\left\{-1, -\frac{3(-b)_+ + 2(-c)_+}{4a}\right\}, \tag{12}$$

where for any $\alpha \in \Re$, $\alpha_+ := \max\{\alpha, 0\}$. It is not difficult to deduce $\psi(t) > 0$ for all $t > \bar{t}_0$ and $\psi(t) < 0$ for all $t < \hat{t}_0$.

Since ϕ is a one-dimensional normal quartic polynomial, it has one or two local minimizers. We distinguish these two different cases in the next proposition and give illustrations of the various situations in Figures 1–4.

PROPOSITION 8. *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a noncritical point of f. Suppose that y is a descent direction satisfying* (5).

(i) *If one of the following three cases occurs, then ϕ has a unique local, hence global minimizer t^*:*

(a) $\Delta_1 \leq 0$*;*

(b) $\Delta_1 > 0$ *and* $\psi(\bar{t}_2) \geq 0$*;*

(c) $\Delta_1 > 0$ *and* $\psi(\bar{t}_1) \leq 0$.

Furthermore, in case (a), *ϕ is convex, and*

$$0 < t^* \leq \bar{t}_0.$$

In case (b), *we have*

$$0 < t^* < \bar{t}_1.$$

In case (c), *we have*

$$(\bar{t}_2)_+ < t^* < \bar{t}_0.$$

(ii) *If $\Delta_1 > 0$ and $\psi(\bar{t}_1) > 0 > \psi(\bar{t}_2)$, then ϕ has two local minimizers t_1^* and t_2^* such that*

$$t_1^* < \bar{t}_1 < \bar{t}_2 < t_2^*.$$

Proof. (i) If case (a) occurs, then $\eta(t) \geq 0$ for all t. This implies that ϕ is convex. Since ϕ is a nonconstant convex polynomial, it can have one local minimizer t^*, which is also a global minimizer. Since $\psi(0) = -d < 0$, $\psi(t^*) = 0$, and $\psi(\bar{t}_0) \geq 0$, we have $0 < t^* \leq \bar{t}_0$.

If case (b) occurs, we have

$$\psi(\bar{t}_1) > \psi(\bar{t}_2) \geq 0.$$

Since $\psi(0) < 0$, we have $t^* \in (0, \bar{t}_1)$ such that $\psi(t^*) = 0$. We now have

$$0 < t^* < \bar{t}_1 < \bar{t} < \bar{t}_2.$$

Since $\eta(t) > 0$ for all $t < \bar{t}_1$, t^* is a local minimizer of ϕ.

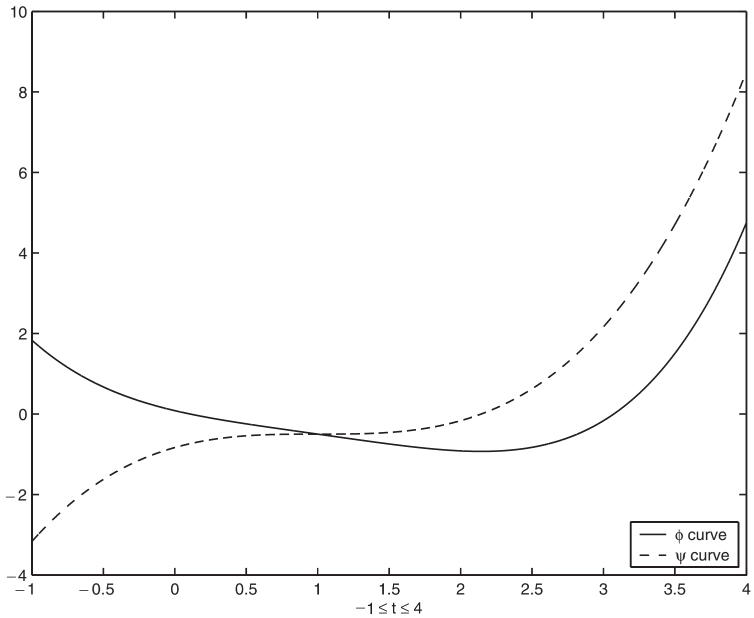

FIG. 1. *Graphs of ϕ and ψ when $\phi(t) = \frac{1}{12}(t-1)^4 - 0.5t$.*

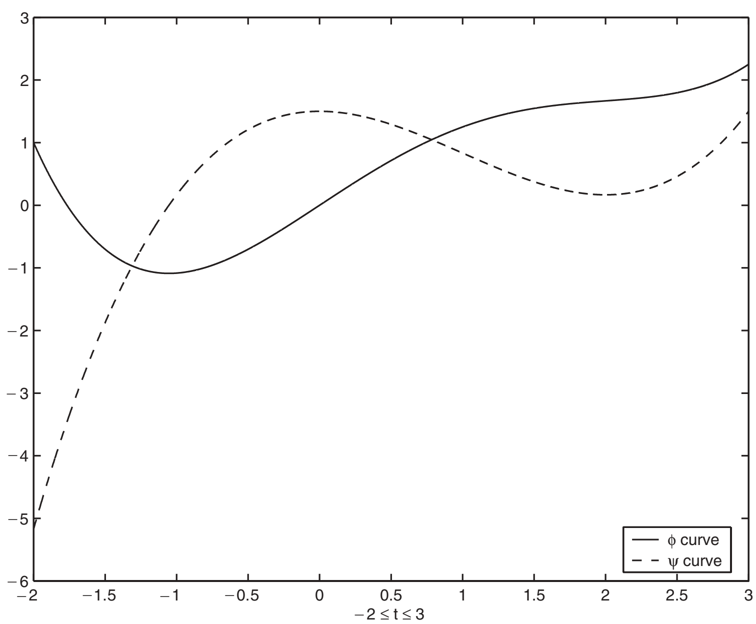

FIG. 2. *Graphs of ϕ and ψ when $\phi(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 1.5t$.*

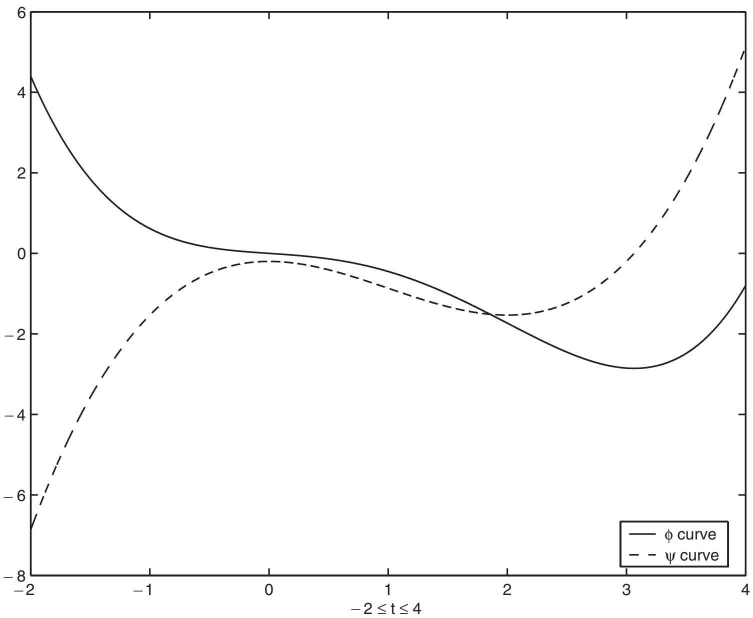

FIG. 3. *Graphs of* ϕ *and* ψ *when* $\phi(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 - 0.2t$.

