Effective properties of piezoelectric composites with periodic structure

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The transformation field theory is developed to investigate the effective properties of piezoelectric composites consisting of anisotropic inclusions having arbitrary geometrical shapes in unit cell. The complicated boundary problem of arbitrary geometrical shapes of anisotropic inclusions is solved by introducing the transformation strain and electric fields. Motivated by theoretical investigation of the effective properties of piezoelectric composites, as an example, the effective dielectric, elastic, and piezoelectric constants of spherical periodic piezoelectric composites are discussed, and a good agreement is obtained by comparing the calculation results with the experimental data in the dilute limit.

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I. INTRODUCTION

Transport properties of piezoelectric composites have attracted much attention because of technological applications and theoretical interest.[1–3] The applications of effective piezoelectric properties include the ultrasonic transducer of underwater acoustics, biomedical imaging, electronic instrumentation, etc. To investigate the effective response mechanism of piezoelectric composites, recently, many authors proposed methods to estimate the effective properties of piezoelectric composites. For example, Furukawa *et al.*[1] and Wong *et al.*[4,5] gave the simple approximation methods for 0–3 piezoelectric composites by means of the analytical stress field and the effective medium approximation. With the eigenstress, the polarization field method, and the virtual work theorems, Benveniste[6,7] proposed universal relations between effective properties of binary and multiphase piezoelectric composites. Based on the assumption of large dielectric constant of the inclusions, Jayasundere *et al.*[2] obtained an analytical expression for the effective piezoelectric constants of 0–3 composites. Jiang *et al.*,[8] Mikata,[9] and Dunn and Wienecke[10] derived the piezoelectric Eshelby's tensor of spheroidal piezoelectric composites by using different approaches, respectively. However, there is not a useful method for estimating the effective responses of the anisotropic piezoelectric composites having complex shapes of inclusions, and up to now, the effective response mechanism is not disclosed in detail. In fact, it is clear that the effective piezoelectric properties are not only related to the physical properties of the materials and volume fraction of inclusions, but also to the shapes of the inclusion material. Different shapes of inclusions induce different internal strain and electric fields, and then these induced fields result in the different effective piezoelectric tensors. Furthermore, the effective dielectric (or elastic) response may be affected by the elastic (or dielectric) properties of composites through the piezoelectric properties, which induce the interactions of the local strain and electric fields. Therefore, it is necessary to develop a method for predicting the effective anisotropic response of piezoelectric composites with arbitrary geometric structure of inclusions and investigating the effective response mechanism.

For complex shapes of anisotropic randomly inclusions, it is difficult to obtain exactly the effective properties of a piezoelectric composite due to the boundary problem of complicated geometrical shapes of inclusions. However, for the randomly piezoelectric composites, the Rayleigh's idea[11,12] can be developed to estimate the bulk effective properties by calculating a larger random sample in the composite, where the larger random sample can be regarded as a unit cell of a periodic composite. For the complex shapes of anisotropic inclusions, the transformation field method can be used to predict the effective response of periodic piezoelectric composites. The idea of the transformation field method was first proposed by Eshelby[13] to investigate the elastic field problem of composites and further developed by Nemat-Nasser and Taya[14] to study the effective modulus of elastic bodies containing voids. Subsequently, Gu *et al.*[15–17] extended this method for estimating the effective electric conductivity of periodic composites with complicated shapes of inclusions, the photonic dispersion relation of periodic lattices of dielectrics, and the effective viscosity of periodic suspensions. In this paper, with Rayleigh's and Eshelby's ideas, the transformation field method is developed to calculate the effective response of periodic piezoelectric composites having arbitrary geometrical shapes of anisotropic inclusions, and the coupled mechanism of the effective piezoelectric, dielectric, and elastic responses is discussed.

In Sec. II, the transformation field method is developed for the constitutive relations of the electric displacement and stress fields for a periodic composite. Two transformation fields of strain and electric fields are introduced, and the integration equations of the transformation strain and electric field are derived. Furthermore, a set of closed algebraic equations is built to determine the unknown coefficients of transformation fields expressed in terms of power series of spatial variables. In Sec. III, based on the transformation strain and electric fields obtained from the transformation field equations of Sec. II, the effective response formulas of periodic piezoelectric composites are given. In Sec. IV, numerical computations are performed to investigate the effective piezoelectric, dielectric, and elastic responses for a piezoelectric composite, and the transformation field method is also discussed by comparing numerical results with experimental data. A brief discussion and conclusion is given in Sec. V.

II. THE TRANSFORMATION FIELD METHOD

We consider the following constitutive relations in the host and inclusion regions of a piezoelectric composite

$$D_i^r = e_{ijk}^r \gamma_{jk} + \varepsilon_{ik}^r E_k, \quad \text{in } \Omega_r, \tag{1}$$

$$\sigma_{ij}^r = C_{ijkl}^r \gamma_{kl} - e_{kij}^r E_k, \quad \text{in } \Omega_r, \tag{2}$$

where the subscripts $i,j,k,l=1,2,3$ (in Cartesian coordinates, x, y, and z are represented by 1, 2, and 3, respectively) and the superscripts $r=i,h$ denote the quantities in host (h) and inclusion (i) regions, respectively. ε_{ik}, e_{ijk}, and C_{ijkl} are dielectric constants, piezoelectric coefficients, and elastic stiffnesses tensors, respectively. The inclusion and host regions are denoted by Ω_r (where subscripts $r=i,h$ denote the inclusion and host regions, respectively). D_i, σ_{ij}, γ_{ij}, and E_i are the electric displacement, stress, strain, and electric fields, respectively. For a state of static equilibrium with no free charges and no body forces, the governing equations in the host and inclusion regions read as, $D_{i,i}^r=0$, $\sigma_{ij,j}^r=0$.

For a cubic periodic composite of infinite extent in three-dimensional space, where the unit cubic cell consists of some anisotropic inclusions of arbitrary geometrical shapes in a matrix, if external uniform strain field γ_{kl}^0 and electric field E_k^0 are applied to the cubic periodic composite, the local strain and electric fields at point $\vec{x}$ in the unit cell are given by $\gamma_{kl}^0+\gamma_{kl}(\vec{x})$ and $E_k^0+E_k(\vec{x})$, respectively, where $\gamma_{kl}(\vec{x})$ and $E_k(\vec{x})$ are the perturbation strain and electric fields because of the presence of inclusions.[15] Thus, under uniform external strain and electric fields, the electric displacement and stress fields in the inclusion and host regions are

$$D_i^r = e_{ijk}^r(\gamma_{jk} + \gamma_{jk}^0) + \varepsilon_{ik}^r(E_k + E_k^0), \quad \text{in } \Omega_r, \tag{3}$$

$$\sigma_{ij}^r = C_{ijkl}^r(\gamma_{kl} + \gamma_{kl}^0) - e_{kij}^r(E_k + E_k^0), \quad \text{in } \Omega_r. \tag{4}$$

