Manuscript

This is the Pre-Published Version

Remarks on minus (signed) total domination in graphs

Erfang Shan[a,b]*, T.C.E. Cheng[a]

[a]Department of Logistics, The Hong Kong Polytechnic University,

Hung Hom, Kowloon, Hong Kong

[b]Department of Mathematics, Shanghai University, Shanghai 200444, China

Abstract

A function $f : V(G) \to \{+1, 0, -1\}$ defined on the vertices of a graph G is a minus total dominating function if the sum of its function values over any open neighborhood is at least one. The minus total domination number $\gamma_t^-(G)$ of G is the minimum weight of a minus total dominating function on G. By simply changing "$\{+1, 0-1\}$" in the above definition to "$\{+1, -1\}$", we can define the signed total dominating function and the signed total domination number $\gamma_t^s(G)$ of G. In this paper we present a sharp lower bound on the signed total domination number for a k-partite graph, which results in a short proof of a result due to Kang et al. on the minus total domination number for a k-partite graph. We also give sharp lower bounds on γ_t^s and γ_t^- for triangle-free graphs and characterize the extremal graphs achieving these bounds.

MSC: 05C69

Keywords: Minus total domination; Signed total domination; k-partite graph; Triangle-free graph; Hypergraph;

1 Introduction

Let $\mathcal{H}$ be a *hypergraph* with *vertex set* S and *edge set* $\{A_1, \ldots, A_m\}$. Let α be an integer and $\mathbf{P}$ an arbitrary subset of integers $\mathbf{Z}$. The function $f : S \to \mathbf{P}$ defines an *α-dominating partition* of the hypergraph $\mathcal{H}$ with respect to $\mathbf{P}$, if

$$f(A) := \sum_{x \in A} f(x) \geq \alpha$$

* *Corresponding author.* Email address: efshan@shu.edu.cn (E.F. Shan), lgtcheng@inet.polyu.edu.hk (T.C.E. Cheng)

for every edge A in $\mathcal{H}$. The *α-domination number* of $\mathcal{H}$ with respect to $\mathbf{P}$ is defined as the minimum of such functions

$$\mathrm{dom}_\alpha(\mathcal{H}) := \min\{f(S) : f \text{ is } \alpha\text{-dominating partition}\}.$$

In particular, when $\mathbf{P} = \{+1, -1\}$ or $\{+1, 0, -1\}$, we obtain the *signed α-domination number* and *minus α-domination number*, denoted by mdom_α and sdom_α, respectively.

Now we consider a *simple graph* $G = (V, E)$ with *vertex set* V and *edge set* E. Let v be a vertex in V. The *open neighborhood* of v, $N_G(v)$, is defined as the set of vertices adjacent to v, i.e., $N_G(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, denote by $G[S]$ the graph induced by S. If $A, B \subseteq V(G)$, $A \cap B = \emptyset$, let $e(A, B)$ be the number of edges between A and B. We write $d_G(v)$ for the *degree* of v in G, and $\Delta(G)$ and $\delta(G)$ denote the *maximum degree* and the *minimum degree* of G, respectively. Let $k \geq 2$ be an integer. A graph $G = (V, E)$ is called *k-partite* if V admits a partition into k classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. Instead of '2-partite' one usually says *bipartite*. A *triangle-free graph* is a graph containing no cycles of length three.

A *signed total dominating function* of a graph G is defined in [12] as a function $f : V(G) \to \{+1, -1\}$ such that for every vertex v, $\sum_{u\in N(v)} f(u) \geq 1$, and the minimum cardinality of the sum $\sum_{v\in V} f(v)$ over all such functions is called the *signed total domination number*, denoted by $\gamma_t^s(G)$, i.e.,

$$\gamma_t^s(G) = \min\{f(V(G)) : f \text{ is a signed dominating function of } G\}.$$

A *minus total dominating function* is defined in [2] as a function of the form $f : V \to \{+1, 0, -1\}$ such that $\sum_{u\in N(v)} f(u) \geq 1$ for all $v \in V$. The *minus total domination number* for a graph G is

$$\gamma_t^-(G) = \min\{f(V(G)) : f \text{ is a minus total dominating function of } G\}.$$

From definitions, every signed total dominating function of G is clearly a minus total dominating function of G, so $\gamma_t^-(G) \leq \gamma_t^s(G)$. Using the notation for hypergraphs, we have that $\gamma_t^s(G) = \mathrm{sdom}_1(\mathcal{N}(G))$ and $\gamma_t^-(G) = \mathrm{mdom}_1(\mathcal{N}(G))$, where $\mathcal{N}$ is the *neighborhood hypergraph* on the vertex set $V(G)$ and its edges are the open neighborhoods $\{N_G(v) : v \in V(G)\}$.