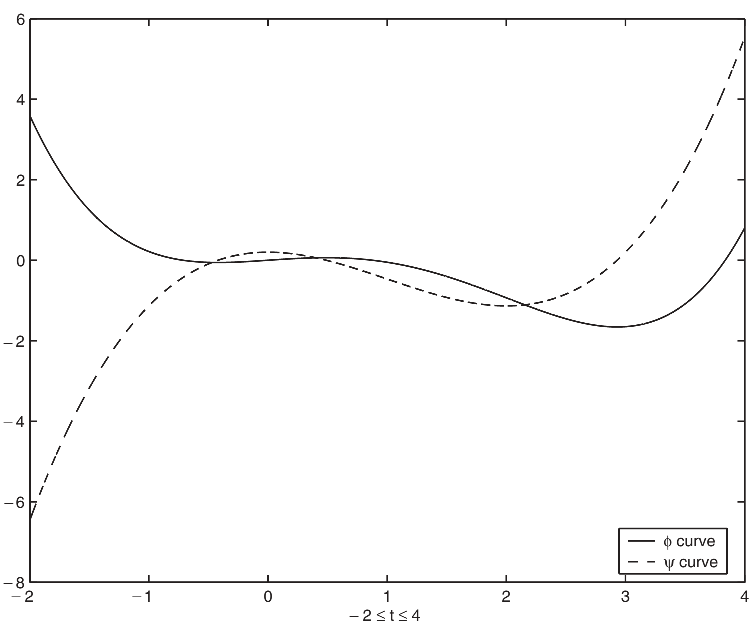

FIG. 4. *Graphs of* ϕ *and* ψ *when* $\phi(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 0.2t$.

If $\psi(\bar{t}_2) > 0$, then t^* is the only zero of ψ. If $\psi(\bar{t}_2) = 0$, then $\bar{t}_2$ is also a zero of ψ. But since $\psi(t) > 0$ when t is close to but not equal to $\bar{t}_2$, in this case $\bar{t}_2$ is a saddle point of ϕ. In both cases, t^* is the unique minimizer of ϕ. Since ϕ is a normal quartic polynomial, it is also the unique global minimizer of ϕ.

If case (c) occurs, we have

$$\bar{t} < (\bar{t}_2)_+ < t^* < \bar{t}_0.$$

The remaining proof is similar to that of the last case.

(ii) Since $a > 0$, $\psi(\bar{t}_1) > 0 > \psi(\bar{t}_2)$, and $\bar{t}_1 < \bar{t}_2$, we have

$$t_1^* < \bar{t}_1 < \bar{t}_2 < t_2^*.$$

Actually, we have

$$\hat{t}_0 \leq t_1^* < \bar{t}_1 < \bar{t} < \bar{t}_2 < t_2^* \leq \bar{t}_0. \qquad \square$$

It follows from the above proposition that if ϕ has only one local minimizer, then it is also the unique global minimizer of ϕ. We now state an algorithm for finding the global minimizer of ϕ in this case.

ALGORITHM 1.

Step 0. *Compute* Δ_1, $\bar{t}$, *and* $\bar{t}_0$ *by* (7), (10), *and* (11), *respectively. Set* $j := 0$.

Step 1. *If* $\psi(\bar{t}_0) = 0$, *then set* $t^* := \bar{t}_0$ *and stop.*

Step 2. *If* $\Delta_1 \leq 0$, *then go to Step* 3; *otherwise go to Step* 5.

Step 3. *If* $\psi(\bar{t}) = 0$, *then set* $t^* := \bar{t}$ *and stop.*

Step 4. *If* $\psi(\bar{t}) > 0$, *then set* $t_0 := 0$*; otherwise, set* $t_0 := \bar{t}_0$. *Go to Step* 6.

Step 5. *If* $\psi(\bar{t}_2) \geq 0$, *then set* $t_0 := 0$*; otherwise set* $t_0 := \bar{t}_0$.

Step 6. *Compute the next iterate point by using the one-dimensional Newton method*

$$t_{j+1} := t_j - \frac{\psi(t_j)}{\eta(t_j)}. \tag{13}$$

Step 7. *Set* $j := j + 1$. *Go to Step* 6.

The following is the convergence statement of the above algorithm.

THEOREM 9 (single minimizer of ϕ). *Let* $f : \Re^n \to \Re$ *be a normal quartic polynomial and* x *be a noncritical point of* f. *Suppose that* y *is a descent direction satisfying* (5) *and that the assumption in Proposition* 8(i) *holds.*

Then Algorithm 1 *is well defined and the generated sequence* $\{t_j\}$ *converges to the unique global minimizer* t^* *of* ϕ *quadratically. Furthermore,* $\{t_j\}$ *strictly increases to* t^* *in the case where we choose* $t_0 = 0$, *and strictly decreases to* t^* *in the case where we choose* $t_0 = \bar{t}_0$.

Proof. We prove only the convergence result when Step 4 in Algorithm 1 is executed. For the other two cases, the proof is similar.

Assume that $\psi(1) \neq 0$ and $\psi(\bar{t}) \neq 0$. Then $\psi(\bar{t}_0) > 0$; otherwise, $\psi(\bar{t}_0) = 0$. It is not difficult to deduce $\bar{t}_0 = 1$, a contradiction. This, together with (i)(a) of the last proposition, implies $0 < t^* < \bar{t}_0$.

If $\psi(\bar{t}) > 0$, then $0 < t^* < \bar{t}$. If $t_j \in [0, t^*)$, then $\psi(t_j) < 0$. We also have $\eta(t_j) > 0$, as $\bar{t}$ is the only possible zero of η (only when $\Delta_1 = 0$). Then

$$t_{j+1} := t_j - \frac{\psi(t_j)}{\eta(t_j)} > t_j.$$

By the mean value theorem, there is an $\xi \in (t_j, t^*)$ such that

$$\eta(\xi)(t^* - t_j) = \psi(t^*) - \psi(t_j) = -\psi(t_j).$$

Since $\eta'(t) < 0$ for all $t < \bar{t}$, as $t_j < \xi$, we have

$$\eta(t_j) > \eta(\xi).$$

Hence,

$$\eta(t_j)(t^* - t_j) > \eta(\xi)(t^* - t_j) = -\psi(t_j).$$

As $\eta(t_j) > 0$, we have

$$t^* - t_j > -\frac{\psi(t_j)}{\eta(t_j)}.$$

This implies that

$$t_j < t_{j+1} < t^*.$$

Hence, if we let $t_0 = 0$, the one-dimensional Newton method (13) is well defined and $\{t_j\}$ strictly increases to a limit. By the property of the Newton method, this limit must be a zero of ψ, which is t^*. As $\eta'(t^*) \neq 0$, the sequence converges quadratically to t^*.