To avoid matching the boundary conditions on the interfaces of different phases, we introduce the transformation strain and electric fields, $\gamma_{jk}^*(\vec{x})$ and $E_k^*(\vec{x})$, so that we obtain the unified constitutive relations of the electric displacement and stress fields. Thus, the transformation strain field $\gamma_{jk}^*(\vec{x})$ and transformation electric field $E_k^*(\vec{x})$ are set to be

$$E_i^*(\vec{x}) = 0, \quad \gamma_{ij}^*(\vec{x}) = 0, \quad \text{in } \Omega_h.$$

In the inclusion region Ω_i, the transformation strain $\gamma_{jk}^*(\vec{x})$ and electric field $E_k^*(\vec{x})$ satisfy the following equations:

$$\begin{aligned} e_{ijk}^h(\gamma_{jk}^0 + \gamma_{jk} - \gamma_{jk}^*) + \varepsilon_{ik}^h(E_k^0 + E_k - E_k^*) \\ = e_{ijk}^i(\gamma_{jk}^0 + \gamma_{jk}) + \varepsilon_{ik}^i(E_k^0 + E_k), \end{aligned} \tag{5}$$

$$\begin{aligned} C_{ijkl}^h(\gamma_{kl}^0 + \gamma_{kl} - \gamma_{kl}^*) - e_{kij}^h(E_k^0 + E_k - E_k^*) \\ = C_{ijkl}^i(\gamma_{kl}^0 + \gamma_{kl}) - e_{kij}^i(E_k^0 + E_k). \end{aligned} \tag{6}$$

Therefore, we obtain two unified constitutive equations of the piezoelectric problem in the unit cell region, $D=\Omega_i+\Omega_h$, respectively

$$D_i = e_{ijk}^h[\gamma_{jk}^0 + \gamma_{jk}(\vec{x}) - \gamma_{jk}^*] + \varepsilon_{ik}^h[E_k^0 + E_k(\vec{x}) - E_k^*], \tag{7}$$

$$\sigma_{ij} = C_{ijkl}^h[\gamma_{kl}^0 + \gamma_{kl}(\vec{x}) - \gamma_{kl}^*] - e_{kij}^h[E_k^0 + E_k(\vec{x}) - E_k^*]. \tag{8}$$

Furthermore, the displacement $u_i(\vec{x})$ and electric potential $\Phi(\vec{x})$ are expressed in terms of the Fourier transformation components because of the periodicity of the unit cells

$$u_i(\vec{x}) = \sum_{|n|\neq 0}^{\infty} u_i(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}),$$

$$\Phi(\vec{x}) = \sum_{|n|\neq 0}^{\infty} \Phi(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}),$$

where the reciprocal vector $\vec{\xi}^n=(\xi_1^n,\xi_2^n,\xi_3^n)=2\pi\left(\frac{n_x}{l_x},\frac{n_y}{l_y},\frac{n_z}{l_z}\right)$ and l_x, l_y, and l_z are the lengths of the unit cell along x, y, and z directions, respectively. Then, the perturbation strain and electric fields, $\gamma_{kl}(\vec{x})$ and $E_k(\vec{x})$, are determined by the Fourier transformation components $u_i(\vec{\xi}^n)$ and $\Phi(\vec{\xi}^n)$,thus

$$\begin{aligned} \gamma_{jk}(\vec{x}) = \frac{1}{2}[\nabla_j u_k(\vec{x}) + \nabla_k u_j(\vec{x})] = \frac{i}{2}\sum_{|n|\neq 0}^{\infty}[\xi_j^n u_k(\vec{\xi}^n) \\ + \xi_k^n u_j(\vec{\xi}^n)]\exp(i\vec{\xi}^n \cdot \vec{x}), \end{aligned}$$

$$E_i(\vec{x}) = -\nabla_i\Phi(\vec{x}) = -i\sum_{|n|\neq 0}^{\infty} \xi_i^n\Phi(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}).$$

Similarly, the transformation strain field $\gamma_{jk}^*(\vec{x})$ and electric field $E_k^*(\vec{x})$ are expanded as Fourier series

$$\gamma_{jk}^*(\vec{x}) = \sum_{|n|\neq 0}^{\infty} \gamma_{jk}^*(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}),$$

$$E_i^*(\vec{x}) = \sum_{|n|\neq 0}^{\infty} E_i^*(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}),$$

where $\gamma_{jk}^*(\vec{\xi}^n)=\frac{1}{V}\int_{\Omega_i}\gamma_{jk}^*(\vec{x})\exp(-i\vec{\xi}^n\cdot\vec{x})d\vec{x}$ and $E_i^*(\vec{\xi}^n)=\frac{1}{V}\int_{\Omega_i}E_i^*(\vec{x})\exp(-i\vec{\xi}^n\cdot\vec{x})d\vec{x}$. V is the volume of the unit cell D. Substituting the above perturbation and transformation fields into Eqs. (7) and (8) and together with the governing equations $D_{i,i}^r=0$ and $\sigma_{ij,j}^r=0$, we determine the perturbation strain $\gamma_{kl}(\vec{x})$ and electric field $E_k(\vec{x})$ by means of the Fourier transformation components $\gamma_{jk}^*(\vec{\xi}^n)$ and $E_i^*(\vec{\xi}^n)$ as follows:

$$E_i(\vec{x}) = -\sum_{|n|\neq 0}^{\infty} i\xi_i^n[e_k^h\delta_{jk}^{-1}(n)\varepsilon_j^*(\vec{\xi}^n) + \alpha^*(\vec{\xi}^n)]\exp(i\vec{\xi}^n \cdot \vec{x}), \tag{9}$$

$$\gamma_{jk}(\vec{x}) = \sum_{|n|\neq 0}^{\infty} \frac{i}{2}[\xi_j^n \delta_{ik}^{-1}(n) + \xi_k^n \delta_{ij}^{-1}(n)]\varepsilon_i^*(\vec{\xi}^n)\exp(i\vec{\xi}^n \cdot \vec{x}), \tag{10}$$

where

$$\varepsilon_j^*(\vec{\xi}^n) = F_{jkl}(n)\gamma_{kl}^*(\vec{\xi}^n) + G_{jk}(n)E_k^*(\vec{\xi}^n),$$

$$\alpha^*(\vec{\xi}^n) = [ie_{ijk}^h \xi_i^n \gamma_{jk}^*(\vec{\xi}^n) + i\varepsilon_{ik}^h \xi_i^n E_k^*(\vec{\xi}^n)]\alpha^h(n),$$

$$e_k^h(n) = \frac{1}{2}\alpha^h(n)[e_{ijk}^h + e_{ikj}^h]\xi_i^n \xi_j^n,$$

$$F_{jkl}(n) = -iC_{ijkl}^h \xi_i^n - ie_{pij}^h \xi_i^n \xi_p^n e_{mkl}^h \xi_m^n \alpha^h(n),$$

$$G_{jk}(n) = ie_{kij}^h \xi_i^n - ie_{pij}^h \xi_i^n \xi_p^n \varepsilon_{lk}^h \xi_l^n \alpha^h(n),$$

$$\delta_{jk}(n) = \frac{1}{2}(C_{ijlk}^h + C_{ijkl}^h)\xi_i^n \xi_l^n + e_{nij}^h \xi_i^n \xi_n^n e_k^h(n),$$