Henning [5] and Harris and Hattingh [2] showed that the decision problems for the signed and minus total domination numbers of a graph are NP-complete respectively, even when the graph is restricted to a bipartite graph or a chordal graph. In [5] many bounds on γ_t^s of graphs were established. Yan et al. [11], and Wang and Shan [10] gave sharp upper bounds on γ_t^- for small-degree regular graphs. The literature on this topic of dominating functions is detailed in [3, 4].

In this paper we first give a sharp lower bound on $\gamma_t^s(G)$ of a k-partite graph G in terms of its order and minimum degree. This implies a short proof of a previous result due to Kang et al. [8], which gave a sharp lower bound on $\gamma_t^-(G)$ for a k-partite graph G. Further, we characterize extremal graphs on Kang et al.'s result. We also obtain sharp lower bounds on $\gamma_t^s(G)$ and $\gamma_t^-(G)$ for triangle-free graphs and characterize the extremal graphs achieving these bounds.

2 Main results

In this section we start with presenting a lower bound on the signed total domination number for k-partite graphs, where $k \geq 2$.

Theorem 1 *Let $G = (V, E)$ be a k-partite graph of order n with $\delta(G) \geq 1$ and let $c = \lceil (\delta(G) + 1)/2 \rceil$. Then*

$$\gamma_t^s(G) \geq \frac{k}{k-1}\left(-(c-1) + \sqrt{(c-1)^2 + 4\frac{k-1}{k}cn}\right) - n$$

and this bound is sharp.

Proof. Let $G = (V, E)$ be a k-partite graph of order n with vertex classes $V_1, V_2, \ldots, V_k$ and no isolated vertex. For $n = 2, 3$ the assertion is trivial, so we may assume that $n \geq 4$. Let $f : V \to \{+1, -1\}$ be a signed total dominating function on G with $f(V(G)) = \gamma_t^s(G)$ and let P and M be the sets of vertices in V that are assigned the value $+1$ and -1, respectively, under f. Further, let $P_i = P \cap V_i$, for $i = 1, \ldots, k$. Then, $n = |P| + |M|$ and $P = \bigcup_{i=1}^k P_i$. For convenience, let $|P| = p$, $|M| = m$, $|P_i| = p_i$ and $\delta(G) = \delta$. For every vertex $v \in M$, v is adjacent to at least $\lfloor d_G(v)/2 \rfloor + 1$ in P since $f(N(v)) \geq 1$, so $|N_G(v) \cap P| \geq \lfloor \delta/2 \rfloor + 1 = \lceil (\delta + 1)/2 \rceil = c$. Hence,

$$e(P, M) = \sum_{v \in M} |N_G(v) \cap P| \geq c(n - p). \tag{1}$$

On the other hand, for every vertex $v \in P_i$, it follows that $|N_G(v) \cap M| \leq |N_G(v) \cap (P - P_i)| - 1 \leq p - p_i - 1$. Hence,

$$e(P, M) = \sum_{v \in P} |N_G(v) \cap M| \leq \sum_{i=1}^k \sum_{v \in P_i} (|N_G(v) \cap (P - P_i)| - 1) \leq \sum_{i=1}^k p_i(p - p_i - 1). \tag{2}$$

Note that

$$k \sum_{i=1}^k p_i^2 \geq p^2. \tag{3}$$

Thus, combining with inequalities (1) and (2), we obtain

$$c(n-p) \leq e(P,M) \leq \frac{k-1}{k}p^2 - p, \tag{4}$$

or equivalently,

$$\frac{k-1}{k}p^2 + (c-1)p - cn \geq 0.$$

Hence,

$$p \geq \left(-(c-1) + \sqrt{(c-1)^2 + 4\frac{k-1}{k}cn}\right) \Big/ 2\left(\frac{k-1}{k}\right).$$

