Hamilton-Connectivity of 3-Domination Critical Graphs with $\alpha = \delta + 1 \geq 5$

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Abstract: A graph $G$ is 3-domination critical if its domination number $\gamma$ is 3 and the addition of any edge decreases $\gamma$ by 1. Let $G$ be a 3-domination critical graph with toughness more than one. It was proved $G$ is Hamilton-connected for the cases $\alpha \leq \delta$ (Discrete Mathematics 271 (2003) 1-12) and $\alpha = \delta + 2$ (European Journal of Combinatorics 23(2002) 777-784). In this paper, we show $G$ is Hamilton-connected for the case $\alpha = \delta + 1 \geq 5$.

Key words: Domination-critical graph, Hamilton-connectivity

1. Introduction

Let $G = (V(G), E(G))$ be a graph. A graph $G$ is said to be $t$-tough if for every cutset $S \subseteq V(G)$, $|S| \geq \omega(G - S)$, where $\omega(G - S)$ is the number of components of $G - S$. The toughness of $G$, denoted by $\tau(G)$, is defined to be $\min\{|S|/\omega(G - S) \mid S$ is a cutset of $G\}$. Let $u, v \in V(G)$ be any two distinct vertices. We denote by $p(u, v)$ the length of a longest path connecting $u$ and $v$. The codiameter of $G$, denoted by $d^*(G)$, is defined to be $\min\{p(u, v) \mid u, v \in V(G)\}$. A graph $G$ of order $n$ is said to be Hamilton-connected if $d^*(G) = n - 1$, i.e., every two distinct vertices are joined by a hamiltonian path. A graph $G$ is called $k$-domination critical, abbreviated as $k$-critical, if $\gamma(G) = k$ and $\gamma(G + e) = k - 1$ holds for any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of $G$. The concept of domination critical graphs was introduced by Sumner and Blitch in [11]. Given three vertices $u, v$ and $x$ such that $\{u, x\}$ dominates $V(G) - \{v\}$ but not $v$, we will write $[u, x] \rightarrow v$. It was observed in [11] that if $u, v$ are any two nonadjacent vertices of a 3-critical graph $G$, then since $\gamma(G + uv) = 2$, there exists a vertex $x$ such that either $[u, x] \rightarrow v$ or $[v, x] \rightarrow u$. If $U, V \subseteq V(G)$ and $U$ dominates $V$, that is, $V$ is contained in the closed neighborhood of $U$, we write $U \succ V$; otherwise we write $U \not\succ V$. For notations not defined here, we follow [5].
It was conjectured in [10] that every connected 3-critical graph of order more than 6 has a hamiltonian path. This was proved by Wojcicka [13] who in turn conjectured that every connected 3-critical graph $G$ with $\delta(G) \geq 2$ has a hamiltonian cycle. Wojcicka’s conjecture has now been proved completely, see [8, 9, 12] or [2]. It is well known that if a graph $G$ has a hamiltonian cycle, then $\tau(G) \geq 1$ and the converse does not hold in general. However, this is not the case when $G$ is 3-critical. Noting that $\tau(G) < 1$ if $G$ is a connected 3-critical graph with $\delta(G) = 1$, we see that the following theorem is a direct consequence of the validity of Wojcicka’s conjecture.

**Theorem 1.** Let $G$ be a connected 3-critical graph. Then $G$ has a hamiltonian cycle if and only if $\tau(G) \geq 1$.

For Hamilton-connectivity, it is known that if a graph $G$ is Hamilton-connected, then $\tau(G) > 1$ and the converse need not hold. However, motivated by Theorem 1, Chen et al. [5] posed the following.

**Conjecture 1** (Chen et al. [5]). A connected 3-critical graph $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

In the same paper, they proved that the conjecture is true when $\alpha(G) \leq \delta(G)$.

**Theorem 2** (Chen et al. [5]). Let $G$ be a connected 3-critical graph with $\alpha(G) \leq \delta(G)$. Then $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

Let $G$ be a 3-connected 3-critical graph. It is shown in [6] that $\tau(G) \geq 1$ and $\tau(G) = 1$ if and only if $G$ belongs to a special infinite family $\mathcal{G}$ described in [6]. Since $\alpha(G) = \delta(G) = 3$ for each $G \in \mathcal{G}$, it is easy to obtain that $\tau(G) > 1$ if $\alpha(G) \geq \delta(G) + 1$.

In [7], Chen et al. showed that the conjecture holds when $\alpha(G) = \delta(G) + 2$.

**Theorem 3** (Chen et al. [7]). Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 2$. Then $G$ is Hamilton-connected.

By a result of Favaron et al. [8] that $\alpha(G) \leq \delta(G) + 2$ for any connected 3-critical graph $G$, we can see the conjecture has only one case $\alpha(G) = \delta(G) + 1$ unsolved. In this paper, we will show that the conjecture is true when $\alpha(G) = \delta(G) + 1 \geq 5$. The main result of this paper is the following.