$\alpha^h(n) = 1/(\varepsilon_{ik}^h \xi_i^n \xi_k^n)$, $\delta_{jk}^{-1}(n)$ is the inverse matrix of $\delta_{jk}(n)$ and $n = (n_x, n_y, n_z)$. Substituting Eqs. (9) and (10) into Eqs. (5) and (6), we have a set of integral equations of the transformation strain field $\gamma_{jk}^*(\vec{x})$ and electric field $E_k^*(\vec{x})$ in the inclusion region Ω_i

$$\begin{aligned} e_{ijk}^h \gamma_{jk}^*(\vec{x}) + \varepsilon_{ik}^h E_k^*(\vec{x}) = {} & (e_{ijk}^h - e_{ijk}^i)\gamma_{jk}^0 + (e_{ijk}^h - e_{ijk}^i)\sum_{|n|\neq 0}^{\infty} \frac{i}{2}[\xi_j^n \delta_{pk}^{-1}(n) + \xi_k^n \delta_{pj}^{-1}(n)] \\ & \times \left\{ \frac{1}{V}\int_{\Omega_i} [F_{pk'l'}(n)\gamma_{k'l'}^*(\vec{x}') + G_{pk'}(n)E_{k'}^*(\vec{x}')]\exp[i\vec{\xi}^n \cdot (\vec{x} - \vec{x}')]d\vec{x}' \right\} + (\varepsilon_{ik}^h - \varepsilon_{ik}^i)E_k^0 - (\varepsilon_{ik}^h - \varepsilon_{ik}^i)\sum_{|n|\neq 0}^{\infty} i\xi_k^n \\ & \times \left\{ \frac{1}{V}\int_{\Omega_i} [F_{k'l'}^1(n)\gamma_{k'l'}^*(\vec{x}') + G_{k'}^1(n)E_{k'}^*(\vec{x}')]\exp[i\vec{\xi}^n \cdot (\vec{x} - \vec{x}')]d\vec{x}' \right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} C_{ijkl}^h \gamma_{kl}^*(\vec{x}) - e_{kij}^h E_k^*(\vec{x}) = {} & (C_{ijkl}^h - C_{ijkl}^i)\gamma_{kl}^0 + (C_{ijkl}^h - C_{ijkl}^i)\sum_{|n|\neq 0}^{\infty} \frac{i}{2}[\xi_k^n \delta_{pl}^{-1}(n) + \xi_l^n \delta_{pk}^{-1}(n)] \\ & \times \left\{ \frac{1}{V}\int_{\Omega_i} [F_{pk'l'}(n)\gamma_{k'l'}^*(\vec{x}') + G_{pk'}(n)E_{k'}^*(\vec{x}')]\exp[i\vec{\xi}^n \cdot (\vec{x} - \vec{x}')]d\vec{x}' \right\} \\ & - (e_{kij}^h - e_{kij}^i)E_k^0 + (e_{kij}^h - e_{kij}^i)\sum_{|n|\neq 0}^{\infty} i\xi_k^n \left\{ \frac{1}{V}\int_{\Omega_i} [F_{k'l'}^1(n)\gamma_{k'l'}^*(\vec{x}') \right. \\ & \left. + G_{k'}^1(n)E_{k'}^*(\vec{x}')]\exp[i\vec{\xi}^n \cdot (\vec{x} - \vec{x}')]d\vec{x}' \right\}, \end{aligned} \tag{12}$$

where $F_{kl}^1(n) = e_p^h \delta_{jp}^{-1}(n)F_{jkl}(n) + i\alpha^h(n)e_{ikl}^h \xi_i^n$

$$G_k^1(n) = e_p^h \delta_{jp}^{-1}(n)G_{jk}(n) + i\alpha^h(n)\varepsilon_{ik}^h \xi_i^n.$$

To solve Eqs. (11) and (12), we express the transformation fields as power series of $(x/l_x)^\alpha (y/l_y)^\beta (z/l_z)^\gamma$, where α, β, γ denote the powers of the series

$$E_i^*(\vec{x}) = \sum_{\alpha,\beta,\gamma}^{\infty} C_i^{\alpha\beta\gamma}(x/l_x)^\alpha (y/l_y)^\beta (z/l_z)^\gamma,$$

$$\gamma_{ij}^*(\vec{x}) = \sum_{\alpha,\beta,\gamma}^{\infty} B_{ij}^{\alpha\beta\gamma}(x/l_x)^\alpha (y/l_y)^\beta (z/l_z)^\gamma,$$

where the unknown coefficients $B_{ij}^{\alpha\beta\gamma}$ and $C_i^{\alpha\beta\gamma}$ will be determined. If we multiply both sides of Eqs. (11) and (12) by power function $(x/l_x)^{\alpha_1}(y/l_y)^{\beta_1}(z/l_z)^{\gamma_1}$ and then integrate the above equations over the inclusion region, a set of linear algebraically closed equations is obtained for determining the unknown coefficients $B_{ij}^{\alpha\beta\gamma}$ and $C_i^{\alpha\beta\gamma}$

$$\sum_{\alpha,\beta,\gamma=0}^{N} B_{k'l'}^{\alpha\beta\gamma}\left[G^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}e_{ik'l'}^{h}-\frac{i}{2}\Delta e_{ijk}W_{jkk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}+i\Delta\varepsilon_{ik}U_{kk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}\right]$$
$$+\sum_{\alpha,\beta,\gamma=0}^{N} C_{k'}^{\alpha\beta\gamma}\left[G^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}\varepsilon_{ik'}^{h}-\frac{i}{2}\Delta e_{ijk}V_{jkk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}+i\Delta\varepsilon_{ik}S_{kk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}\right]=G^{\alpha_1\beta_1\gamma_1}(\Delta e_{ijk}\gamma_{jk}^{0}+\Delta\varepsilon_{ik}E_k^0), \tag{13}$$

$$\sum_{\alpha,\beta,\gamma=0}^{N} B_{k'l'}^{\alpha\beta\gamma}\left[-G^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}C_{ijk'l'}^{h}+\frac{i}{2}\Delta C_{ijkl}W_{klk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}+i\Delta e_{kij}U_{kk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}\right]$$
$$+\sum_{\alpha,\beta,\gamma=0}^{N} C_{k'}^{\alpha\beta\gamma}\left[G^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}e_{k'ij}^{h}+\frac{i}{2}\Delta C_{ijkl}V_{klk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}+i\Delta e_{kij}S_{kk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}\right]=G^{\alpha_1\beta_1\gamma_1}(\Delta e_{kij}E_k^0-\Delta C_{ijkl}\gamma_{kl}^{0}), \tag{14}$$