If $\psi(\bar{t}) < 0$, then $\bar{t} < t^* < \bar{t}_0$. By taking $t_0 = \bar{t}_0$, similar to the above proof, we can deduce the corresponding convergence result. ∎

When ϕ has two local minimizers t_1^* and t_2^*, we can use a similar method to find them. The differences lie in that

(i) we also calculate $\hat{t}_0$ by (12);

(ii) if $\psi(\hat{t}_0) = 0$, we have $t_1^* = \hat{t}_0$. If $\psi(\bar{t}_0) = 0$, we have $t_2^* = \bar{t}_0$. Otherwise, we use the one-dimensional Newton method (13) to find t_1^* and t_2^* with different choices of the starting point t_0. Specifically speaking, we may use $t_0 = \bar{t}_0$ to find t_2^*. If $0 < \bar{t}_1$, we may use $t_0 = 0$ to find t_1^*; otherwise, we use $t_0 = \hat{t}_0$ to find t_1^*.

We call such a modified algorithm Algorithm 2. The next theorem gives its convergence statement.

THEOREM 10 (two local minima of ϕ). *Let $f : \Re^n \to \Re$ be a normal quartic polynomial and x be a noncritical point of f. Suppose that y is a descent direction satisfying* (5) *and that the assumption in Proposition* 8(ii) *holds.*

Then Algorithm 2 *is well defined, and the two sequences generated by* (13) *converge to two local minimizers t_1^* and t_2^* of ϕ quadratically, with one sequence strictly increasing to t_1^* and another sequence strictly decreasing to t_2^*.*

The proof of the above theorem is omitted since it is similar to that of Theorem 9.

We may also find one of t_1^* and t_2^*, say t_1^*, first. Then we may find another of them, say t_2^*, by solving the quadratic polynomial

$$\frac{d}{dt}\left(\frac{\phi(t)}{t - t_1^*}\right).$$

After finding t_1^* and t_2^*, comparing the values of $\phi(t_1^*)$ and $\phi(t_2^*)$, we may find a global minimizer t^* of ϕ.

Since minimizing ϕ is only a subproblem of minimizing f, we only need to find an approximate global minimizer t^* of f such that $\phi(t^*) < 0$ and $|\psi(t^*)| \leq \epsilon$ for some given $\epsilon > 0$.

5. Global descent directions at a local minimizer. At a noncritical point of f, we may use the method in section 4 to find a one-dimensional global minimizer of f along this descent direction and its opposite direction. At a saddle point or the unique local maximizer of f, we may identify a global descent direction by Theorems 4 and 5 and use the formula in section 3 to calculate a one-dimensional global minimizer of f along this global descent direction. Actually, if we proceed in this way, we will not meet the unique local maximizer of f, unless we by chance use it as the starting point, since each time, the iterate is a one-dimensional global minimizer along a line. Now, the only difficult points are local minimizers of f. Theorem 3 provides a criterion for checking if a given direction is a global descent direction of f at a local minimizer, but does not provide a constructive way for finding or identifying a global descent direction.

In this section, we discuss this problem. Assume that x is a local minimizer of f. We use the same notation as in section 3. By Theorem 3, we may solve the following problem to obtain a global descent direction y:

$$\max \Delta(y) \quad \text{subject to } y \in S. \tag{14}$$

If we use the infinity norm, then the maximization problem (14) can be converted to $2n$ $(n-1)$-dimensional maximization problems:

$$\max \Delta(y) \quad \text{subject to } y_i = 1, |y_j| \leq 1 \text{ for } j \neq i, \tag{15}$$

and

$$\max \Delta(y) \quad \text{subject to } y_i = -1, |y_j| \leq 1 \text{ for } j \neq i \tag{16}$$

for $i = 1, \ldots, n$.

When $n = 2$, (15) and (16) are four one-dimensional sixth degree polynomial maximization problems. They are practically solvable. In fact, it follows from Theorem 3 that the following result holds.

THEOREM 11. *Assume that x is a local minimizer of f and $n = 2$. If one of the four one-dimensional sixth degree polynomial maximization problems in* (15) *and* (16) *has a feasible point y such that $\Delta(y) > 0$, then y is a global descent direction of f at x. Otherwise, i.e., when all of them have no positive optimal function values, x is a global minimizer of f.*

When $n \geq 3$, it seems that there are no advantages to solving (15) and (16) other than solving the normal quartic minimization problem

$$\min\{Ay^4 + Ny^3 + Qy^2 : y \in \Re^n\}, \tag{17}$$

which is equivalent to the original problem. If (17) has a feasible solution y such that

$$Ay^4 + Ny^3 + Qy^2 < 0,$$

then y is a global descent direction of f at x. Otherwise, x is already a global minimizer of f.

For $n \geq 3$, we propose a constrained nonlinear equation approach for finding a global descent direction at a local minimizer. This approach is also valid for the general global optimization. Hence, in the following proposition only, f is a general nonconvex function.

Let $\epsilon > 0$. We say x^* is an ϵ-global minimizer of a nonconvex function $f : \Re^n \to \Re$ if for all $x \in \Re^n$,

$$f(x) > f(x^*) - \epsilon.$$

The next proposition gives an approach to compute a global descent direction at a local minimizer.

PROPOSITION 12. *x^* is an ϵ-global minimizer of a nonconvex function $f : \Re^n \to \Re$ if and only if the bound constrained system of nonlinear equations*

$$\left\{ \begin{array}{ccc} F(x) & = & 0, \\ f(x) - x_{n+1} & = & 0, \\ x_{n+1} - f(x^*) + \epsilon & \leq & 0 \end{array} \right. \tag{18}$$

has no feasible solution. If (18) *has a feasible solution $\bar{x}$, then $y = \bar{x} - x^*$ is a global descent direction of f at x^*.*

This proposition is easy to prove. In the literature, there are many methods for solving bound constrained systems of nonlinear equations [1, 7, 8, 9, 11, 13, 19, 24].

6. Criteria for identifying a global minimizer. Analogous to local optimality conditions in the ordinary nonlinear programming, it is very important to provide appropriate computable termination rules for the algorithms to compute a global minimizer. We call such a termination rule the *global optimality condition*.

Although Theorem 6 presents a sufficient and necessary condition for a local minimizer to be global, it is usually not checkable in the general case. For $n = 2$, Theorem 11 shows that a local minimizer is global if the problems (15) and (16) have no positive optimal function value; however, Theorem 11 is not usable for the case $n \geq 3$. In this section, we present some checkable criteria to judge whether a local minimizer is global when $n \geq 3$.

Throughout this section, we assume that f is a normal quartic polynomial and x is a local minimizer of f. In the next proposition we give a sufficient condition for x to be a global minimizer of f.

PROPOSITION 13. *Assume that f is a normal quartic polynomial and x is a local minimizer of f. If $\|N\|^2 \leq 4[A][Q]$, then x is a global minimizer of f.*

This follows directly from Theorem 3.

The following corollary is a direct consequence of Propositions 13 and 2.

COROLLARY 14. *Assume that f is a normal quartic polynomial and x is a local minimizer of f. If $N = 0$, then x is a global minimizer of f.*

As mentioned in section 2, $[Q]$ is the smallest eigenvalue of the symmetric matrix Q when we use the 2-norm. We also have

$$\|N\|_2 \leq \|N\|_\infty \leq \sum_{i,j,k=1}^{n} |N_{ijk}|.$$

But this upper bound for $\|N\|_2$ may be too large. We may improve it.

Let $N_{i\cdot\cdot}$ denote the second order totally symmetric tensor with elements N_{ijk}, where i is fixed. By means of $N_{i\cdot\cdot}$, we obtain an upper bound of $\|N\|_2$.