**Theorem 4.** Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 \geq 5$. Then $G$ is Hamilton-connected.

Noting that $\tau(G) > 1$ implies $\delta(G) \geq 3$, we can see that the conjecture is still open for the case $\alpha(G) = \delta(G) + 1 = 4$.

Now, we restate a result due to Chen et al. for later use.
Theorem 5 (Chen et al. [3]). Let $G$ be a 3-connected 3-critical graph of order $n$. Then $d^*(G) \geq n - 2$.

2. Properties of Maximum Independent Set

In order to prove Theorem 4, we need to use a classical tool — closure operation in hamiltonian theory. In 1976, Bondy and Chvátal defined a (Hamilton-connected) closure operation of a graph.

Theorem 6 (Bondy and Chvátal [1]). Let $G$ be a graph of order $n$. Let $a$ and $b$ be nonadjacent vertices of $G$ such that $d(a) + d(b) \geq n + 1$. Then for any two distinct vertices $x, y$, $p(x, y) = n - 1$ in $G$ if and only if $p(x, y) = n - 1$ in $G + ab$.

Now, given a graph $G$ of order $n$, repeat the following recursive operation, named Bondy-Chvátal closure operation, as long as possible: For each pair of nonadjacent vertices $a$ and $b$, if $d(a) + d(b) \geq n + 1$, then add the edge $ab$ to $G$. We denote by $cl(G)$ the resulting graph and call it the Bondy-Chvátal (Hamilton-connected) closure of $G$.

By Theorem 6 we get the following.

Theorem 7 (Bondy and Chvátal [1]). Let $G$ be a graph of order $n$. Then for any two distinct vertices $x, y$, $p(x, y) = n - 1$ in $G$ if and only if $p(x, y) = n - 1$ in $cl(G)$.

Let $G$ be a 3-critical graph of order $n$, $\alpha(G) = \delta(G) + 1$ and $v_0 \in V(G)$ with $d(v_0) = \delta(G) = k \geq 3$. Suppose $N(v_0) = \{v_1, \ldots, v_k\}$ and $I = \{v_0, w_1, \ldots, w_k\}$ is an independent set. In this section, we will give some properties of $I$ in $G$ and $G^* = cl(G)$.

The following lemma restates a lemma due to Sumner and Blitch [11], which has proven to be of considerable use in dealing with 3-critical graphs. In [11] they considered the case $l \geq 4$, which guarantees $P(U) \cap U = \emptyset$. For the cases $l = 2$ and $l = 3$, Lemma 2.1 can be easily verified since $G$ is a 3-critical graph.

Lemma 2.1. Let $G$ be a connected 3-critical graph and $U$ an independent set of $l \geq 2$ vertices. Then there exist an ordering $u_1, u_2, \ldots, u_l$ of the vertices of $U$ and a sequence $P(U) = (y_1, y_2, \ldots, y_{l-1})$ of $l - 1$ distinct vertices such that $[u_i, y_i] \rightarrow u_{i+1}$, $1 \leq i \leq l - 1$.

The next lemma is a useful consequence of Lemma 2.1.

Lemma 2.2 (Favaron et al. [8]). Let $U$ be an independent set of $l \geq 3$ vertices of a 3-critical graph $G$ such that $U \cup \{v\}$ is independent for some $v \notin U$. Then the sequence $P(U)$ defined in Lemma 2.1 is contained in $N(v)$.

Since $I$ is an independent set of order at least 4, by Lemmas 2.1 and 2.2, we may assume without loss of generality that
\[ [w_i, v_i] \rightarrow w_{i+1} \text{ for } 1 \leq i \leq k - 1. \] (2-1)

By (2-1), it is easy to obtain the following.

\[ v_j v_{j+1} \in E(G) \text{ for } 1 \leq j \leq k - 2. \] (2-2)

**Lemma 2.3.** If \( w_i v_k \notin E(G) \) with \( i \neq 1 \), then \( G[N(v_0) - \{v_{i-1}, v_k\}] \) is a clique. If \( w_1 v_k \notin E(G) \), then \( G[N(v_0) - \{v_k\}] \) is a clique.

**Proof.** Let \( v_1, v_m \in N(v_0) - \{v_{i-1}, v_k\} \) with \( l \leq m - 1 \). If \( l = m - 1 \), then \( v_l v_m \in E(G) \) by (2-2). If \( l \leq m - 2 \), then since \( w_{l+1} w_{m+1} \notin E(G) \), there is some vertex \( z \) such that \( [w_{l+1}, z] \rightarrow w_{m+1} \) or \( [w_{m+1}, z] \rightarrow w_{l+1} \). Since \( k \geq 3 \), by Lemma 2.2 we have \( z \in N(v_0) \). Since \( w_l v_k \notin E(G) \), we have \( z \neq v_k \). By (2-1), either \( [w_{l+1}, v_m] \rightarrow w_{m+1} \) or \( [w_{m+1}, v_l] \rightarrow w_{l+1} \). In both cases, we have \( v_l v_m \in E(G) \) and hence \( G[N(v_0) - \{v_{i-1}, v_k\}] \) is a clique. As for the latter part, the proof is similar. \( \Box \)

**Lemma 2.4.** If \( w_i v_k \notin E(G) \) with \( i \neq 1 \), then \( [w_1, v_{j-1}] \rightarrow w_j \) for \( j \geq 3 \) and \( j \neq i \).