where the subscripts $i,j,k,l,k',l'=1,2,3$; the superscripts $\alpha,\beta,\gamma=0,1,2,3,\ldots,N$ and $\alpha_1,\beta_1,\gamma_1=0,1,2,\ldots,M$; N is the approximation order of the transformation strain and electric fields; the value of M can be selected so that Eqs. (13) and (14) are closed

$$\Delta e_{ijk}=e_{ijk}^h-e_{ijk}^i,\quad \Delta C_{ijkl}=C_{ijkl}^h-C_{ijkl}^i,\quad \Delta\varepsilon_{ij}=\varepsilon_{ij}^h-\varepsilon_{ij}^i,$$

$$W_{jkk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}=\sum_{|n|\neq 0}^{\infty}T_{jkp}(n)F_{pk'l'}(n)q^{\alpha\beta\gamma}(n)R^{\alpha_1\beta_1\gamma_1}(n),$$

$$U_{kk'l'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}=\sum_{|n|\neq 0}^{\infty}\xi_k^nF_{k'l'}^1(n)q^{\alpha\beta\gamma}(n)R^{\alpha_1\beta_1\gamma_1}(n),$$

$$S_{kk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}=\sum_{|n|\neq 0}^{\infty}\xi_k^nG_{k'}^1(n)q^{\alpha\beta\gamma}(n)R^{\alpha_1\beta_1\gamma_1}(n),$$

$$V_{jkk'}^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}=\sum_{|n|\neq 0}^{\infty}T_{jkp}(n)G_{pk'}(n)q^{\alpha\beta\gamma}(n)R^{\alpha_1\beta_1\gamma_1}(n),$$

$$T_{ijk}(n)=\xi_i^n\delta_{kj}^{-1}(n)+\xi_j^n\delta_{ki}^{-1}(n),$$

$$G^{\alpha\beta\gamma,\alpha_1\beta_1\gamma_1}=\int_{\Omega_i}\left(\frac{x}{l_x}\right)^{\alpha+\alpha_1}\left(\frac{y}{l_y}\right)^{\beta+\beta_1}\left(\frac{z}{l_z}\right)^{\gamma+\gamma_1}dV,$$

$$F_{kl}^1(n)=e_p^h\delta_{jp}^{-1}(n)F_{jkl}(n)+i\alpha^h(n)e_{ikl}^h\xi_i^n,$$

$$G_k^1(n)=e_p^h\delta_{jp}^{-1}(n)G_{jk}(n)+i\alpha^h(n)\varepsilon_{ik}^h\xi_i^n,$$

$$q^{\alpha\beta\gamma}(n)=\frac{1}{V}\int_{\Omega_i}\left(\frac{x}{l_x}\right)^{\alpha}\left(\frac{y}{l_y}\right)^{\beta}\left(\frac{z}{l_z}\right)\exp(-i\vec{\xi}^n\cdot\vec{x})dV,$$

$$R^{\alpha_1\beta_1\gamma_1}(n)=\int_{\Omega_i}\left(\frac{x}{l_x}\right)^{\alpha_1}\left(\frac{y}{l_y}\right)^{\beta_1}\left(\frac{z}{l_z}\right)^{\gamma_1}\exp(i\vec{\xi}^n\cdot\vec{x})dV.$$

Therefore, the transformation fields are obtained by solving Eqs. (13) and (14) and the perturbation fields are also given by Eqs. (9) and (10). When we solve Eqs. (13) and (14), the symmetry of the transformation strain fields, $\gamma_{jk}^*(\vec{x})=\gamma_{kj}^*(\vec{x})$, should be considered. Here, it is noted that the transformation field Eqs. (13) and (14) are suitable to investigate an anisotropic piezoelectric composite having complex shapes of multiparticle inclusions because there is not any limitation on the shape, the physical properties, the number of particles, and spatial distributions of the inclusions in the unit cell.[14,15] In addition, if the piezoelectric coefficients e_{ijk}^r are set to be zero in Eqs. (13) and (14), we can obtain transformation field equations of the elastic and dielectric problems, which were studied in detail in Refs. 14 and 15.

III. EFFECTIVE RESPONSE FORMULAS

Considering Eqs. (3) and (4), we define the effective piezoelectric constant tensors e_{ijk}^*, dielectric constant tensors ε_{ik}^*, and elastic constant tensors C_{ijkl}^* as follows:

$$e_{ijk}^*\gamma_{jk}^0+\varepsilon_{ik}^*E_k^0=\langle D_i\rangle, \tag{15}$$

$$C_{ijkl}^*\gamma_{kl}^0-e_{kij}^*E_k^0=\langle\sigma_{ij}\rangle, \tag{16}$$

where $\langle A\rangle=V^{-1}\int_D AdV$. Here, we should note that the volume averages of the perturbation strain field $\gamma_{kl}(\vec{x})$ and electric field $E_k(\vec{x})$ are zero because of the periodicity of composites.[15] Similarly, taking volumetric average over the unit cell to Eqs. (7) and (8), we have

$$\langle D_i\rangle=e_{ijk}^h(\gamma_{jk}^0-\langle\gamma_{jk}^*(\vec{x})\rangle)+\varepsilon_{ik}^h(E_k^0-\langle E_k^*(\vec{x})\rangle), \tag{17}$$

$$\langle\sigma_{ij}\rangle=C_{ijkl}^h(\gamma_{kl}^0-\langle\gamma_{kl}^*(\vec{x})\rangle)-e_{kij}^h(E_k^0-\langle E_k^*(\vec{x})\rangle), \tag{18}$$

where

$$\langle\gamma_{jk}^*(\vec{x})\rangle=\frac{1}{V}\int_D\gamma_{jk}^*(\vec{x})dV=\frac{1}{V}\int_{\Omega_i}\gamma_{jk}^*(\vec{x})dV\equiv\langle\gamma_{jk}^*(\vec{x})\rangle_i,$$

$$\langle E_{jk}^*(\vec{x})\rangle=\frac{1}{V}\int_D E_{jk}^*(\vec{x})dV=\frac{1}{V}\int_{\Omega_i}E_{jk}^*(\vec{x})dV\equiv\langle E_{jk}^*(\vec{x})\rangle_i.$$

From Eqs. (15)–(18), we have the following equations:

$$e^*_{ijk}\gamma^0_{jk} + \varepsilon^*_{ik}E^0_k = e^h_{ijk}(\gamma^0_{jk} - \langle \gamma^*_{jk}(\vec{x})\rangle_i) + \varepsilon^h_{ik}(E^0_k - \langle E^*_k(\vec{x})\rangle_i), \tag{19}$$

$$C^*_{ijkl}\gamma^0_{kl} - e^*_{kij}E^0_k = C^h_{ijkl}(\gamma^0_{kl} - \langle \gamma^*_{kl}(\vec{x})\rangle_i) - e^h_{kij}(E^0_k - \langle E^*_k(\vec{x})\rangle_i). \tag{20}$$