PROPOSITION 15. *For any third order tensor N, we have*

$$\|N\|_2 \leq \|z\|_2,$$

where $z \in \Re^n$ and

$$z_i = \|N_{i\cdot\cdot}\|_2.$$

Proof.

$$\begin{aligned}\|N\|_2 &= \max\{\textstyle\sum_{i,j,k=1}^n |N_{ijk}x_ix_jx_k| : \|x\|_2 = 1\}\\ &= \max\{\textstyle\sum_{i=1}^n |x_iN_{i\cdot\cdot}x^2| : \|x\|_2 = 1\}\\ &\le \max\{\textstyle\sum_{i=1}^n |x_i|\|N_{i\cdot\cdot}\|_2 : \|x\|_2 = 1\}\\ &\le \|z\|_2.\end{aligned} \qquad \square$$

Let $P^{(0)}$ and $Q^{(0)}$ be two second order totally symmetric tensors. Then we use $P^{(0)} \times Q^{(0)}$ to denote the outproduct of $P^{(0)}$ and $Q^{(0)}$. In particular, we have

$$P^{(0)} \times Q^{(0)}x^4 = P^{(0)}x^2Q^{(0)}x^2.$$

We may also have the addition of two fourth order totally symmetric tensors. It is easy to prove the following proposition, which provides a lower bound of $[A]$.

PROPOSITION 16. *If $P^{(0)}$ and $Q^{(0)}$ are two second order positive definite totally symmetric tensors, and $P^{(i)}$ and $Q^{(i)}$ for $i = 1, \dots, k$ are second order positive semidefinite totally symmetric tensors, then*

$$A = \sum_{i=0}^{k} P^{(i)} \times Q^{(i)} \tag{19}$$

is a fourth order positive definite totally symmetric tensor, and

$$[A] \ge \sum_{i=0}^{k} [P^{(i)}][Q^{(i)}].$$

Based on Propositions 13–16, we deduce a checkable sufficient condition for a local minimizer to be global as follows.

COROLLARY 17. *Assume that f is a normal quartic polynomial and x is a local minimizer of f. If the leading term coefficients tensor A can be expressed by* (19) *and*

$$\|z\|_2^2 \le 4\sum_{i=0}^{k} [P^{(i)}][Q^{(i)}][Q] \tag{20}$$

holds, where z is defined as in Proposition 15, *$P^{(i)}$, $Q^{(i)}$ come from* (19), *and Q comes from* (2), *then x is a global minimizer of f.*

Note that (19) also gives a way to generate normal quartic polynomials. Actually, we may generate A by (19) and generate M, P, p, and p_0 randomly.

7. Global descent algorithms.

We use x_i to denote the ith component of $x \in \Re^n$ and $x^{(k)}$ to denote different points in $\Re^n$.

We now state an algorithm for minimizing f when $n = 2$.

ALGORITHM 3.

Step 0. *Have an initial point $x^{(0)} \in \Re^n$, $\rho > 2$, $\epsilon_0 \ge 0$, $\epsilon_1, \epsilon_2 > 0$. Usually we may choose $x^{(0)} = 0$. Let $k = 0$.*

Step 1 *(noncritical point). If*

$$\|F(x^{(k)})\| \le \epsilon_0,$$

go to Step 2. *Otherwise, calculate the Newton direction*

$$d = -(\nabla F(x^{(k)}))^{-1}F(x^{(k)}).$$

If the Newton direction d does not exist or if

$$d^T F(x^{(k)}) \geq -\epsilon_1 \|d\|^\rho,$$

then let $y = -F(x^{(k)})$. Otherwise, let $y = d$. Then we use Algorithm 1 or Algorithm 2 to calculate t^* as an approximate global minimizer of ϕ defined by (6) such that $\phi(t^*) < 0$ and $|\psi(t^*)| \leq \epsilon_2$. Go to Step 6.

Step 2 (negative eigenvalue). If Q has a negative eigenvalue, let y be an eigenvector corresponding to this eigenvalue. Go to Step 5.

Step 3 (zero eigenvalue). If Q has a zero eigenvalue and there is an eigenvector y corresponding to this eigenvalue such that $b \neq 0$, go to Step 5.

Step 4 (local minimizer). Solve four maximization problems in (15) and (16) to find a feasible point y of (15) and (16) such that $\Delta(y) > 0$. If such a y can be found, go to the next step. If no such y can be found, then $x^{(k)}$ is a global minimizer. Stop.

Step 5. Use (3) to calculate t^*.

Step 6. Let

$$x^{(k+1)} = x^{(k)} + t^* y.$$

Let

$$k := k + 1$$

and go to Step 1.

At Step 1 of the above algorithm we determine whether the current iterate x^k is a critical point of f.

In the case where x^k is a critical point of f, when x^k is a saddle point or a local maximizer, it follows from Theorems 4 and 5 that y obtained from Step 2 or Step 3 is a global descent direction. When x^k is a local minimizer, by Theorem 11, y obtained from Step 4 is a global descent direction, and if all of the problems in (15) and (16) have no positive optimal function value, then $x^{(k)}$ is a global minimizer.

In the case where x^k is not a critical point of f, we first compute the Newton direction. If it exists and is acceptable, then it is used as a global descent direction; otherwise, the negative gradient direction of f is used. Then we use Algorithm 1 or Algorithm 2 to calculate the global minimizer of ϕ. Theorems 9 and 10 provided convergence results for these two algorithms.

Combining the above analysis with the discussion in sections 2–5, we have the following convergence theorem.

THEOREM 18. *Let $\epsilon_0 = 0$ in Algorithm 3. Assume that $f : \Re^2 \to \Re$ is a normal quartic polynomial. Then the sequence generalized by Algorithm 3 is globally convergent to a global minimizer x^* of f. If $\nabla F(x^*)$ is nonsingular, the convergence is quadratic.*

Proof. By Proposition 1, the sequence generalized by Algorithm 3 is globally convergent to a global minimizer x^* of f.

The second assertion follows directly from the properties of the Newton method. $\square$

When $n \geq 3$, we may use some sufficient condition given in section 6 to determine whether a local minimizer x of f is global. If x is not a global minimizer of f, we may solve (18) to find a global descent direction y.

Denote $z = (x^T, x_{n+1})^T$, $\Omega = \{z : x_{n+1} \le f(x^*) - \epsilon\}$,

$$\Phi(z) = \begin{pmatrix} F(x) \\ f(x) - x_{n+1} \end{pmatrix}, \quad \text{and} \quad \Psi(z) = \frac{1}{2}\|\Phi(z)\|^2.$$

In the following, we describe an algorithm for solving the bound constrained system of nonlinear equations (18), which is similar to that in [13].

ALGORITHM 4.

Step 0. *Given constants* $\sigma \in (0,1)$, $\theta \in [0,1)$, $\eta \in (0,1)$, $\varepsilon > 0$ *and* $M > 0$. *Let* $\gamma = 1 - \theta^2$, $\alpha_0 = 1$, *and* $z^{(0)} = (x^{*T}, f(x^*) - 1.1\epsilon)^T \in \Omega$. *Let* $l = 0$.

