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WEAK SHARP MINIMA IN MULTICRITERIA LINEAR PROGRAMMING*

SIEN DENG† AND X. Q. YANG‡

Abstract. In this short note, we study the existence of weak sharp minima in multicriteria linear programming. It is shown that weak sharp minimality holds for certain residual functions and gap functions.

Key words. multicriteria linear optimization, weak sharp minima, natural residual function, gap function

AMS subject classifications. 90C29, 90C31, 90C05

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1. Introduction.

Consider the following multicriteria linear programming (MCLP) problem:

$$(\mathcal{P}) \qquad \begin{array}{ll} \min & Cx \\ \text{s.t.} & x \in X, \end{array}$$

where $Cx = (c_1^T x, \dots, c_m^T x)^T$, and $X \subset \mathbb{R}^n$ is a nonempty polyhedral convex set.

Let $W = \mathbb{R}^m \backslash (-\text{int}\, \mathbb{R}^m_+)$, where int denotes interior. A vector $\bar{x} \in X$ is a weakly efficient solution of problem $(\mathcal{P})$ if and only if

$$Cx - C\bar{x} \in W \quad \forall x \in X.$$

Denote by E_w the set of all weakly efficient solutions to problem $(\mathcal{P})$.

Weak sharp minima play important roles in mathematical programming. They have been well studied for scalar minimization problems. See [2, 3, 9] and references therein. In this short note, we study weak sharp minima for MCLP problems. In scalar convex optimization, as is well known, weak sharp minimality holds for linear programming, certain quadratic programming, and linear complementarity problems [2]; it is shown that convexity and polyhedrality of solution sets are very important for the existence of weak sharp minima. Unlike the scalar case, for MCLP problems, E_w is not convex in general. However, E_w is a finite union of polyhedral convex sets. By examining such structures carefully, we are able to show that weak sharp minimality holds for certain natural residual functions associated with the underlying MCLP problems. Specifically, we obtain weak sharp minima of the solution set for the natural residual function $\text{dist}\,(Cx \mid CE_w)$ and for the associated gap function, respectively. To the best of our knowledge, these results are new and should be useful in sensitivity analysis and in designing algorithms for solving multicriteria optimization problems.

The notation that we employ is for the most part the same as that in [7, 8]. A partial list is provided below for the reader's convenience.

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†Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115 (deng@math.niu.edu).
‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong (mayangxq@polyu.edu.hk).

We denote the set $\{1, 2, \ldots, m\}$ by $[1, m]$. For any $J \subset [1, m]$, we define C_J to be the matrix obtained from C by removing rows whose indexes are in $[1, m]\backslash J$. For any given $J \subset [1, m]$, the subproblem $\min C_J x$ s.t. $x \in X$, and its solution set, are denoted by $(\mathcal{P}(\mathcal{J}))$ and $E_w(J)$, respectively.

2. Weak sharp minima. In this section, we discuss two existence results of weak sharp minima. We begin by reviewing a basic result on structures of solution sets for MCLP that was given in [1]. See also [6, Thm. 3.3, p. 96].

THEOREM 2.1 (see [1]). *Let Λ be the canonical simplex in $\mathbb{R}^m$ [8, p. 318]. Then there are finitely many vectors $\lambda(1), \ldots, \lambda(r)$ of Λ such that $E_w = \cup_{k=1}^r S_{\lambda(k)}$, where $S_{\lambda(k)} = \arg\min_{x\in X} \lambda(k)^T Cx$.*

2.1. Weak sharp minima for dist $(Cx \mid CE_w)$. For $y \in \mathbb{R}^n$, define

$$\operatorname{dist}(Cy \mid CE_w) = \inf_{x\in E_w} \max_{i\in[1,m]} |c_i y - c_i x|.$$

It is easy to verify that $\operatorname{dist}(Cx \mid CE_w) = 0$ and $x \in X$ if and only if $x \in E_w$. So $\operatorname{dist}(Cx \mid CE_w)$ serves as a natural residual function for problem $(\mathcal{P})$. When $m = 1$ (the scalar linear programming case), $\operatorname{dist}(Cx \mid CE_w) = Cx - f_{\min}$ for $x \in X$, where $f_{\min}$ is the optimal value of $(\mathcal{P})$. We say that E_w is a set of weak sharp minima for the function $\operatorname{dist}(Cx \mid CE_w)$ if there is some positive constant τ such that

$$\operatorname{dist}(x \mid E_w) \le \tau \operatorname{dist}(Cx \mid CE_w) \quad \forall x \in X,$$

where $\operatorname{dist}(x \mid E_w) = \inf_{z\in E_w} \max_{i\in[1,m]} |z_i - x_i|$.

The first main result of this note follows.