If we let external fields $\gamma^0_{kl}=0$ and $E^0_k \neq 0$ (for another case: $E^0_k=0$ and $\gamma^0_{kl}\neq 0$) in Eqs. (19) and (20), the effective responses e^*_{ijk}, ε^*_{ik}, and C^*_{ijkl} are obtained as follows:

$$e^*_{ijk}\gamma^0_{jk} = e^h_{ijk}(\gamma^0_{jk} - \langle \gamma^*_{jk}(\vec{x})\rangle_i) - \varepsilon^h_{ik}\langle E^*_k(\vec{x})\rangle_i, \tag{21}$$

$$\varepsilon^*_{ik}E^0_k = - e^h_{ijk}\langle \gamma^*_{jk}(\vec{x})\rangle_i + \varepsilon^h_{ik}(E^0_k - \langle E^*_k(\vec{x})\rangle_i), \tag{22}$$

$$C^*_{ijkl}\gamma^0_{kl} = C^h_{ijkl}(\gamma^0_{kl} - \langle \gamma^*_{kl}(\vec{x})\rangle_i) + e^h_{kij}\langle E^*_k(\vec{x})\rangle_i, \tag{23}$$

$$e^*_{kij}E^0_k = C^h_{ijkl}\langle \gamma^*_{kl}(\vec{x})\rangle_i + e^h_{kij}(E^0_k - \langle E^*_k(\vec{x})\rangle_i). \tag{24}$$

For formulas (21) and (24) of effective piezoelectric coefficients e^*_{ijk}, with governing equations and external applied fields, Benveniste and Dvorak[18] demonstrated that effective piezoelectric coefficients e^*_{ijk} obtained from Eq. (21) is the same as those of Eq. (24) by means of virtual work theorems, and our numerical calculations confirmed this conclusion. Thus, we have obtained the effective response formulas (21)–(23) of piezoelectric composites. From the above equations, we can predict the coupled effective piezoelectric, dielectric, and elastic responses of periodic piezoelectric composites.

IV. NUMERICAL ANALYSIS

As an example of the crystal class 2 mm of orthorhombic isotropic simple-center-cubic spherical periodic piezoelectric composites, where the matrix is also a 2 mm crystal material, and an inclusion sphere is located at the center of the unit cell, numerical results will be performed to discuss the transformation field method and analyze the effective dielectric, elastic, and piezoelectric responses, respectively. In Fig. 1, we have shown the zeroth, the first, and the second order approximations of transformation field method for predicting the effective piezoelectric constants of periodic composites with various radius of spherical particle. It is clear that the zero-order approximation is enough to calculate the effective response of composites in the dilute limit and the higher-order approximations are suitable to predict bulk effective responses of composites having high concentration of inclusions. In the dilute limit, because interactions between inclusions are negligible under external strain and electric fields, the zero-order approximation (constant transformation fields) of the transformation field is suitable to calculate local strain and electric fields of composites. Of course, for high concentration of inclusions, the local strain and electric fields are dependent of inclusion spatial position because of the enhanced interactions of inclusion particles. Therefore, the high-order approximations can be used to calculate effective response of composites having high concentration of inclusions. Furthermore, in Fig. 2, for piezoelectric composites

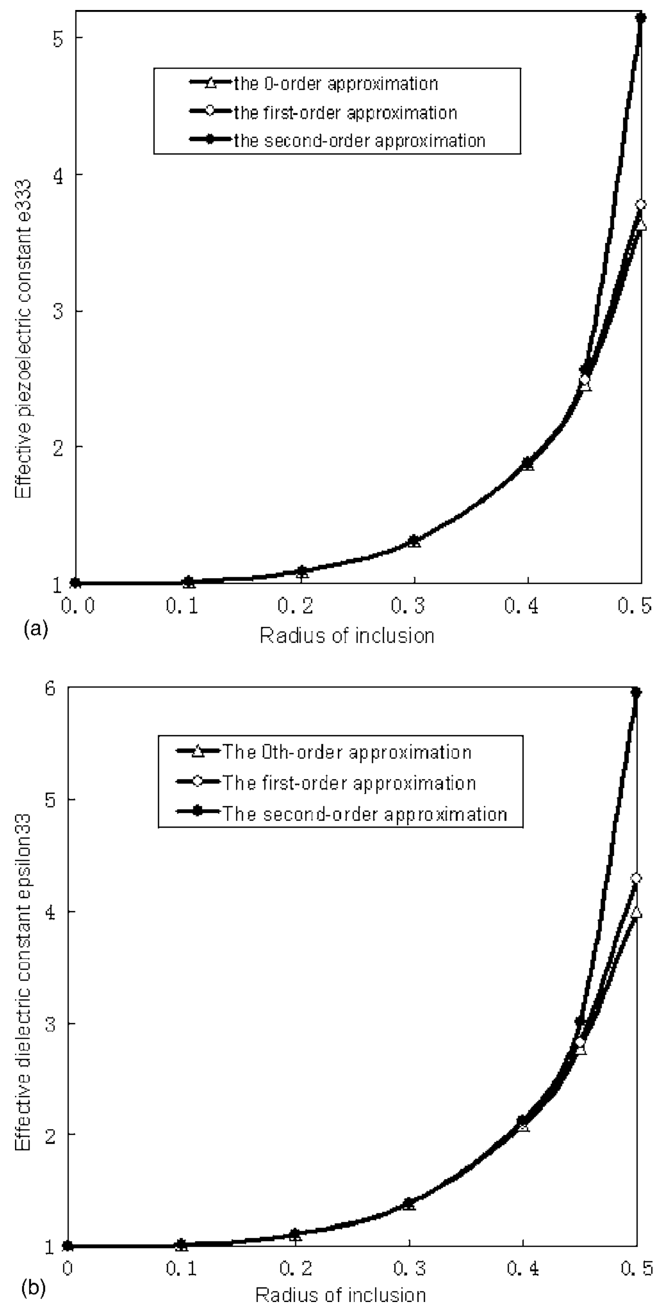

FIG. 1. The effective piezoelectric and dielectric constants e^*_{333} and ε^*_{33} [(a), e^*_{333} is denoted by e_{333}, in (b) ε^*_{33} is noted by "epsilon33"] of the isotopic piezoelectric periodic composites are calculated for different radii of spherical inclusion, where the unit cell is a cube of side 1. The dimensionless parameters of Lame constants $\lambda^h=\mu^h=2$, $e^h_{311}=e^h_{322}=e^h_{333}=e^h_{223}=e^h_{113}=1$, $\varepsilon^h_{ii}=1$ for the host region and Lame constants $\lambda^i=\mu^i=20$, $e^i_{311}=e^i_{322}=e^i_{333}=e^i_{223}=e^i_{113}=50$, $\varepsilon^i_{ii}=10$ for the inclusion sphere.