Step 1. *If* $\|\Phi(z^{(l)})\| \le \varepsilon$, *stop.*

Step 2. *Choose* $d^{(l)} \in \Re^{n+1}$ *such that*

$$\begin{aligned} &z^{(l)} + d^{(l)} \in \Omega, \quad \|d^{(l)}\|_\infty \le M \quad \textit{and} \\ &\|\Phi'(z^{(l)})d^{(l)} + \Phi(z^{(l)})\| \le \theta\|\Phi(z^{(l)})\|. \end{aligned} \tag{21}$$

If such a choice is not possible, stop.

Step 3. *If*

$$\Psi(z^{(l)} + \alpha_l d^{(l)}) < \Psi(z^{(l)}),$$

then set $z^{l+1} = z^{(l)} + \alpha_l d^{(l)}$. *Otherwise, let* $\alpha_l := \eta\alpha_l$ *and go to Step* 3.

Step 4. *If*

$$\Psi(z^{(l)} + \alpha_l d^{(l)}) \le (1 - \sigma\gamma\alpha_l)\Psi(z^{(l)}),$$

then set $\alpha_{l+1} := 1$. *Otherwise, set* $\alpha_{l+1} := \eta\alpha_l$.

Let $l := l + 1$ *and go to Step* 1.

At Step 2 of Algorithm 4, any reasonable approach for computing $d^{(l)}$ can be used. In this paper we consider the auxiliary subproblem

$$\min \|\Phi'(z^{(l)})d + \Phi(z^{(l)})\|^2 \quad \text{subject to} \quad z^{(l)} + d \in \Omega, \quad \|d\|_\infty \le M. \tag{22}$$

Many methods for solving the above problem have been proposed in the literature; see, e.g., [7, 8]. Following the proof of Theorem 2.2 in [13], it is not difficult to deduce the following convergence result.

THEOREM 19. *Assume that* $\{z \in \Omega : \Psi(z) \le \Psi(z^{(0)})\}$ *is bounded. Let* $\{z^{(l)}\}$ *be a sequence generated by Algorithm* 4. *Then every limit point of* $\{z^{(l)}\}$ *is a solution of* (18).

If Algorithm 4 stops at Step 1, then $y = x^{(l)} - x^*$ can be used as a global descent direction of f at x^*.

If Algorithm 4 stops at Step 2, i.e., there is no direction $d^{(l)}$ satisfying (21), then when we impose that θ is close to 1 and M is large, $z^{(l)}$ is close to a stationary point of Ψ, which is not a solution of $\Phi(z) = 0$. In this case, we may use the method proposed in [15] to go further to find a solution of $\Phi(z) = 0$. The method in [15] always works.

Actually, since we need only find a global descent direction of f at x^*, we do not need to solve (18) completely. Hence, we may change Step 1 of Algorithm 4 to the following:

Step 1′. *If* $|\Phi_{n+1}(z^{(l)})| \le 0.5\varepsilon$, *stop.*

In Step 4 of Algorithm 3, we may use Algorithm 4 instead of solving (15) and (16) to find a global descent direction of f. We call such an algorithm Algorithm 5 and use it to find an ϵ-global minimizer of f when $n \geq 3$. Sufficient conditions given in section 6 can be used as the termination condition for identifying the global minimizer.

THEOREM 20. *Let $\epsilon_0 = 0$ in Algorithm 5. Assume that $f : \Re^n \to \Re$ is a normal quartic polynomial. Then the sequence generalized by Algorithm 5 is globally convergent to a ϵ-global minimizer x^* of f. If $\nabla F(x^*)$ is nonsingular, the convergence is quadratic.*

The proof of this theorem is the same as the proof of Theorem 18. Note that without the property indicated in Proposition 1, both Theorems 18 and 20 do not hold. This exploited the characteristics of polynomials.

8. An application in signal processing and numerical tests. In this section, we first describe an application of the proposed method in signal processing. Then we report some results of numerical experiments of the algorithms described in the last section.

In the area of broadband antenna array signal processing, the following global optimization problem often arises (see [22, 23]):

$$\begin{array}{ll} \min\limits_{w} & f_0(w) \\ \text{subject to} & g_l(w) = 0, l = 1, \ldots, (N_l + N_q), \end{array} \tag{23}$$

where $f_0 : \Re^n \to \Re$ is a strictly convex multivariate polynomial of degree 2, $g_l : \Re^n \to \Re, l = 1, \ldots, (N_l + N_q)$ are multivariate polynomials of degree at most 2, N_l denotes the number of linear constraints, N_q denotes the number of quadratic constraints, and $w \in \Re^n$ is the n-tuple real weight vector.

In [23], Thng, Cantoni, and Leung showed that the problem (23) is equivalent to the global minimization of a quartic multivariate polynomial. Instead of finding all common zeros of a set of multivariate cubic polynomials as in [23], we use the algorithms proposed in this paper to compute the global minimizer of the quartic polynomial.

As an example, we apply our method to solve a 70-tuple problem listed in Appendix C in [23]. As shown in [23], this problem can be transformed into globally minimizing a bivariate quartic polynomial as follows:

$$\begin{aligned} \mathcal{X}(\alpha_1, \alpha_2) = \quad & 0.337280011659804177 - 0.122071359035091510\alpha_1^2 \\ & + 0.077257128600040819\alpha_1^4 - 0.217646697603541049\alpha_1\alpha_2 \\ & + 0.233083387816363887\alpha_1^3\alpha_2 - 0.129244611969892874\alpha_2^2 \\ & + 0.286227131697582205\alpha_1^2\alpha_2^2 + 0.175571952500361967 3\alpha_1\alpha_2^3 \\ & + 0.056769191379277343 3\alpha_2^4. \end{aligned}$$

Before solving the above problem, we give a general computable criteria to judge whether or not any quartic polynomial is normal for the case $n = 2$.

Given a quartic polynomial $f : \Re^n \to \Re$ in the form of (1), f is normal if and only if $[A] := \min\{Ax^4 : x \in S\} > 0$. If we also use the infinity norm, then we can compute $[A]$ for $n = 2$ by solving the following four minimization problems:

$$\min\{Ax^4 : x_1 = 1, |x_2| \leq 1\}, \qquad \min\{Ax^4 : x_2 = 1, |x_1| \leq 1\}, \tag{24}$$

and

$$\min\{Ax^4 : x_1 = -1, |x_2| \leq 1\}, \qquad \min\{Ax^4 : x_2 = -1, |x_1| \leq 1\}. \tag{25}$$

Since (24) and (25) are four at most fourth degree univariate polynomial minimization problems on the interval $[-1, 1]$, it is easy to find their global minimum

TABLE 1
The numerical results when $n = 2$.

Question	GMP (x^*, y^*)	MOF	IN
Q21	$(1.11753212587544, 0.17682220411240)$	-10.78897112065468	10
Q22	$(-0.20087198302720, 0.39084435467521)$	-17.36200487514549	5
Q23	$(0.09135514315091, -0.84352027358099)$	-31.14019175696217	4
Q24	$(-0.49521459227644, -0.74362935904160)$	-92.28342983346143	7
Q25	$(0.30109168886, -1.96636703726)$	1.55127331116065	5
Q26	$(6.50340840, -9.42272282)$	1288925.94605849	7
Q27	$(-1.348191548098, 0.675076327936)$	-103.344964605849	7
Q28	$(0.34648622547009, 0.4125003776761)$	56.94847262588482	4
Q29	$(0.71784344987734, 0.25545015129099)$	-83.95500103898701	5
Q210	$(0.3397935539523, -0.15034069197633)$	-15.38564238623246	4

solutions. Thus, we can determine the value of $[A]$ by choosing the smallest minimum value of the above four minimization problems.