**Proof.** Since \( w_1 w_j \notin E(G) \), by Lemma 2.2, there is some \( z \in N(v_0) \) such that \( [w_1, z] \rightarrow w_{j+1} \) or \( [w_{j+1}, z] \rightarrow w_1 \). By (2-1) and the assumption, we can see that \( [w_j, z] \rightarrow w_1 \) is impossible for any \( z \in N(v_0) \) and hence \( [w_1, v_{j-1}] \rightarrow w_j \). \( \Box \)

**Lemma 2.5.** If \( [v_0, z] \rightarrow w_i \) for some \( i \) with \( 1 \leq i \leq k - 1 \), then \( z \notin N(v_0) \) and if \( [v_0, v_i] \rightarrow w_k \) for some \( v_i \in N(v_0) \), then \( l = k - 1 \).

**Proof.** If \( i = 1 \) and \( z \in N(v_0) \), then \( z = v_k \) by (2-1). Thus, we have \( \{v_2, v_k\} \succ V(G) \) by Lemma 2.3, a contradiction. If \( i \geq 2 \) and \( z \in N(v_0) \), then by (2-1) we have \( z = v_{i-1} \) or \( v_k \) and \( N(v_0) - \{v_{i-1}, v_k\} \subseteq N(v_i) \). If \( z = v_{i-1} \), then \( w_i v_k \notin E(G) \) for otherwise \( \{v_{i-1}, w_i\} \succ V(G) \). Since \( [w_i, v_i] \rightarrow w_{i+1} \), \( v_i v_k \in E(G) \). By Lemma 2.4, we have \( [w_1, v_i] \rightarrow w_{i+1} \), which implies \( v_i w_i \in E(G) \). Thus by Lemma 2.3, we have \( \{v_{i-1}, v_i\} \succ V(G) \), a contradiction. If \( z = v_k \) and \( i \neq 2 \), then by Lemma 2.3 we have \( \{v_{i-2}, v_k\} \succ V(G) \), a contradiction. If \( z = v_k \) and \( i = 2 \), then by Lemma 2.4 we have \( [w_1, v_2] \rightarrow w_3 \), which implies \( v_2 w_2 \in E(G) \) and hence \( \{v_2, v_k\} \succ V(G) \) by Lemma 2.3, also a contradiction. Thus, \( z \notin N(v_0) \).

If \( [v_0, v_i] \rightarrow w_k \) for some \( v_i \in N(v_0) \), then by (2-1), we have \( l = k - 1 \) or \( k \). If \( l = k \), then by Lemma 2.3, we have \( \{v_{k-2}, v_k\} \succ V(G) \), a contradiction. \( \Box \)

**Lemma 2.6.** If \( [v_0, v_{k-1}] \rightarrow w_k \), then \( N(v_k) \cap \{v_1, \ldots, v_{k-1}, w_k\} = \emptyset \) and \( \{v_1, \ldots, w_{k-1}\} \subseteq N(v_k) \).

**Proof.** By (2-1), we have \( N(v_0) - \{v_{k-1}, v_k\} \subseteq N(v_k) \). If \( w_k v_k \notin E(G) \), then since \( [v_0, v_{k-1}] \rightarrow w_k \), we have \( \{v_{k-1}, w_k\} \succ V(G) \) and hence \( w_k v_k \notin E(G) \). By Lemma 2.3, \( G[N(v_0) - \{v_{k}, v_k\}] \) is a clique. Thus, if \( v_{k-1} v_k \in E(G) \), then \( \{v_{k-1}, v_1\} \succ V(G) \) and if \( v_1 v_k \in E(G) \) for some \( i \) with \( 1 \leq i \leq k - 2 \), then \( \{v_{k-1}, v_i\} \succ V(G) \), a contradiction. Since \( N(v_k) \cap \{v_1, \ldots, v_{k-1}\} = \emptyset \), by (2-1) we have \( \{w_1, \ldots, w_{k-1}\} \subseteq N(v_k) \). \( \Box \)
Lemma 2.7. If \([v_0, v_{k-1}] \rightarrow w_k\), then \(G[N(v_0) - \{v_k\}]\) is a clique and \(N(w_k) \cap N(v_k) = \emptyset\).