Therefore,

$$\gamma_t^s(G) = 2p - n \geq \frac{k}{k-1}\left(-(c-1) + \sqrt{(c-1)^2 + 4\frac{k-1}{k}cn}\right) - n.$$

That the bound is sharp may be seen as follows: For integers $k \geq 2$, let H_i be a complete bipartite graph with vertex classes V_i and U_i, where $|V_i| = k$ and $|U_i| = k^2 - k - 1$, for $i = 1, 2, \ldots, k$. We let $H(k)$ be the graph obtained from the disjoint union of $H_1, H_2, \ldots, H_k$ by joining each vertex of V_i in H_i with all the vertices of $\bigcup_{j=1, j\neq i}^{k} V_j$, and adding $(k-1)(k^2-k-1)$ edges between U_i with $\bigcup_{j=1, j\neq i}^{k} U_j$ so that each vertex of U_i has exactly $k-1$ neighbors in $\bigcup_{j=1, j\neq i}^{k} U_j$ while each vertex of $\bigcup_{j=1, j\neq i}^{k} U_j$ has exactly one neighbor in U_i for all $i = 1, 2, \ldots, k$. Let $Y_i = V_i \cup U_{i+1}$, where $i+1 \pmod{k}$. Then $H(k)$ is a k-partite graph of order $n = k(k^2-1)$ with vertex classes $Y_1, Y_2, \ldots, Y_k$ and $|Y_i| = k^2 - 1$. The graph $H(3)$ is shown in Fig. 1. Note that each vertex of U_i has minimum degree $2k-1$. Assigning to each vertex of $\bigcup_{i=1}^{k} V_i$ the value $+1$ and to each vertex of $\bigcup_{i=1}^{k} U_i$ the value -1, we produce a total signed dominating function f of H with weight

$$\begin{aligned} f(V(H(k)) &= k^2 - k(k^2 - k - 1) \\ &= k(-k^2 + 2k + 1) \\ &= \frac{k}{k-1}\left(-(c-1) + \sqrt{(c-1)^2 + 4\frac{k-1}{k}cn}\right) - n. \end{aligned}$$

Consequently,

$$\gamma_t^s(H(k)) = \frac{k}{k-1}\left(-(c-1) + \sqrt{(c-1)^2 + 4\frac{k-1}{k}cn}\right) - n.$$

□

Henning [5] showed that for a bipartite graph G, $\gamma_t^s(G) \geq 2\sqrt{2n} - n$. From Theorem 1, we can easily extend the result to k-partite graphs and characterize the extremal graphs achieving this bound. For this purpose, we recall a family $\mathcal{T}$ of graphs due to Kang et al. [8] as follows.

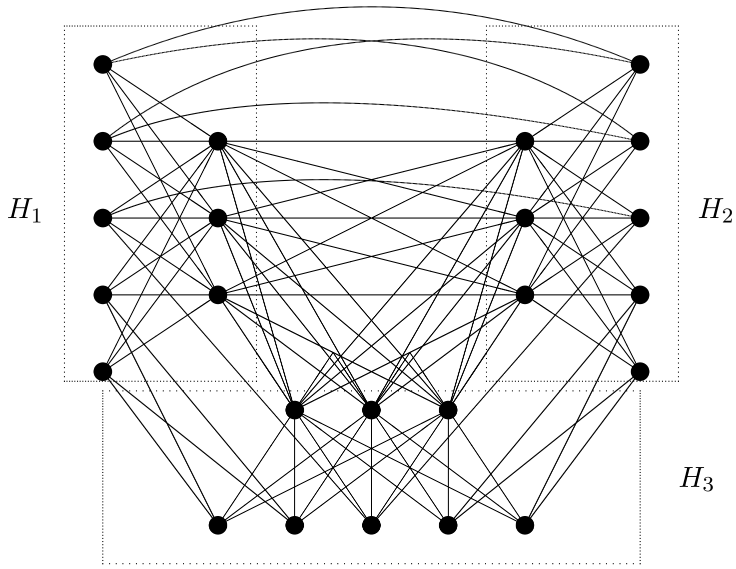

Figure 1: The graph $H(3)$.