THEOREM 2.2. *Suppose that E_w is nonempty. Then E_w is a set of weak sharp minima for $\operatorname{dist}(Cx \mid CE_w)$.*

Proof. For any $y \notin E_w$, there is some $\bar{x} \in E_w$ such that $\operatorname{dist}(Cy \mid CE_w) = \max_{i\in[1,m]} |c_i y - c_i \bar{x}|$. By Theorem 2.1, there is some $\lambda(j) \in \{\lambda(1), \ldots, \lambda(r)\}$ such that

$$\bar{x} \in \arg\min_{x\in X} \lambda(j)^T Cx.$$

Hence, by Hoffman's lemma [5], we have

$$\begin{aligned}
\operatorname{dist}(y \mid E_w) &\le \operatorname{dist}(y \mid S_{\lambda(j)}) \\
&\le \tau(j)\lambda(j)^T (Cy - C\bar{x}) \\
&\le \tau ||\lambda(j)||_1 ||Cy - C\bar{x}||_\infty \\
&\le \tau \operatorname{dist}(Cy \mid CE_w),
\end{aligned}$$

where $||\cdot||_1$ and $||\cdot||_\infty$ are l_1 and l_∞ norms, respectively, and $\tau = \max_{i=1}^r \tau(i)$. The last inequality follows from $||\lambda(j)||_1 = 1$. ∎

2.2. Weak sharp minima for the gap function. For any $y \in X$, define the gap function as follows:

$$g(y) = \max_{x\in X} \left\{ \min_{i\in[1,m]} (c_i y - c_i x) \right\}.$$

The gap function was first introduced in [4]. The gap function g_J for the subproblem $(\mathcal{P}(\mathcal{J}))$ is defined accordingly. It is easy to see that $g(y) \ge 0$ for any $y \in X$, and

$g(y) = 0$ if and only if $y \in E_w$. This gap function can also be expressed as follows:

$$\begin{aligned} g(y) &= \max_{x \in X} \left\{ \min_{i \in [1,m]} (c_i y - c_i x) \right\} \\ &= \max_{x \in X} \left\{ - \max_{i \in [1,m]} (c_i x - c_i y) \right\} \\ &= - \min_{x \in X} \left\{ \max_{i \in [1,m]} (c_i x - c_i y) \right\}. \end{aligned}$$

The following proposition gives some basic properties of gap functions.

PROPOSITION 2.3. *Suppose that E_w is nonempty. Then the following is true for subproblems $(\mathcal{P}(\mathcal{J}))$ and gap functions:*

(a) *g is a finite concave function.*

(b) *$\min_{J \subset [1,m]} g_J(y) = g(y)$ for all $y \in X$, and the following inequality holds:*

$$g(y) \geq \min_{J \subset [1,m]} \operatorname{dist} (C_J y \mid C_J(E_w(J))) \quad \forall y \in X. \tag{1}$$

Proof. (a) Since $c_i x - c_i y$ is jointly convex in x and y, so is $\max_{i \in [1,m]}(c_i x - c_i y)$. This, along with the nonemptiness of E_w, implies that

$$\min_{x \in X} \max_{i \in [1,m]} (c_i x - c_i y)$$

is a finite convex function in y. So g is a finite concave function.

(b) For any $J \subset [1,m]$, by definition, we always have $g_J(y) \geq g(y)$ for all $y \in X$. On the other hand, for any given $y \in X$, let $f(x) = \max_{i \in [1,m]}(c_i x - c_i y)$. Then f is bounded below on X since $E_w \neq \emptyset$. But f is piecewise linear. So $\arg\min_{x \in X} f(x)$ is nonempty. Suppose that $\hat{x} \in \arg\min_{x \in X} f(x)$. Then

$$0 \in \partial f(\hat{x}) + N_X(\hat{x}),$$

where $\partial f(\hat{x}) = co_{i \in I}\{c_i\}$ and $I = \{i \in [1,m] \mid c_i \hat{x} - c_i y = f(\hat{x})\}$. It follows that there is some $\lambda \in \Lambda(I)$ (the canonical simplex in $\mathbb{R}^{|I|}$) such that $\hat{x} \in \arg\min_{x \in X} \lambda^T C_I x$. This implies that $\hat{x} \in E_w(I) \subset E_w$, which, in turn, implies that

$$g(y) = - \min_{x \in E_w} \left\{ \max_{i \in [1,m]} (c_i x - c_i y) \right\}.$$

Let $f_I(x) = \max_{i \in I}(c_i x - c_i y)$. Then

$$0 \in \partial f_I(\hat{x}) + N_X(\hat{x}).$$

So $g(y) = g_I(y)$. This shows that

$$g(y) = \min_{J \subset [1,m]} g_J(y) \quad \forall y \in X.$$

With this given $\hat{x}$ and the choice of I, we have $g(y) = c_i y - c_i \hat{x}$ for any $i \in I$. Thus, for any $\lambda \in \Lambda(I)$,

$$g(y) = \lambda^T (C_I y - C_I \hat{x}) = ||C_I y - C_I \hat{x}||_\infty.$$

Inequality (1) follows from

$$||C_I y - C_I \hat{x}||_\infty \geq \text{dist}\,(C_I y \mid C_I(E_w(I))) \geq \min_{J\subset[1,m]} \text{dist}\,(C_J y \mid C_J(E_w(J))). \qquad \square$$

We say that E_w is a set of weak sharp minima for the gap function g if there is some positive constant γ such that

$$\text{dist}\,(x \mid E_w) \leq \gamma g(x) \quad \forall x \in X.$$

To prove the second main result of this note, we need to use the following result on structures of solution sets for multicriteria convex programming problems.