Proof. By Lemma 2.6, \(v_kw_k \notin E(G)\). By Lemma 2.3, \(G[N(v_0) - \{v_k, v_k\}]\) is a clique. By (2-1), \(v_{k-2}v_{k-1} \in E(G)\). For \(1 \leq i \leq k - 3\), there is some \(z \in N(v_0)\) such that \([w_{i+1}, z] \rightarrow w_k\) or \([w_k, z] \rightarrow w_{i+1}\) by Lemma 2.2. By (2-1) and Lemma 2.6, we can see that \(\{w_{i+1}, v_k\} \neq v_i\) and \(\{w_k, v_k\} \neq v_{k-1}\), which implies \(z \neq v_k\) and hence \(z = v_i\) or \(v_{k-1}\). In both cases, we have \(v_iv_{k-1} \in E(G)\), which implies \(G[N(v_0) - \{v_k\}]\) is a clique. If \(N(w_k) \cap N(v_k) \neq \emptyset\), then since \([v_0, v_{k-1}] \rightarrow w_k\) and \(G[N(v_0) - \{v_k\}]\) is a clique, we can see that \(\{v_{k-1}, z\} \rightarrow V(G)\) for any \(z \in N(w_k) \cap N(v_k)\), a contradiction. 

Lemma 2.8. If \(k \geq 4\), \([v_0, v_{k-1}] \rightarrow w_k\) and for each \(w_i\) with \(1 \leq i \leq k - 1\), there is no vertex \(z\) such that \([v_0, z] \rightarrow w_i\), then \(N^*[w_1] = N_G, [w_1] = V(G)\).

Proof. Let \(U = V(G) - (I \cup N(v_0))\), \(N(w_1) \cap U = U_1\) and \(U_2 = U - U_1\). In order to prove the result, we need the following claims.

Claim 2.1. \(N(w_i) \cap N(v_i) \cap U \neq \emptyset\) for \(1 \leq i \leq k - 2\).

Proof. By the assumption, there is some vertex \(z\) such that \([w_{i+1}, z] \rightarrow v_0\). Obviously \(z \in U\). By (2-1), we have \(z \in N(w_i) \cap N(v_i)\) and hence \(z \in N(w_i) \cap N(v_i) \cap U\).

By Lemmas 2.4 and 2.6, we have \([w_1, v_i] \rightarrow w_{i+1}\) for \(2 \leq i \leq k - 2\) and hence
\[w_iw_i \in E(G)\] for \(2 \leq i \leq k - 2\). (2-3)

Claim 2.2. \(d(w_2) \geq \delta + 1\) and if \(d(w_2) = \delta + 1\), then \(d(v_2) \geq n - \delta\).

Proof. By the assumption, we may assume \([w_3, z] \rightarrow v_0\), which implies \(z \in N(v_2) \cap N(w_2) \cap U\). If \(d(w_2) = \delta\), then \(N_U(w_2) = \{z\}\) by (2-3). Since \([w_3, z] \rightarrow v_0\), by (2-1) and Lemma 2.7 we have \(V(G) - \{w_3, v_k\} \subseteq N[v_2]\). By Lemma 2.6, \(w_3v_k \in E(G)\).

Thus, \(v_2, v_3\) \(\rightarrow V(G)\), a contradiction. Since \(k \geq 4\) and \([w_2, v_2] \rightarrow w_3\), by (2-1) and Claim 2.1, we have \(|N(w_2) \cap N(v_2)| \geq 2\). By (2-3), \(w_2v_2 \in E(G)\). Thus, we have \(d(w_2) + d(v_2) \geq n + 1\) and the conclusion follows.

Claim 2.3. For any \(u \in N_U(w_k)\), either \(uw_2 \in E(G)\) or \(uw_3 \in E(G)\).

Proof. Suppose \(u \in N_U(w_k)\) and \(w_2, w_3 \notin N(u)\). By Lemma 2.2, there is some vertex \(z \in N(v_0)\) such that \([w_3, z] \rightarrow u\) or \([u, z] \rightarrow w_3\). If \([u, z] \rightarrow w_3\), then we must have \(z = v_2\), which is impossible since \(u, v_2 \neq v_k\) by Lemmas 2.6 and 2.7. If \([w_3, z] \rightarrow u\), then since \([w_2, v_2] \rightarrow w_3\) and \(uw_2 \notin E(G)\), we have \(z \neq v_2\). By (2-1) and Lemma 2.6, we can see \(z \in N(v_0) - \{v_2\}\) is also impossible, a contradiction.

Claim 2.4. \(v_{k-1} \in N^*(w_k)\).