For integers $r \geq 1$, $k \geq 2$, let H_i $(i = 1, 2, \ldots, k)$ be the graph obtained from the disjoint union of r stars $K_{1,(k-1)r-1}$ (the graph $K_{1,0}$ is regarded as K_1 when $r = 1$ and $k = 2$) with centers $V_i = \{x_{i,j} \mid j = 1, 2, \ldots, r\}$. Furthermore, let U_i denote the set of vertices of degree 1 in H_i that are not central vertices of stars and write $X_i = V_i \cup U_{i+1}$, where $i + 1 \pmod{k}$. We let $G_{k,r}$ be the k-partite graph obtained from the disjoint union of $H_1, H_2, \ldots, H_k$ by joining each center of H_i $(i = 1, 2, \ldots, k)$ with all the centers of $\bigcup_{j=1, j\neq i}^{k} H_j$. By construction, we know that $G_{k,r}$ is a k-partite graph of order $n = k(k-1)r^2$ with vertex classes $X_1, X_2, \ldots, X_k$ and $|X_i| = (k-1)r^2$. Let $\mathcal{T} = \{G_{k,r} \mid r \geq 1, k \geq 2\}$.

Theorem 2 *If $G = (V, E)$ is a k-partite graph of order n with no isolated vertex, then*

$$\gamma_t^s(G) \geq 2\sqrt{\frac{k}{k-1}n} - n,$$

where equality holds if and only if $G \in \mathcal{T}$.

Proof. Let $g(x) = \frac{k}{k-1}\left(-x + \sqrt{x^2 + 4\frac{k-1}{k}(x+1)n}\right) - n$. It is easy to check that $g'(x) > 0$ if $n \geq 2$, so $g(x)$ is a strictly monotone increasing function when $x \geq 0$. Note that $c \geq 1$, hence $\gamma_t^s(G) \geq g(c-1) \geq g(0)$, which implies the desired bound.

If $\gamma_t^s(G) = 2\sqrt{kn/(k-1)} - n$, then $c = 1$ as $g(x)$ is a strictly monotone function, and thus $\delta = 1$. Further, all the equalities hold in (1), (2) and (3). The equality in (3) implies that

$p_1 = p_2 = \ldots = p_k := r$. The equalities in (1) and (2) imply that each vertex of M has degree 1 and is exactly adjacent to a vertex of P, while each vertex of P_i has degree $p - p_i = kr - r$ in $G[P]$ and has exactly $p - p_i - 1 = r(k-1) - 1$ neighbors in M. It follows that $G \in \mathcal{T}$.

On the other hand, suppose $G \in \mathcal{T}$. Thus, there exist integers $r \geq 1$, $k \geq 2$ such that $G = G_{k,r}$. Assigning to all kr central vertices of the stars the value $+1$, and to all other vertices the value -1, we produce a signed total dominating function f of weight $f(V(G)) = kr - kr(2k-1) = 2kr - 2k^2r = 2\sqrt{kn/(k-1)} - n$. $\Box$

Now we present a short proof of a result due to Kang et al. [8] and here we further give a characterization of the extremal graphs.

Corollary 3 *If $G = (V, E)$ is a k-partite graph of order n with no isolated vertex, then*

$$\gamma_t^-(G) \geq 2\sqrt{\frac{k}{k-1}n} - n,$$

where equality holds if and only if $G \in \mathcal{T}$.