Using the above method, we can judge that the quartic polynomial $\mathcal{X}(\alpha_1, \alpha_2)$ is normal, and $[A] = 0.007286157241353 > 0$.

Since $n = 2$, we use Algorithm 3 to compute the global minimizer of $\mathcal{X}(\alpha_1, \alpha_2)$ in eight iterations.

We further test Algorithm 3 for $n = 2$, and test Algorithm 5 for $n = 3, 4$ and 6. All computer procedure codes are written in MATLAB language and are implemented on a Pentium III 550e PC. Some parameters are taken as

$$\rho = 3, \quad \varepsilon = \varepsilon_1 = \varepsilon_2 = 10^{-8}, \quad \sigma = 0.5, \quad \theta = 0.25, \quad \eta = 0.5, \quad M = 1.$$

In addition, in our numerical experiments, if $\|F(x^{(k)})\| \le 10^{-8}$ at iteration k, we view $x^{(k)}$ as a critical point. For this time, we find a global descent direction at $x^{(k)}$ by Step 2, 3, or 4 (instead of Step 1) of Algorithms 3 and 5.

When $n = 2$, using Proposition 16 and the routine RANDPOLY in MAPLE, we randomly generate 10 normal quartic polynomials and find their global minimizers by Algorithm 3. The results show that the maximal iteration number is 10 and the minimal iteration number is 4. The computer time for each example is no more than one second. Table 1 lists the global minimal point (GMP), the corresponding minimal value of the objective function (MOF), and the iteration number (IN) of Algorithm 3 for each problem. We recorded the times Algorithm 3 visited Step 2–4 before it obtained a global minimizer for the first 10 examples in the appendix. We found that Algorithm 3 visited Step 2–4 three times before it obtained a global minimizer for test example Q21, and only once for the other 9 test examples.

In Tables 2 and 3, we give numerical results by implementing Algorithm 5 for the cases $n = 3$ and 4, respectively. Here, 15 normal quartic polynomials are generated randomly in a similar fashion to that for the case $n = 2$, of which ten minimization problems are for $n = 3$, and five are for $n = 4$. This time, the maximum IN is 21, the minimum is 6. The computer time for each example is approximately one second.

The termination criterion when $n \ge 3$ is Corollary 17 following Proposition 16. Hence, the obtained global minimizers satisfy Proposition 13.

When n further increases, the number of coefficients of a quartic polynomial increases rapidly. This poses a storage problem. Also, in practice, when n is large, the coefficient tensors may be sparse. Hence, at last in this section, we use Algorithm 5 to solve a class of special normal quartic polynomials with $n = 6$ and sparse coefficient tensors. These results further verify the effectiveness of the proposed algorithm.

TABLE 2
The numerical results when $n = 3$.

Question	GMP (x^*, y^*, z^*)	MOF	IN
Q31	$(-1.2618991, 0.481777475, 0.511554542)$	-112.374935	7
Q32	$(0.45198184, 0.28181568, -0.04439019)$	-22.11006871	6
Q33	$(0.42542661, 1.21252312, -1.858553196)$	-228.406952	8
Q34	$(1.52134791, 1.95307499, -1.95758625)$	-748.700592	21
Q35	$(1.63015225, 0.79920839, -1.37311600)$	-280.1038486	8
Q36	$(7.715035687, -1.54753605, -2.52152795)$	-17687.1075	12
Q37	$(0.8565932583, -0.26891694, 0.162867234)$	-70.1167646	8
Q38	$(0.752808377, 0.287362024, -0.822919492)$	-102.236381	6
Q39	$(0.814835158, -0.308996855, -1.38411083)$	-147.45786	8
Q310	$(-0.693137416, 3.75111948, -11.5578088)$	-31876.9903	10

TABLE 3
The numerical results when $n = 4$.

Question	GMP (x^*, y^*, z^*, u^*)	MOF	IN
Q41	$(-2.16567286, -1.05504009, 1.91472660, 3.19936935)$	-1395.23409	17
Q42	$(-0.05721938, -1.34223106, 0.71604593, 1.29859919)$	-222.071582	9
Q43	$(-1.2292586, 0.63703738, 0.73368052, -0.74318163)$	-117.921805	9
Q44	$(-0.33345841, -0.20877798, 1.50120003, -0.19617979)$	-122.44731	16
Q45	$(-0.67299109, -0.10065256, 0.83060242, -0.01484504)$	-201.373676	6

It is seen that the following polynomial is normal:

$$f(x) = \sum_{i=1}^{6} a_i x_i^4 + x^T P x + p^T x,$$

where $a_i, i = 1, \ldots, 6$, are six randomly generated positive numbers, P is a randomly generated 6×6 matrix, and p is a randomly generated six-dimensional vector. Furthermore, if we use the infinity norm, then we have $[A] = \min\{a_i\}$ (here A represents the fourth order coefficient tensor of the leading degree term of the polynomial).

We first compute the global minimizer of a simple example:

$$x^4 + y^4 + z^4 + u^4 + v^4 + w^4 + x^2 + y^2 - z^2 - u - v + w.$$

It is not very difficult to verify that

$$\begin{aligned}(x, y, z, u, v, w) = (0, 0, &0.70710678118655, 0.62996052494744,\\ &0.62996052494744, -0.62996052494744)\end{aligned}$$

is a global minimizer by directly solving the KKT equations. By implementing Algorithm 5, we get the above solution after five iterations.

Table 4 lists the numerical results when we randomly generate the coefficients.

All 29 randomly generated tested examples in this section are listed in the appendix.

9. Concluding remarks. In this paper, we introduced the concept of global descent directions. For a normal quartic polynomial, we give ways to find a global descent direction at a noncritical point, saddle point, or local maximizer in the general case and at a local minimizer when $n = 2$. For $n \geq 3$, we propose a constrained nonlinear equation approach to find a global descent direction at a local minimizer. We also give a formula at a critical point and a method at a noncritical point to find a

TABLE 4
The numerical results when $n = 6$.

Question	Q61	Q62	Q63	Q64
x^*	0.54521881	−0.65466417	−0.363974062	−1.350391459
y^*	−1.46441019	−1.86951600	−1.028303631	−1.483150332
z^*	−0.72060665	−0.368135071	0.563190492	−1.369006772
u^*	1.17814427	0.819086646	0.9486927097	−1.10594118
v^*	0.79406511	0.77562232	−0.707559149	1.54353024
w^*	−0.465794119	−0.53132279	0.470942714	2.33088412
MOF	28.94281730	−23.00564783	−16.27241853	−70.87818171
IN	25	23	21	23

one-dimensional global minimizer along a global descent direction. Based upon these, two global descent algorithms were proposed. For $n = 2$, the proposed algorithm can find a global minimizer of a normal quartic polynomial. For $n \geq 3$, the proposed algorithm can find an ϵ-global minimizer of the objective function when some global optimality condition holds.