Proof. Since \([v_0, v_{k-1}] \rightarrow w_k\), by Lemma 2.7 we have \(d(v_{k-1}) = n - 3\). Noting that
\[d(w_k) \geq \delta \geq 4, \text{ we have } d(v_{k-1}) + d(w_k) \geq n + 1 \text{ and hence } v_{k-1} \in N^*(w_k).\]

**Claim 2.5.** If \(d(w_2) = \delta + 1\) and \(d(w_3) = \delta\), then \(v_k \in N^*(w_k)\).

**Proof.** Let \(N(w_k) \cap U = U_3\) and \(U_4 = U - U_3\). By (2-1) and Lemma 2.6, we have \(v_{k-1}, v_k \notin N(w_k)\) and hence \(|U_3| \geq 2\). By the assumption, there are some \(z_i \in U\) such that \([w_i, z_i] \to v_0\) for \(i = 1, 2\). If \(z_1 \neq z_2\), then \(d_U(w_3) \geq 2\). If \(k = 4\), then \(w_3v_3 \in E(G)\) by the assumption and if \(k \geq 5\), then \(w_3v_3 \in E(G)\) by (2-3). By (2-1) and Lemma 2.6, \(N(v_0) \setminus \{v_2, v_3\} \subseteq N(w_3)\). Thus we have \(d(w_3) \geq \delta + 1\) and hence we may assume \(z_1 = z_2 = u_1\). Obviously, \(u_1 \in U_3\). Since \(d(w_2) = \delta + 1\) and \(d(w_3) = \delta\), by Claim 2.3, we have \(|U_3| = 2\) and \(U_4 = U_3\). Since \([w_2, u_1] \to v_0, v_{k-1} \in N(w_2) \cap N(u_1)\) and \(w_2u_1 \in E(G)\), we have \(d(u_1) + d(w_2) \geq n\), which implies \(d(u_1) \geq n - \delta - 1\). We now show \([w_k, v_k] \to v_{k-1}\). If \(U_4 = \emptyset\), then by (2-1) and Lemma 2.6, \([w_k, v_k] \to v_{k-1}\). If \(U_4 \neq \emptyset\), then since \(u_1w_3 \in E(G)\) and \(d(w_3) = \delta\), we have \(N(w_3) \cap U_4 = \emptyset\). For any \(u \in U_4\), by Lemma 2.2, there is some vertex \(z \in N(v_0)\) such that \([u, z] \to w_3\) or \([w_3, z] \to u\). If \([w_3, z] \to u\), then since \([w_2, v_2] \to w_3\) and \(u \notin N(w_2)\), we have \(z \neq v_2\). By (2-1) and Lemma 2.6, \(z \notin N(v_0) \setminus \{v_2\}\), a contradiction. If \([u, z] \to w_3\), then by (2-1) and Lemma 2.6, \(z = v_2\). Since \(v_2v_k \notin E(G)\) by Lemma 2.6, we have \(v_ku \in E(G)\) and hence \(U_4 \subseteq N(v_k)\). Thus, \([w_k, v_k] \to v_{k-1}\). Since \(d(v_{k-1}) = n - 3\), \(d(v_2) \geq n - \delta\) by Claim 2.2 and \(d(v_1) \geq n - \delta - 1\), we have \(v_{k-1}, v_2, u_1 \in N^*(v_k)\). By Claim 2.4, \(v_{k-1} \in N^*(w_k)\). By Lemmas 2.6 and 2.7, \(v_{k-1}, v_2, u_1 \notin N(v_k)\). Thus, we have \(d^*(w_k) + d^*(v_k) \geq n + 1\) and hence \(v_k \in N^*(w_k)\).

**Claim 2.6.** For any \(u \in U_2\), we have \([u, v_1] \to w_1\).

**Proof.** Since \(uw_1 \notin E(G)\), there exists some vertex \(z\) such that \([w_1, z] \to u\) or \([u, z] \to w_1\). In order to dominate \(v_0\), we have \(z \in N[v_0]\). Thus by (2-1) and Lemma 2.6, it is easy to see \([w_1, z] \to u\) is impossible. If \([u, z] \to w_1\), then by the assumption we have \(z \neq v_0\). By (2-1) and Lemma 2.6, we have \(z = v_1\), that is, \([u, v_1] \to w_1\).

**Claim 2.7.** For any \(u \in U_2\), \(N(v_0) \subseteq N(u)\).

**Proof.** Since \([w_1, v_1] \to w_2\) and \(u \in U_2\), we have \(v_1 \in N(u)\). By Lemmas 2.4 and 2.6, we have \(v_i \in N(u)\) for \(2 \leq i \leq k - 2\). By Lemma 2.6 and Claim 2.6, we have \(v_k \in N(u)\). We now show \(v_{k-1} \in N(u)\). Since \(w_1w_k \notin E(G)\), by Lemma 2.2, there exists some vertex \(z \in N(v_0)\) such that \([w_1, z] \to w_k\) or \([w_k, z] \to w_1\). By (2-1) and Lemma 2.6, we can see \([w_k, z] \to w_1\) is impossible. Thus we have \([w_1, z] \to w_k\). By Claim 2.6 we have \(w_1v_1 \notin E(G)\). By Lemma 2.6, we have \(z \neq v_k\) since \(\{w_1, v_k\} \neq v_1\). By (2-1), we have \(z = v_{k-1}\) which implies \(v_{k-1} \in N(u)\).