Proof. Let $f : V \to \{+1, 0, -1\}$ be a minus total dominating function on G with $f(V(G)) = \gamma_t^-(G)$ and let Q be the set of vertices in $V(G)$ that are assigned the value 0. Further, Let $G' = G - Q$ and suppose that G' is a k'-partite graph of order n'. Then $2 \leq k' \leq k$ and $2 \leq n' \leq n$. Clearly, $f' = f|_{G'}$ is a signed total dominating function on G', so $\gamma_t^s(G') \leq f'(V(G')) = f(V(G))$. By Theorem 2, we have

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{\frac{k'}{k'-1}n'} - n'.$$

Let $h(x, y) = 2\sqrt{yx/(y-1)} - x$. It is easy to see that $\partial h(x, y)/\partial x < 0$ and $\partial h(x, y)/\partial y < 0$ for $x, y \geq 2$, so $h(x, y)$ is a strictly monotone decreasing function on variables x and y, respectively. This implies that

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{\frac{k}{k-1}n} - n.$$

The following theorem implies the fact that the equality holds if and only if $G \in \mathcal{T}$. $\Box$

Finally, by Theorem 2 and Corollary 3, we obtain the following extremal result on the minus total domination and signed total domination of a k-partite graph.

Theorem 4 *If $G = (V, E)$ is a k-partite graph of order n with no isolated vertex, then the following statements are equivalent.*

(i) $\gamma_t^s(G) = 2\sqrt{\frac{k}{k-1}n} - n$;

(ii) $\gamma_t^-(G) = 2\sqrt{\frac{k}{k-1}n} - n$;

(iii) $G \in \mathcal{T}$.

Proof. By Theorem 2 and Corollary 3, we have $\gamma_t^s(G) = \gamma_t^-(G) \geq 2\sqrt{kn/(k-1)} - n$, so it suffices to prove that (ii)⇒(iii). We use the notation introduced in the proof of Corollary 3. If $\gamma_t^-(G) = 2\sqrt{kn/(k-1)} - n$, then $h(k', n') = h(k, n)$. Observe the fact that $h(x, y)$ is a strictly monotone function on variables x and y, respectively, when $x, y \geq 2$, which implies $k' = k$, $n' = n$. Hence $Q = \emptyset$. Thus f is also a minimum signed total dominating function, i.e., $\gamma_t^s(G) = 2\sqrt{kn/(k-1)} - n$. The result immediately follows from Theorem 2. $\Box$

Recall a subclass $\mathcal{F} = \{G_{2,r} \mid r \geq 1\}$, constructed by Henning [5], of $\mathcal{T}$. Clearly, each $G_{2,r}$ of $\mathcal{F}$ is a bipartite graph of order $n = 2r^2$ with vertex classes X_1, X_2 and $|X_i| = r^2$. As a special case of Theorem 4, we obtain the following result due to Henning [5].

Corollary 5 ([5]) *If G is a bipartite graph of order n with $\delta(G) \geq 1$, then $\gamma_t^s(G) \geq 2\sqrt{2n} - n$, where equality holds if and only if $G \in \mathcal{F}$.*

We recall that $\gamma_t^s(G) \geq \gamma_t^-(G)$ for any graph G. Next we show that the minus total domination number of a triangle-free graph has the above lower bound and we characterize the extremal graphs attaining this bound.

The following result is well-known and useful.

Lemma 6 ([1]) *For any triangle-free graph G of order p, $|E(G)| \leq p^2/4$, where equality holds if and only if $G = K_{\frac{p}{2},\frac{p}{2}}$ and $K_{\frac{p}{2},\frac{p}{2}}$ is a balance complete bipartite graph.*

To achieve our goal, we first give a sharp lower bound on $\gamma_t^s(G)$ for a triangle-free graph G.

Theorem 7 *Let G be a triangle-free graph of order n with $\delta(G) \geq 1$ and let $c = \lceil(\delta(G)+1)/2\rceil$. Then*

$$\gamma_t^s(G) \geq 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n.$$

Proof. We first prove that $\gamma_t^s(G) \geq 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n$ for a triangle-free graph G. Let $f : V \to \{+1, -1\}$ be a signed total dominating function of G with $f(V(G)) = \gamma_t^s(G)$

and let $P = \{v \in V(G) \mid f(v) = +1\}$, $M = \{v \in V(G) \mid f(v) = -1\}$. Further, let $|P| = p$ and $|M| = m$. Obviously, $P \cup M$ is a partition of $V(G)$. Then $\gamma_t^s(G) = |P| - |M| = 2p - m$. Similar to the argument given in Theorem 1, by estimating the number of edges between P and M, we get

$$e(P, M) = \sum_{v \in M} |N_G(v) \cap P| \geq cm \tag{5}$$

and

$$e(P, M) = \sum_{v \in P} |N_G(v) \cap M| \leq \sum_{v \in P} (|N_G(v) \cap P| - 1) = \sum_{v \in P} d_{G[P]}(v) - p. \tag{6}$$