PROPOSITION 2.4. *For any $J \subset [1,m]$, let $E_w(J)$ be the set of weakly efficient solutions to minimizing $C_J x$ s.t. $x \in X$. Then*

$$\cup_{J\subset[1,m]} E_w(J) = E_w.$$

The second main result now follows.

THEOREM 2.5. *Suppose that E_w is nonempty. Then E_w is a set of weak sharp minima for the gap function g.*

Proof. For any given $I \subset [1,m]$ with $E_w(I)$ nonempty, by Theorem 2.2 there is a $\tau(I) > 0$ such that

$$\text{dist}\,(y \mid E_w(I)) \leq \tau(I)\text{dist}\,(C_I y \mid C_I E_w(I)) \quad \forall y \in X.$$

By Proposition 2.4, we have $E_w = \cup_{I\subset[1,m]} E_w(I)$, and it follows that for any $y \in X$,

$$\text{dist}\,(y \mid E_w) \leq \min_{I\subset[1,m]} \text{dist}\,(y \mid E_w(I)).$$

So

$$\begin{aligned}
\text{dist}\,(y \mid E_w) &\leq \min_{I\subset[1,m]} \text{dist}\,(y \mid E_w(I)) \\
&\leq \min_{I\subset[1,m]} (\tau(I)\text{dist}\,(C_I y \mid C_I E_w(I))) \\
&\leq \left(\max_{I\subset[1,m]} \tau(I)\right) \left(\min_{I\subset[1,m]} \text{dist}\,(C_I y \mid C_I E_w(I))\right) \\
&\leq \left(\max_{I\subset[1,m]} \tau(I)\right) g(y) \quad \textit{(by (1) in Proposition 2.3)}.
\end{aligned}$$

Setting $\hat{\tau} = \max_{I\subset[1,m]} \tau(I)$ yields the desired result. $\square$

Note. It is easy to see that $\text{dist}\,(Cy \mid CE_w) \geq \min_{I\subset[1,m]} \text{dist}\,(C_I y \mid C_I E_w(I))$. However, we don't know how $g(y)$ and $\text{dist}\,(Cy \mid CE_w)$ are related to each other.

We conclude this section with the following example, which illustrates that the technique of parametric linear programming will not yield the sharp results in Theorems 2.2 and 2.5.

Consider the following MCLP:

$$(\mathcal{Q}) \qquad \begin{aligned} \min \quad & Cx \\ \text{s.t.} \quad & Ax \leq 0, \end{aligned}$$

where $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. Its associated scalar linear programming problems are to minimize $c_\mu^T x$ s.t. $Ax \le 0$, where

$$c_\mu = \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1-\mu) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\mu - 1 \end{pmatrix},$$

and $0 \le \mu \le 1$. Then

$$S_\mu = \begin{cases} \mathbb{R}_+(1,1) & \text{if } \mu = 0, \\ (0,0) & \text{if } 0 < \mu < 1, \\ \mathbb{R}_+(1,-1) & \text{if } \mu = 1. \end{cases}$$

For $0 < \mu < 1$, consider the distance between the vector $\bar{x} = (a,a)^T$, where $a > 0$ and $S_\mu = (0,0)$. We have $\operatorname{dist}(\bar{x} \mid S_\mu) = a$, and $c_\mu^T \bar{x} = 2\mu a$. So

$$\tau(\mu) \ge \frac{\operatorname{dist}(\bar{x} \mid S_\mu)}{c_\mu^T \bar{x}} = (2\mu)^{-1} \to \infty \quad \text{as } \mu \downarrow 0.$$

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REFERENCES

[1] K. J. Arrow, E. W. Barankin, and D. Blackwell, *Admissible points of convex sets,* in Contributions to the Theory of Games, Vol. 2, H. W. Kuhn and A. W. Tucker, eds., Princeton University Press, Princeton, NJ, 1953, pp. 87–91.

[2] J. V. Burke and M. C. Ferris, *Weak sharp minima in mathematical programming*, SIAM J. Control Optim., 31 (1993), pp. 1340–1359.

[3] J. V. Burke and S. Deng, *Weak sharp minima revisited.* I. *Basic theory*, Control Cybernet., 31 (2002), pp. 439–469.

[4] C. J. Goh and X. Q. Yang, *On the solution of vector variational inequality*, in Proceedings of the Fourth International Conference on Optimization Techniques and Applications, L. Caccetta et al., eds., Curtin University of Technology, Perth, Australia, 1998, pp. 1158–1164.

[5] A. J. Hoffman, *On approximate solutions of systems of linear inequalities*, J. Research Nat. Bur. Standards, 49 (1952), pp. 263–265.

[6] D. T. Luc, *Theory of Vector Optimization*, Springer-Verlag, Berlin, 1989.

[7] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[8] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[9] M. Studniarski and D. E. Ward, *Weak sharp minima: Characterizations and sufficient conditions*, SIAM J. Control Optim., 38 (1999), pp. 219–236.