**Claim 2.8.** If \(U_2 \neq \emptyset\), then \(N_U(w_k) \subseteq N(w_1) \cap N(w_2)\).

**Proof.** Let \(u \in N_U(w_k)\) and \(w \in \{w_1, w_2\}\). If \(uw \notin E(G)\), then there is some vertex \(z\) such that \([u, z] \to w\) or \([w, z] \to u\). If \([w, z] \to u\), then \(z \in N(v_0)\). By Claim 2.6,
\(v_1w_1 \notin E(G)\), which implies \([w_2, v_1] \to u\) cannot occur. Thus, by (2-1) and Lemma 2.6 we see that \([w, z] \to u\) is impossible. If \([u, z] \to w\), then by the assumption, \(z \neq v_0\).

By Lemma 2.6, \(z \neq v_k\). If \(z \in N(v_0) - \{v_k\}\), then \(\{u, z\} \neq v_k\) by Lemmas 2.6 and 2.7. Thus, \(z \notin N(v_0)\), a contradiction.

We first show that \(w_1v_1 \in E(G^*)\).

If \(w_1v_1 \in E(G)\), then \(w_1v_1 \in E(G^*)\). If \(\delta \geq 5\), then by Lemma 2.7, Claim 2.1 and \([w_1, v_1] \to w_2\), we have \(d(w_1) + d(v_1) \geq n + 1\) and hence \(w_1v_1 \in E(G^*)\). Thus, we may assume that \(w_1v_1 \notin E(G)\) and \(\delta = 4\).

If \(|N(w_1) \cap N(v_1) \cap U| \geq 2\), then by Lemma 2.7 and \([w_1, v_1] \to w_2\), we have \(d(w_1) + d(v_1) \geq n + 1\) and hence \(w_1v_1 \in E(G^*)\). Thus by Claim 2.1 we may assume

\[N(w_1) \cap N(v_1) \cap U = \{u_1\}\]  
(2-4)

By the assumption, we let \([w_1, z] \to v_0\). If \(z \neq u_1\), then \(z \in U_2\) by (2-4). This is impossible since \(\{w_1, z\} \neq w_k\) by Claim 2.8 and hence we have

\([w_1, u_1] \to v_0\).  
(2-5)

If \(U_2 \neq \emptyset\), we let \(u \in U_2\). If \(u' \in U_2\) and \(uu' \notin E(G)\), then there is some vertex \(z\) such that \([u, z] \to u'\) or \([u', z] \to u\). By symmetry we may assume \([u, z] \to u'\). By Claim 2.7, \(z \notin N(v_0)\). If \(z = v_0\), then \(\{u, z\} \neq w_1\), a contradiction. Hence \(U_2\) is a clique. If \(u' \in U_1\) and \(uu' \notin E(G)\), then by Claim 2.6 we have \(u' \in N(v_1)\), which implies \(u' = u_1\) by (2-4). By (2-5), \(u_1u \in E(G)\). Thus, \(U \subseteq N[u]\) for any \(u \in U_2\). By Claim 2.6, \(U_2 \subseteq N(w_2)\). Thus by Claim 2.7, we have \(d(u) \geq n - \delta - 1\). If \(d(w_1) \geq \delta + 2\), then \(uu_1 \in E(G^*)\), which implies \(w_1v_1 \in E(G^*)\). If \(d(w_1) \leq \delta + 1\), then by (2-1) and Lemma 2.6 we have \(|U_1| \leq 2\). By Lemma 2.6 and the assumption, we have \(d_U(w_k) \geq 2\). Thus by Claim 2.8 we have \(U_1 = N_U(w_k) \subseteq N(w_2)\) and hence \(U \subseteq N(w_2)\). In this case, we have \([v_1, w_2] \to w_1\). By Lemma 2.7, Claim 2.7 and (2-4), \(|N(v_1) \cap N(w_2)| \geq 4\). Thus we have \(v_1w_2 \in E(G^*)\) and hence \(w_1v_1 \in E(G^*)\).

If \(U_2 = \emptyset\), then since \(w_1v_1 \notin E(G)\), there is some vertex \(z\) such that \([w_1, z] \to v_1\) or \([v_1, z] \to w_1\). If \([w_1, z] \to v_1\), then \(z \neq v_0\) and hence \(z \in N(v_0)\). By Lemma 2.7, \(z = v_k\). This is impossible since \(\{v_1, v_k\} \neq w_1\) by Lemma 2.6. Thus we have \([v_1, z] \to w_1\). Since \(U_2 = \emptyset\) and \(N(v_0) - \{v_k\} \subseteq N(w_1)\), we have \(z \in \{w_2, \ldots, w_k\}\). In this case, \(z = w_2\), that is, \([w_2, v_1] \to w_1\). By (2-5), \(u_1w_2 \in E(G)\). Thus by (2-4), we have \(U \subseteq N(w_2)\). By (2-1) and Lemmas 2.4 and 2.6, \(v_2, v_3, v_4 \in N(w_1) \cap N(w_2)\). Thus, if \(|U| \geq 4\), then \(d(w_1) + d(w_2) \geq n + 1\), which implies \(w_1w_2 \in E(G^*)\) and hence \(w_1v_1 \in E(G^*)\). If \(|U| \leq 3\), then \(n \leq 12\). After an easy but tedious check, we can show \(w_1v_1 \in E(G^*)\).