having the inclusion and matrix materials of 6 mm symmetry, we have compared the numerical results of transversely isotropic piezoelectric spherical simple- center-cubic periodic composites with the experimental data of PZT-P (VDF/TrFE) 0–3 random piezoelectric composites by means of the dielectric, elastic, and piezoelectric parameters for the inclusion and matrix materials given in the experiment Refs. 4 and 19. In the dilute limit, our results are in good agreement

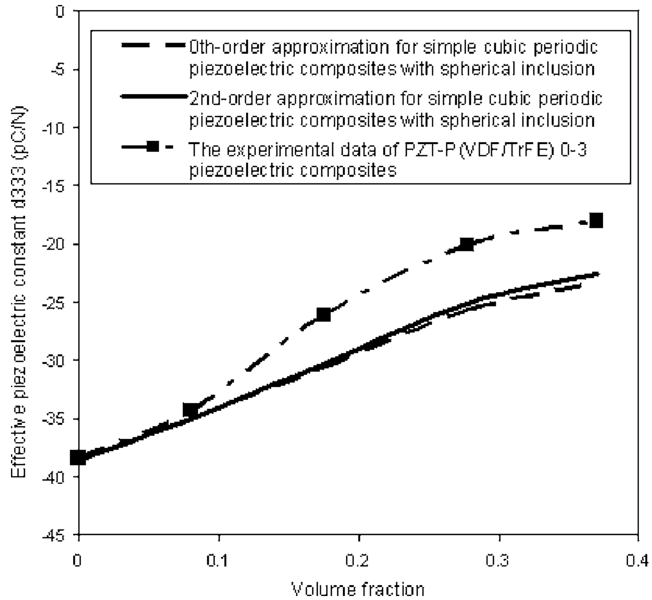

FIG. 2. The numerical results of simple center cubic periodic composites are compared with the experimental data of PZT-P (VDF/TrFE) 0–3 composites without periodic structure (Refs. 4 and 19), where physical properties parameters of inclusion and host were given in Table 1 and Fig. 3 of Ref. 4, note that the effective piezoelectric constant matrix d of the first kind constitutive relation is obtained from the effective elastic constant matrix C^* and effective piezoelectric constant matrix e^* of the second constitutive relation through the formula $d=e^*C^{*-1}$.

with the experimental data since the microstructures of dilute concentration inclusions do not affect bulk effective properties of composites. Next, as numerical analyses, without loss of generality, we shall discuss the effective piezoelectric, dielectric, and elastic responses of this kind of 2 mm crystal symmetry periodic piezoelectric composites, where the volume fraction of inclusion is 0.1, except for Figs. 3(b) and 4(b) where the volume fraction is 0.35. In these two figures, second-order approximation was employed in the calculation.

As numerical discussions of the effects of elastic and dielectric constants of inclusions on the effective piezoelectric response under external electric or strain fields, we have performed numerical results in Figs. 3(a) and 4(a), respectively. In Fig. 3(a), the larger Lame constants of inclusion do not clearly give rise to the change of effective piezoelectric response, and the effective piezoelectric constant sharply increases with Lame constant increase within the range of smaller Lame constants. From Fig. 4(a), we can see that, within the range of smaller dielectric constants, for larger piezoelectric constant of inclusion, the dielectric constant of inclusion results in the effective piezoelectric constant decrease with increasing dielectric constants of inclusions, and, for smaller piezoelectric constant of inclusion, the effective piezoelectric constant increases with particle dielectric constant increase. However, within the range of larger dielectric constants, the effective piezoelectric constant is not affected by the dielectric constant of inclusion. These numerical results show that the effective piezoelectric constant is a coupled response of the piezoelectric, dielectric, and elastic

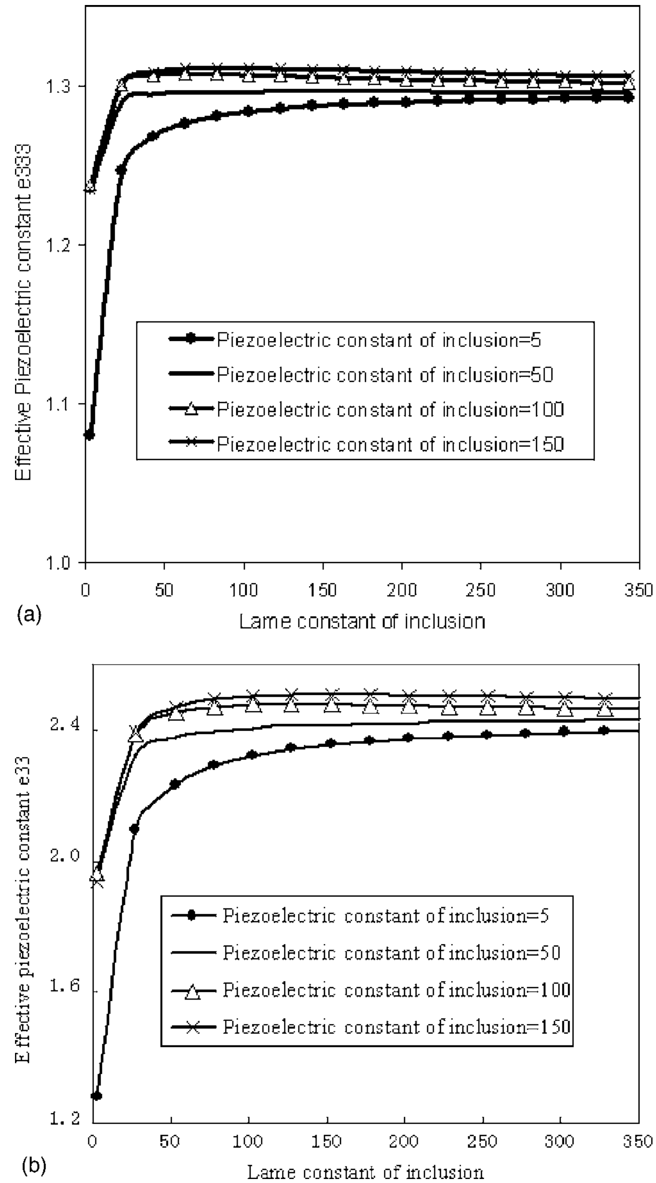

FIG. 3. The effective piezoelectric constants e_{333}^* (in this figure e_{333}^* is denoted by e_{333}) against the Lame constants of inclusion spheres, where the dimensionless parameters in host region are given in Fig. 1 and dielectric constants $\varepsilon_{ii}^i=60$ and Lame constant $\lambda^i=\mu^i$ for the inclusion sphere. The volume fractions of the inclusions are 0.1 and 0.35 in (a) and (b), respectively.

constants of composite materials. For larger volume fractions of the inclusions, the elastic and dielectric constants of inclusions may affect the values of the effective piezoelectric properties because of the enhancing interactions among the inclusion particles. However, as shown in Figs. 3(b) and 4(b), there are not much qualitative changes in the effective piezoelectric properties, changing with the elastic and dielectric constants of the inclusions.