By Lemma 6, we further obtain

$$c(n - p) \leq e(P, M) \leq 2|E(G[P])| - p \leq \frac{p^2}{2} - p, \tag{7}$$

this implies that $p \geq -(c-1) + \sqrt{(c-1)^2 + 2cn}$. Hence,

$$\gamma_t^s(G) = 2p - n \geq 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n.$$

That the bound is sharp may be seen as follows: For integers $k \geq 2$, let $K_{k,k}$ be a complete graph with bipartition (X_1, X_2) and J a $(k-2)$-regular bipartite graph with bipartition (Y_1, Y_2). Let G_k be the graph obtained from the disjoint union of $K_{k,k}$ and J by adding $k(k-1)$ edges between X_i and Y_i for $i = 1, 2$ so that each vertex of X_i is adjacent to $k-1$ vertices of Y_i while each vertex of Y_i is also adjacent to $k-1$ vertices of X_i. Then G_k is a bipartite graph of order $n = 4k$ with bipartition $(X_1 \cup Y_2, X_2 \cup Y_1)$. The graph G_3 is shown in Fig. 2. Note that G_k has minimum degree $2k-3$, so $c = k-1$. Let $f : V(G_k) \to \{+1, -1\}$ be defined as follows: Let $f(v) = 1$ if $v \in X_1 \cup X_2$ and let $f(v) = +1$ otherwise. Then f is a signed total dominating function of G_k with weight $0 = 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n$. So $\gamma_t^s(G) = 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n$. $\square$

Applying Theorem 7, we obtain the following result.

Theorem 8 *If G is a triangle-free graph of order n with $\delta(G) \geq 1$, then*

$$\gamma_t^-(G) \geq 2\sqrt{2n} - n,$$

where equality holds if and only if $G \in \mathcal{F}$.

Proof. Define

$$h_1(x) = 2\left(-x + \sqrt{x^2 + 2(x+1)n}\right) - n.$$

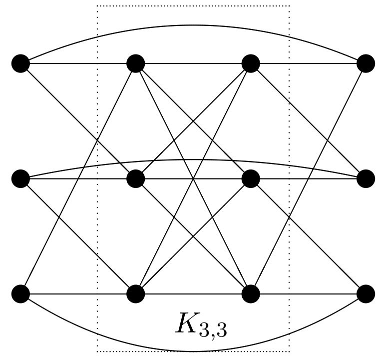

Figure 2: The graph G_3.

It is easy to check that $h_1(x)$ is a strictly monotone increasing function when $x \geq 0$ and $n \geq 2$. Since $c \geq 1$, $h_1(c-1) \geq h_1(0)$. Thus, by Theorem 7, we have $\gamma_t^s(G) \geq 2\sqrt{2n} - n$.

We now show that $\gamma_t^-(G) \geq 2\sqrt{2n} - n$. Let $f : V \to \{+1, 0, -1\}$ be a minus total dominating function on G with $f(V(G)) = \gamma_t^-(G)$ and let Q be the set of vertices in $V(G)$ that are assigned the value 0. Further, Let $G' = G - Q$ and $|V(G')| = n'$. Then G' is triangle-free. Clearly, $f' = f|_{G'}$ is a signed total dominating function on G', so $\gamma_t^s(G') \leq f'(V(G')) = f(V(G))$. Observe that $h_2(x) = 2\sqrt{2x} - x$ is a strictly monotone decreasing function when $x > 1$. Hence,

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{2n'} - n' \geq 2\sqrt{2n} - n.$$