10. Appendix. In this appendix, we list all 29 randomly generated tested examples:

Q21: $32x^4+54x^3y+127x^2y^2+78xy^3+73y^4+75-2x^3-45x^2y-74x^2-60xy^2-xy-31x-43y^3+73y^2-82y$;

Q22: $108x^4+142x^3y+266x^2y^2+160xy^3+96y^4-4-50x^3-47x^2y+75x^2+67xy^2-79xy+53x+63y^3-24y^2-29y$;

Q23: $32x^4+32x^3y+150x^2y^2+124xy^3+81y^4+21+68x^3-84x^2y+80x^2+23xy^2-20xy-7x+4y^3-77y^2+40y$;

Q24: $16x^4+12x^3y+92x^2y^2+18xy^3+127y^4-49+22x^3-8x^2y+53x^2+84xy^2-98xy-5x+20y^3-72y^2+10y$;

Q25: $86x^4+140x^3y+170x^2y^2+76xy^3+39y^4-15-21x^3-75x^2+93xy^2+23xy-16x+83y^3-17y^2+51y$;

Q26: $42x^4+102x^3y+128x^2y^2+76xy^3+24y^4+63-85x^3-55x^2y-37x^2-35xy^2+97xy+50x+79y^3+56y^2+49y$;

Q27: $57x^4+126x^3y+222x^2y^2+172xy^3+119y^4-1+x^3-47x^2y-91x^2-47xy^2-61xy+41x-58y^3-90y^2+53y$;

Q28: $121x^4+218x^3y+277x^2y^2+168xy^3+76y^4+88+43x^3-66x^2y-53x^2-61xy^2-23xy-37x+31y^3-34y^2-42y$;

Q29: $74x^4+128x^3y+182x^2y^2+126xy^3+52y^4-32-76x^3-65x^2y+25x^2+28xy^2-61xy-60x+9y^3+29y^2-66y$;

Q210: $80x^4+172x^3y+260x^2y^2+176xy^3+80y^4+5+78x^3+39x^2y+94x^2+68xy^2-17xy-98x-36y^3+40y^2+22y$;

Q31: $117x^4+86x^3y+182x^3z+185x^2y^2+241x^2z^2+204x^2yz+44xy^3+172xy^2z+114xyz^2+136xz^3+57y^4+84y^3z+159y^2z^2+56yz^3+50z^4+88+66x^3-29x^2y-91x^2z-53x^2-19xy^2-47xyz+68xy-72xz^2-87xz+79x+43y^3-66y^2z-53y^2-61yz^2-23yz-37y+31z^3-34z^2-42z$;

Q32: $105x^4+96x^3y+90x^3z+223x^2y^2+209x^2z^2+148x^2yz+102xy^3+168xy^2z+176xyz^2+114xz^3+98y^4+136y^3z+275y^2z^2+142yz^3+108z^4+5-76x^3-65x^2y+25x^2z+28x^2-61xy^2-60xyz+9xy+29xz^2-66xz-32x+78y^3+39y^2z+94y^2+68yz^2-17yz-98y-36z^3+40z^2+22z$;

Q33: $61x^4+42x^3y+130x^3z+104x^2y^2+225x^2z^2+114x^2yz+44xy^3+114xy^2z+180xyz^2+162xz^3+36y^4+48y^3z+142y^2z^2+138yz^3+112z^4+81-88x^3-43x^2y-73x^2z+25x^2+4xy^2-59xyz+62xy-55xz^2+25xz+9x+40y^3+61y^2z+40y^2-78yz^2+62yz+11y+88z^3+z^2+30z$;

Q34: $96x^4+38x^3y+262x^3z+169x^2y^2+414x^2z^2+202x^2yz+30xy^3+256xy^2z+238xyz^2+282xz^3+75y^4+138y^3z+253y^2z^2+146yz^3+110z^4-73-5x^3-28x^2y+4x^2z-11x^2+10xy^2+57xyz-82xy-48xz^2-11xz+38x-7y^3+58y^2z-94y^2-68yz^2+14yz-35y-14z^3-9z^2-51z$;

Q35: $80x^4+130x^3y+206x^3z+248x^2y^2+302x^2z^2+258x^2yz+130xy^3+356xy^2z+258xyz^2+210xz^3+108y^4+140y^3z+216y^2z^2+94yz^3+82z^4+45-73x^3-91x^2y+x^2z+5x^2-86xy^2+43xyz-4xy-50xz^2+50xz+67x-39y^3+8y^2z-49y^2+11yz^2+93yz-14y-99z^3-67z^2+68z$;

Q36: $22x^4+46x^3y+66x^3z+151x^2y^2+161x^2z^2+110x^2yz+126xy^3+148xy^2z+180xyz^2+126xz^3+90y^4+84y^3z+187y^2z^2+70yz^3+87z^4+21-84x^3+46x^2y+59x^2z-56x^2-83xy^2-91xyz+92xy-93xz^2+91xz-54x+10y^3-77y^2z-63y^2-90yz^2+61yz-3y-82z^3+16z^2-40z$;

Q37: $104x^4+88x^3y+128x^3z+136x^2y^2+239x^2z^2+208x^2yz+36xy^3+120xy^2z+152xyz^2+106xz^3+26y^4+56y^3z+146y^2z^2+124yz^3+102z^4-17-79x^3-27x^2y+32x^2z-24x^2-46xy^2+12xyz+81xy+63xz^2-85xz-36x-35y^3+11y^2z+90y^2+31yz^2+47yz-50y-54z^3+71z^2-71z$;

Q38: $76x^4+172x^3y+176x^3z+285x^2y^2+247x^2z^2+360x^2yz+204xy^3+342xy^2z+420xyz^2+236xz^3+93y^4+182y^3z+293y^2z^2+182yz^3+126z^4+6+76x^3-57x^2y-80x^2z-92x^2+81xy^2+77xyz-87xy+50xz^2+74xz-60x+19y^3-68y^2z+78y^2+34yz^2+66yz-53y+59z^3+28z^2+38z$;

Q39: $111x^4+126x^3y+126x^3z+256x^2y^2+221x^2z^2+162x^2yz+122xy^3+214xy^2z+186xyz^2+138xz^3+69y^4+78y^3z+166y^2z^2+82yz^3+89z^4-15-50x^3-47x^2y+75x^2z+67x^2-79xy^2+53xyz+63xy-24xz^2-29xz-4x-21y^3-75y^2+93yz^2+23yz-16y+83z^3-17z^2+51z$;

Q310: $59x^4+136x^3y+56x^3z+205x^2y^2+84x^2z^2+120x^2yz+144xy^3+130xy^2z+88xyz^2+14xz^3+80y^4+88y^3z+91y^2z^2+38yz^3+11z^4-37-95x^3-47x^2y+51x^2z+47x^2+5xy^2+33xyz+9xy+26xz^2-55xz+37x-94y^3-65y^2z+90y^2-38yz^2-46yz+28y+88z^3+64z^2-22z$;