The effective dielectric responses of piezoelectric composites are shown in Fig. 5 for performing effects of piezoelectric and elastic constants under external electric field. Generally, the effective dielectric constant increases while the inclusion piezoelectric constant increase and decreases

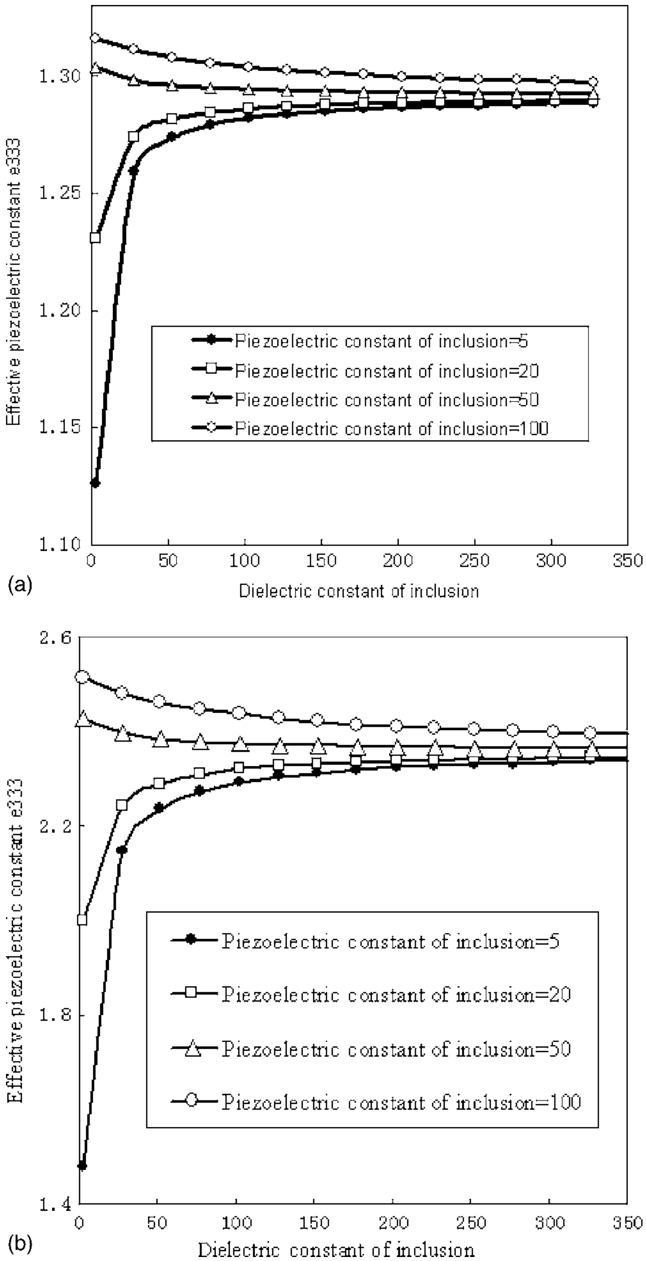

FIG. 4. The effective piezoelectric constant e_{333}^* (in this figure e_{333}^* is denoted by e_{333}) against the dielectric constants of inclusions, where the dimensionless parameters in host region are given in Fig. 1 and the Lame constant $\lambda^i = \mu^i = 60$ for the inclusion sphere. Furthermore, the volume fractions of inclusions are 0.1 and 0.35 in (a) and (b), respectively.

with the inclusion elastic constant increases. However, for the smaller piezoelectric constants of inclusions, the elastic constants do not induce the change of effective dielectric responses. This phenomenon indicates that, for piezoelectric composites, the piezoelectric constants play an important role in inducing the interchange from electric displacement field (or the stress field) to stress field (or electric displacement field). Meanwhile, the effective dielectric constant is also a coupled response related to the piezoelectric and elastic constants, and the larger elastic constant slightly reduces

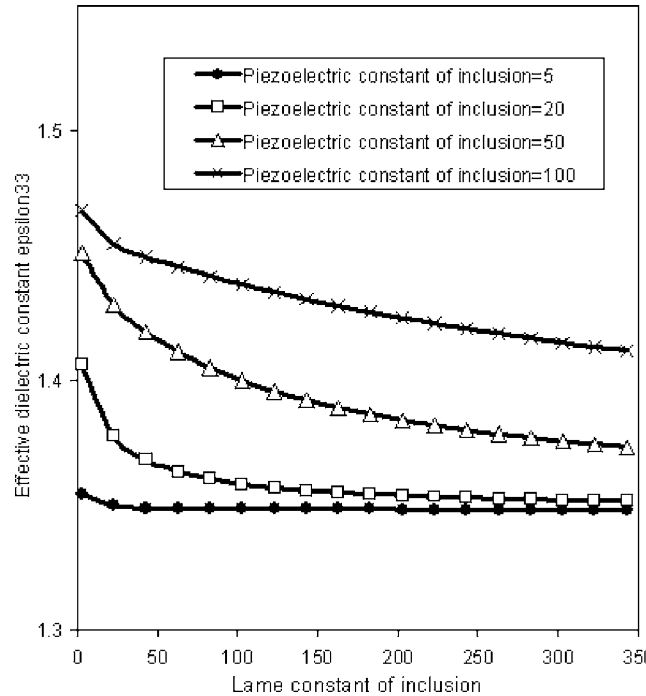

FIG. 5. The effective dielectric constants ε_{33}^* (in this figure ε_{33}^* is denoted by "epsilon33") against the Lame constant of inclusions, where the dimensionless parameters in host region are shown in Fig. 1 and the dielectric constants of inclusions, $\varepsilon_{ii}^i = 10$.

the effective dielectric response at larger dielectric constant of inclusion, see Fig. 6.

For the external strain field, the effective elastic response is similar to the effective dielectric response under external electric field. In Fig. 7, for smaller elastic constants of inclusions, the piezoelectric constants enhance the effective elastic responses, and the dielectric constants of inclusions re-

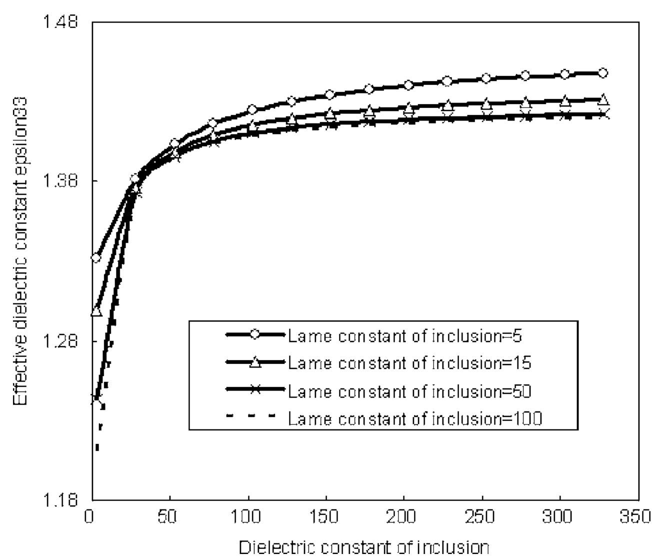

FIG. 6. The effective dielectric constants ε_{33}^* (in this figure ε_{33}^* is denoted by "epsilon33") against dielectric constants of inclusions, where the dimensionless parameters in host region are shown in Fig. 1 and the piezoelectric constants of inclusions, $e_{311}^i = e_{322}^i = e_{333}^i = e_{223}^i = e_{113}^i = 20$.