If $\gamma_t^-(G) = 2\sqrt{2n} - n$, then $n' = n$ as $h_2(x)$ is a strictly monotone function. This implies that $Q = \emptyset$. Hence f is a signed total dominating function on G, and thus $\gamma_t^s(G) \leq \gamma_t^-(G)$. So $\gamma_t^s(G) = \gamma_t^-(G) = 2\sqrt{2n} - n$. This means that $\gamma_t^s(G) = h_1(0)$, so $c = 1$ and equalities hold for the inequalities (5), (6) and (7) in the proof of Theorem 7. The chain of equalities in (7) implies that $|E(G[P])| = p^2/4$. By Lemma 6, $G[P]$ is a (balance) complete bipartite graph $K_{\frac{p}{2},\frac{p}{2}}$. Further, the chain of equalities implies that each vertex of M has degree 1 and is precisely adjacent to a vertex of P, while each vertex of P has degree $p-1$ and is precisely adjacent to $p/2 - 1$ vertices of M. Then G is a bipartite graph. By Corollary 5, $G \in \mathcal{F}$. On the other hand, suppose $G \in \mathcal{F}$. Then, by Corollary 5 again, $\gamma_t^s(G) = 2\sqrt{2n} - n$. Since $\gamma_t^s(G) \geq \gamma_t^-(G) \geq 2\sqrt{2n} - n$, we have $\gamma_t^-(G) = 2\sqrt{2n} - n$. $\square$

As an immediate consequence of Corollary 5 and Theorems 7 and 8, we have

Theorem 9 *If G is a triangle-free graph of order n with $\delta(G) \geq 1$, then the following statements are equivalent.*

(*i*) $\gamma_t^s(G) = 2\sqrt{2n} - n$,

(*ii*) $\gamma_t^-(G) = 2\sqrt{2n} - n$,

(*iii*) $G \in \mathcal{F}$.

3 Conclusion

The minus (reps. signed) total domination problem can be seen as a proper generalization of the classical total domination problem and minus (reps. signed) domination problem. In this paper we studied lower bounds on minus and signed total domination numbers of k-partite graphs and triangle-free graphs and extremal graphs achieving these bounds. We do not know whether the minus total domination number of a triangle-free graph has the same lower bound as described in Theorem 7. Moreover, Kang et al. [7] and Wang et al. [9] independently gave sharp lower bound on the minus domination number for bipartite graphs. Kang et al. [6] further extented the result to k-partite graphs. The method in this paper may be used to characterize the extremal graphs of k-partite graphs attaining the lower bound.

Acknowledgements

This research was partially supported by The Hong Kong Polytechnic University under grant number G-YX69, the National Nature Science Foundation of China under grant 10571117 and the Development Foundation of Shanghai Education Committee under grant 05AZ04.

References

[1] B. Bollobás, *Modern Graph Theory*, Springer, New York, 1998.

[2] L. Harris, J.H. Hattingh, The algorithm complexity of certain functional variations of total domination in graphs, *Australasian J. Combinatorics* 29 (2004) 143-156.

[3] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] M.A. Henning, Dominating function in graphs, *Domination in Graphs: Advanced Topics*, Vol.II, Marcel Dekker Inc., New York, 1998, pp. 31-62.

[5] M.A. Henning, Signed total domination in graphs, *Discrete Math.* 278 (2004) 109-125.

[6] Liying Kang, Hye Kyung Kim, Moo Young Sohn, Minus domination number in k-partite graphs, *Discrete Math.* 277 (2004) 295-300.

[7] Liying Kang, H. Qiao, Erfang Shan, and D.Z. Du, Lower bounds on the minus domination and k-subdomination numbers, *Theoret. Comput. Sci.* 296 (2003) 89-98.

[8] Liying Kang, Erfang Shan, L. Caccett, Total minus domination in k-partite graphs, *Discrete Math.* 306 (2006) 1771-1775.

[9] C. Wang, J. Mao, A proof of a conjecture of minus domination in graphs, *Discrete Math.* 256 (2002) 519-521.

[10] Haichao Wang, Erfang Shan, Upper minus total domination of a 5-regular graph, *Ars Combin.* (accepted).

[11] H. Yan, X.Q. Yang, Erfang Shan, Upper minus total domination in small-degree regular graphs, *Discrete Math.* (accepted).

[12] B. Zelinka, Signed total domination number of a graph, *Czechoslovak Math. J.* 51 (2001) 225-229.