Next, we show \(U \subseteq N^*(w_1)\). If \(U_2 = \emptyset\), then \(U \subseteq N(w_1) \subseteq N^*(w_1)\) and hence we assume \(U_2 \neq \emptyset\). Let \(u \in U_2\). Suppose \(u' \in V(G - N[v_0])\) and \(u' \notin N^*(u)\). Obviously, \(uu' \notin E(G)\) and hence there is some \(z\) such that \([u', z] \to u\) or \([u, z] \to u'\). If \([u', z] \to u\), then \(z \notin N(v_0)\) by Claim 2.7 and hence \(z = v_0\). In this case, \(u' \notin U\).
Since \([v_0, v_{k-1}] \rightarrow w_k, v_{k-1} \in N(u')\). By Claim 2.6, \(v_1u' \in E(G)\). Thus we have \(d(u') \geq n - \delta - 1\). By the assumption, there exists some \(z'\) such that \([u_1, z'] \rightarrow v_0\). By Lemma 2.7 and Claim 2.7, \(z' \in U_1\) and hence \(N_{U_1}(u) \neq \emptyset\). By Claim 2.6, \(w_2 \in N(u)\). Thus, by Claim 2.7 we have \(d(u) \geq \delta + 2\), which implies \(u' \in N^*(u)\) and hence \([u', z] \rightarrow u\) is impossible. Thus we always have \([u, z] \rightarrow u'\). By Claim 2.8, \(w_k \notin N(u)\). Thus we have \(z \neq v_0\) since \([u, v_0] \neq \{w_1, w_k\}\) and hence \(z \in N(v_0)\). If \(V(G) - N[v_0]\) contains \(\delta\) vertices, say \(u'_1, u'_2, \ldots, u'_k\), that are not adjacent to \(u\) in \(G^*\), then there are \(z_{u'_i} \in N(v_0)\) such that \([u, z_{u'_i}] \rightarrow u'_i\) for \(1 \leq i \leq k\). Clearly, if \(i \neq j\), then \(z_{u'_i} \neq z_{u'_j}\) since \(u'_i \neq u'_j\). This is impossible since \([u, v_{k-1}] \neq w_k\) and \([u, v_k] \neq w_k\). Therefore, \(V(G) - N[v_0]\) contains at most \(\delta - 1\) vertices that are not adjacent to \(u\) in \(G^*\) and hence \(d^*(u) \geq n - \delta - 1\) since \(N(v_0) \subseteq N(u)\) by Claim 2.7. By Claim 2.6, \(w_1v_1 \notin E(G)\). By Lemma 2.6 and the assumption, \(d_U(w_k) \geq 2\) which implies \(d_U(w_1) \geq 2\) by Claim 2.8. Thus by (2-1) and Lemma 2.6 we have \(d(w_1) \geq \delta + 1\) and hence \(d^*(w_1) \geq \delta + 2\) since \(w_1v_1 \in E(G^*)\). This implies \(d^*(w_1) + d^*(u) \geq n + 1\) and thus \(U \subseteq N^*(w_1)\).

Finally, we show \(N^*[w_1] = V(G)\). Since \(w_1v_1 \in E(G^*)\) and \(U \subseteq N^*(w_1)\), by (2-1), we have \(d^*(w_1) \geq n - \delta - 1\). By Claim 2.2, \(d(w_2) \geq \delta + 1\). If \(d(w_2) \geq \delta + 2\), then by Claim 2.4, we have \(w_2, w_k \in N^*(w_1)\), which implies \(d^*(w_1) \geq n - \delta + 1\) and hence \(N^*[w_1] = V(G)\). If \(d(w_2) = \delta + 1\) and \(d(w_3) \geq \delta + 1\), then by Claim 2.2 we have \(d^*(w_3) \geq \delta + 2\). Thus \(w_3, w_2 \in N^*(w_1)\) and hence \(N^*[w_1] = V(G)\). If \(d(w_2) = \delta + 1\) and \(d(w_4) = \delta\), then \(d^*(w_4) \geq \delta + 2\) by Claims 2.4 and 2.5. Thus, \(w_k, w_2 \in N^*(w_1)\) and hence \(N^*[w_1] = V(G)\).