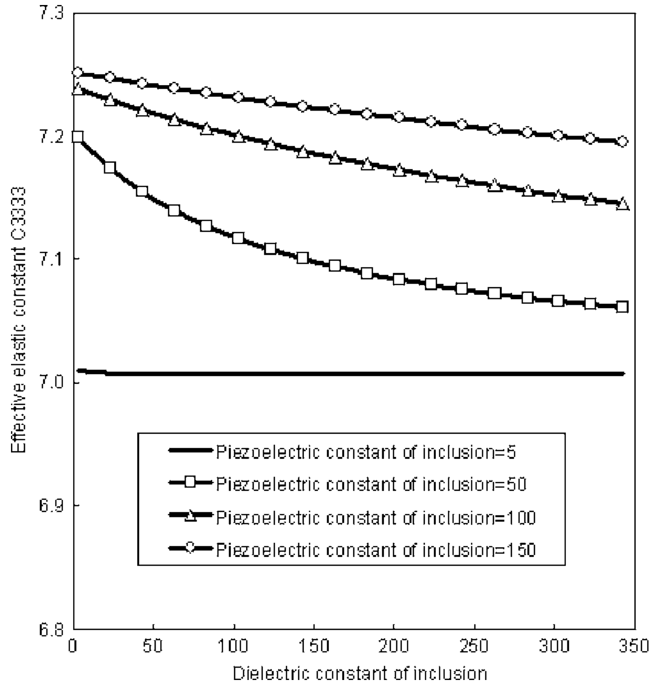

FIG. 7. The effective elastic constant C^*_{ijkl} (C^*_{3333} is denoted by C_{3333}) against dielectric constants of inclusions, where the dimensionless parameters in host region are shown in Fig. 1 and the Lame constant $\lambda^i=\mu^i=10$ for the inclusion spheres.

duce the effective elastic response at larger piezoelectric constants. However, if the piezoelectric constants are very small, the dielectric constants do not affect the effective elastic response. In Fig. 8, for larger elastic and dielectric constants of inclusions, the effective elastic responses are not clearly affected by the piezoelectric constants of inclusions.

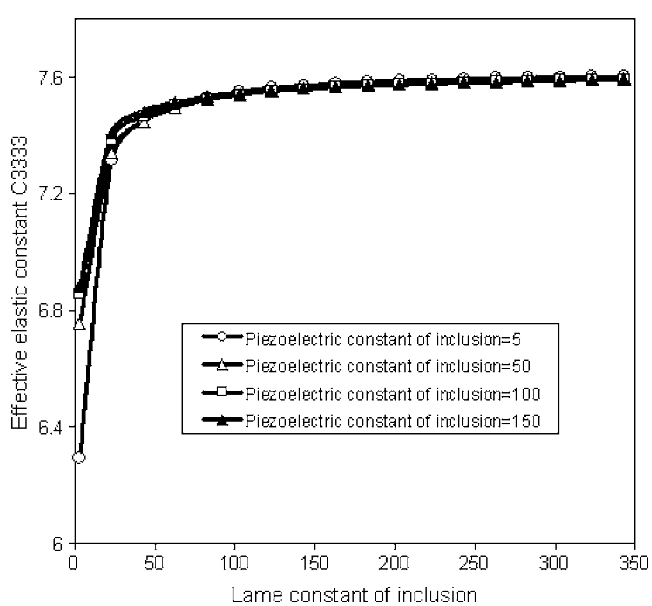

FIG. 8. The effective elastic constant C^*_{ijkl} (in this figure, $C^*_{3333}$is denoted by C_{3333}) against Lame constants of inclusions, where the dimensionless parameters in host region are shown in Fig. 1 and the dielectric constants $\varepsilon^i_{ii}=60$ for the inclusion spheres.

These numerical results show that the effective elastic response is also related to the piezoelectric, dielectric, and elastic constants.

From the above numerical analyses, it is found that effective properties of piezoelectric composites are coupled responses of the piezoelectric, dielectric, and elastic constants of host and inclusion materials. The piezoelectric constant is the key physical property parameter of inducing the stress and displacement fields under external strain and electric fields. Therefore, the effective dielectric and elastic responses of piezoelectric composites are related to the piezoelectric physical properties of materials, and they interact on each other through the piezoelectric constants of composites.

V. DISCUSSIONS AND CONCLUSIONS

The transformation field method is developed to investigate the effective piezoelectric, dielectric, and elastic properties of periodic piezoelectric composites having complicated shapes of anisotropic inclusions. In order to overcome the difficulties of matching complex boundary conditions of complicated structure inclusions, the transformation strain and electric fields are introduced into constitutive relations of electric displacement and stress fields, and a set of linear algebraically closed equations is built to solve the introduced transformation strain and electric fields. Numerical results show that the zero-order approximation and the high-order approximations can be used to estimate the effective responses of composites having low and high concentrations of inclusions, respectively. Effective responses of piezoelectric composites are clearly different from those of dielectric (or elastic) composites due to the piezoelectric parameters. In fact, effective dielectric and elastic coefficients of piezoelectric composites are coupled responses of piezoelectric, dielectric, and elastic physical properties of materials. The coupled effective piezoelectric property is also related to the dielectric and elastic properties of composites since piezoelectric properties of materials are important parameters of interchanging strain field and electric field. Therefore, the effective elastic (or dielectric) constant can be controlled by the dielectric (or elastic) and piezoelectric constants. This mechanism can be used to improve the elastic (or dielectric) properties of piezoelectric composites in practical applications.

Because this method does not impose any limitation on the shapes, the physical properties, the number, and the spatial distribution of inclusions in periodic cell, it is suitable to study the effective response of composites having spatial random distribution inclusions embedded in the host, where an available sample of random piezoelectric composites can be regarded as a unit cell so that the transformation field method is applied to estimate the bulk effective response of random composites. Furthermore, our method has developed Rayleigh's idea of periodic unit cell applied to estimate effective response of random composites and Eshelby's transformation field idea for solving the problem of complex shapes of inclusions. Thus, we have proposed a method for estimating the effective responses of random piezoelectric composites having the complex shapes of inclusions. Here, we should note that this method can be extended to study the piezoelectric composites having the nonlinear dielectric constituent,[20,21] and we believe that some interesting effective piezoelectric responses will be investigated, which is different from those of the linear piezoelectric composites. In the next work, we shall discuss effects of random piezoelectric composites containing inclusion shapes, inclusion arrangements, and anisotropic properties on the effective piezoelectric responses.

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