Q41: $96y^2zu + 380xuz^2 + 338xzu^2 + 476x^2zu + 408xyu^2 + 388xyz^2 + 456xy^2u + 470xy^2z + 464x^2yu + 420x^2yz + 208yuz^2 + 138yzu^2 + 240x^3y + 408x^3z + 444x^3u + 430x^2y^2 + 554x^2z^2 + 522x^2u^2 + 192xy^3 + 362xz^3 + 326xu^3 + 144y^3z + 192y^3u + 352y^2z^2 + 341y^2u^2 + 156yz^3 + 184yu^3 + 297z^2u^2 + 216x^4 + 143y^4 + 165z^4 + 132u^4 + 408xzyu - 61 - 93x - 50y - 62y^2u + 63xz^2 + 50xy^2 + 45xu^2 + 92y^3 + 43y^2z + 66yz^2 + z - 47u + 97x^2 + 49xy - 59xz - 8xu + 77y^2 - 5yz - 61yu + 31z^2 - 91u^2 + 79xyz + 56xyu + 57xzu + 54yzu - 55x^2y - 37x^2z - 35x^2u - 62zu + 99yu^2 - 18z^2u - 26zu^2 - 85x^3 - 12z^3 - 47u^3$;

Q42: $760xyzu + 832x^2yz + 600x^2yu + 454x^2zu + 722xy^2z + 578xy^2u + 720xyz^2 + 576xyu^2 + 422xz^2u + 476xzu^2 + 614y^2zu + 408x^3y + 442x^3z + 306x^3u + 604x^2y^2 + 619x^2z^2 + 514x^2u^2 + 344xy^3 + 354xz^3 + 252xu^3 + 456y^3z + 328y^3u + 617y^2z^2 + 486y^2u^2 + 436yz^3 + 288yu^3 + 616yzu^2 + 289x^4 + 195y^4 + 536yz^2u + 182z^4 + 160u^4 + 190z^3u + 396z^2u^2 + 188zu^3 - 61 - 93x - 50y + z - 47u + 97x^2 + 49xy - 59xz - 8xu + 77y^2 - 5yz - 61yu + 31z^2 - 62zu - 91u^2 + 79xyz + 56xyu + 57xzu + 54yzu - 55x^2y - 37x^2z - 35x^2u + 50xy^2 + 63xz^2 + 45xu^2 + 43y^2z - 62y^2u + 66yz^2 + 99yu^2 - 18z^2u - 26zu^2 - 85x^3 + 92y^3 - 12z^3 - 47u^3$;

Q43: $380xyzu + 308x^2yz + 336x^2yu + 210x^2zu + 358xy^2z + 302xy^2u + 324xyz^2 + 296xyu^2 + 182xz^2u + 310xzu^2 + 336y^2zu + 196x^3y + 192x^3z + 86x^3u + 429x^2y^2 + 284x^2z^2 + 327x^2u^2 + 260xy^3 + 144xz^3 + 114xu^3 + 162y^3z + 362y^3u + 367y^2z^2 + 564y^2u^2 + 132yz^3 + 294yu^3 + 282yzu^2 + 130x^4 + 272y^4 + 316yz^2u + 90z^4 + 180u^4 + 138z^3u + 345z^2u^2 + 192zu^3 + 79 + 85x + 72y - 47z - 87u - x^2 + 23xy - 50xz - 53xu + 72y^2 - 86yz + 80yu - 91z^2 - 19zu - 72u^2 + 83xyz - 86xyu + 19xzu - 85yzu - 58x^2y - 90x^2z + 53x^2u + 94xy^2 - 84xz^2 + 88xu^2 + 78y^2z + 17y^2u - 99yz^2 + 30yu^2 - 29z^2u - 53zu^2 + 41x^3 + 49y^3 + 66z^3 + 68u^3$.

Q44: $308xyz^2 + 518xy^2u + 400xy^2z + 556x^2zu + 548x^2yu + 406x^2yz + 322y^2zu + 632xzu^2 + 450xz^2u + 468xyu^2 + 324yzu^2 + 236yz^2u + 428xyzu + 316x^3y + 248x^3z + 520x^3u + 481x^2y^2 + 343x^2z^2 + 735x^2u^2 + 288xy^3 + 144xz^3 + 530xu^3 + 210y^3z + 108y^3u + 303y^2z^2 + 445y^2u^2 + 144yz^3 + 128yu^3 + 162z^3u + 344z^2u^2 + 298zu^3 + 240x^4 + 180y^4 + 81z^4 + 255u^4 + 4 - 61x - 17y - 36z^2u + 78yzu - 76xzu - 34xyu + 31xyz + 25u + 94yu^2 + 9y^2z + 29y^2u - 32yz^2 - 53x^2z + 22zu^2 - 37xy^2 - 88z - 23x^2 - 42xy - 65xz + 28xu - 66y^2 + 39yz + 68yu + 40z^2 + 5zu - 73u^2 - 66x^2y - 61x^2u + 88xz^2 + 25xu^2 + 43x^3 - 60y^3 - 98z^3 - 43u^3$;

Q45: $440xyz^2 + 610xy^2u + 392xy^2z + 654x^2zu + 526x^2yu + 370x^2yz + 610y^2zu + 732xzu^2 + 744xz^2u + 628xyu^2 + 628yzu^2 + 596yz^2u + 720xyzu + 190x^3y + 250x^3z + 354x^3u + 399x^2y^2 + 468x^2z^2 + 598x^2u^2 + 242xy^3 + 300xz^3 + 456xu^3 + 242y^3z + 332y^3u + 474y^2z^2 + 562y^2u^2 + 260yz^3 + 364yu^3 + 444z^3u + 668z^2u^2 + 456zu^3 + 144x^4 + 195y^4 + 224z^4 + 224u^4 - 73 + 81x + 38y + 58z^2u + 57yzu + 11xzu + 40xyu + 61xyz - 51u - 48yu^2 - 28y^2z + 4y^2u + 10yz^2 - 55x^2z - 68zu^2 + 40xy^2 - 35z + 9x^2 - 78xy + 88xz + 30xu - 11y^2 - 82yz - 11yu - 94z^2 + 14zu - 9u^2 + 62x^2y + 25x^2u + 62xz^2 + xu^2 - 59x^3 - 5y^3 - 7z^3 - 14u^3$;

Q61: $9x^4 + 2y^4 + 6z^4 + 4u^4 + 8v^4 + 7w^4 + 2x + 6y + 5z + 2w + 4x^2 + 8xy + 18xz + 6xu + 8xv + 2xw + 3y^2 + 14yz + 18yu + 18yv + 4yw + 4z^2 + 14zu + 12zv + 12zw + 4u^2 + 4uv + 12uw + 8v^2 + 6vw + 5w^2$;

Q62: $4x^4 + y^4 + 8z^4 + 4u^4 + 6v^4 + 7w^4 + 4x^2 + 6xu + 6xw + 12yu + 12yv + 5z^2 + 6zv + 12zw + 4u^2 + 8uv + 6uw + 4v^2 + 10vw + 2w^2 + 8x + 7y + 7z + 8u + 6v + 2w$;

Q63: $9x^4+7y^4+z^4+4u^4+9v^4+9w^4+8x^2+2xz+6xu+18xv+18xw+18yz+10yu+4yv+12yw+4z^2+2zu+2zv+16zw+16uv+2v^2+2vw+8w^2+5x+8y+6z+9u+9v$;

Q64: $x^4+2y^4+z^4+6u^4+2v^4+w^4+8x+7y+6z+7v+4u+6w+4x^2+2xy+8xz+4xu+8xv+8xw+y^2+8yz+2yv+14yw+4z^2+12zu+12zv+14zw+6u^2+14uv+18uw+3v^2+3w^2$.

Acknowledgments. The authors thank Professor Sheng-Li Tan and Dr. Guohui Zhao for sharing their knowledge of algebraic varieties, and thank a referee for helpful comments.

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