3. Some Lemmas

Let \(G\) be a graph of order \(n\), and \(x, y\) vertices of \(G\) such that the longest \((x, y)\)-path is of length \(n - 2\). Let \(P = P_{xy}\) be an \((x, y)\)-path of length \(n - 2\) and suppose the orientation of \(P\) is from \(x\) to \(y\). We denote by \(x_P\) the only vertex not in \(P\) and let \(d(x_P) = k \geq 2\) with

\[
\begin{align*}
N(x_P) &= X = \{x_1, x_2, \ldots, x_k\}, \\
A &= X^+ = \{a_1, a_2, \ldots, a_s\}, \quad \text{indices following the orientation of } P; \\
B &= X^- = \{b_1, b_2, \ldots, b_t\}, \quad \text{where } a_i = x_i^+, x_i^- \in V(P) \text{ and } s \geq k - 1; \\
P_i &= a_iPb_{i+1}, \quad \text{where } 1 \leq i \leq k - 1.
\end{align*}
\]

Furthermore, we let \(P_0 = xPb_1\) if \(x \notin X\) and \(P_k = a_KPy\) if \(y \notin X\). In this section, we will establish some lemmas. It is worth noting that all lemmas in this section except the last one do not depend on the 3-critical property of \(G\).

**Definition.** A vertex \(v \in P_i\) (\(1 \leq i \leq k\)) is called an \(A\)-vertex if \(G[V(P_i) \cup \{x_{i+1}\}]\) contains a hamiltonian \((v, x_{i+1})\)-path, and \(v \in P_i\) (\(0 \leq i \leq k - 1\)) a \(B\)-vertex if \(G[V(P_i) \cup \{x_i\}]\) contains a hamiltonian \((x_i, v)\)-path, where \(x_{k+1} = y\) and \(x_0 = x\).
From the definition, we can see that each $a_i$ is an $A$-vertex and each $b_i$ is a $B$-vertex. Let $u_i \in P_i$ be an $A$-vertex and $Q_i$ a given Hamiltonian $(u_i, x_{i+1})$-path in $G[V(P_i) \cup \{x_{i+1}\}]$. Suppose the orientation of $Q_i$ is from $u_i$ to $x_{i+1}$. We have the following two lemmas.

**Lemma 3.1.** If $u_i \in P_i$ and $u_j \in P_j$ are two $A$-vertices ($B$-vertices, respectively) with $i \neq j$, then $x_P u_i \notin E(G)$ and $u_i u_j \notin E(G)$. In particular, both $A \cup \{x_P\}$ and $B \cup \{x_P\}$ are independent sets.

**Proof.** If $x_P u_i \in E(G)$, then $x_P x_i x_P u_i \bar{Q}_i x_{i+1} \bar{P} y$ is a Hamiltonian $(x, y)$-path. Assume $i < j$. If $u_i u_j \in E(G)$, then the $(x, y)$-path $x_P x_i x_P x_j \bar{P} x_{i+1} \bar{Q}_i u_i u_j \bar{Q}_j x_{j+1} \bar{P} y$ is Hamiltonian, a contradiction.

**Lemma 3.2.** Let $u_i \in P_i$, $u_j \in P_j$ be $A$-vertices with $i < j$, $Q = u_i \bar{Q}_i x_{i+1} \bar{P} x_j$ and $R = u_j \bar{Q}_j x_{j+1} \bar{P} y$. If $v \in N_Q(u_i)$, then $v^{-} \notin N(u_j)$ and if $v \in N(u_i) \cap (x_P x_i \cup R)$, then $v^{+} \notin N(u_j)$. In particular, let $a_i, a_j \in A$ with $i < j$ and $v \in N(a_i)$, then $v^{-} \notin N(a_j)$ if $v \in a_i \bar{P} x_j$ and $v^{+} \notin N(a_j)$ if $v \in x_P x_i \cup a_j \bar{P} y$.

**Proof.** If $v \in N_Q(u_i)$ and $v^{-} \notin N(u_j)$, then the $(x, y)$-path $x_P x_i x_P x_j \bar{Q} v u_i \bar{Q} v^{-} u_j \bar{R} y$ is Hamiltonian, a contradiction. As for the latter case, the proof is similar.

By symmetry of $A$ and $B$, Lemma 3.2 still holds if we exchange $A$ and $B$.

**Lemma 3.3.** Let $u, v \in a_i \bar{P} b_j$ with $j \geq i + 1$ and $G[a_i \bar{P} b_j]$ contain a Hamiltonian $(u, v)$-path $Q$. Suppose that $w \in x_P x_i \cup x_j \bar{P} y$ and $uw \in E(G)$. Then $w^{-} v \notin E(G)$ if $w^{-} \in x_P x_i \cup x_j \bar{P} y$, and $w^{+} v \notin E(G)$ if $w^{+} \in x_P x_i \cup x_j \bar{P} y$. In particular, let $a_i \in A$ and $b_j \in B$ with $j \geq i + 1$. Suppose that $v \in x_P x_i \cup x_j \bar{P} y$ and $a_i v \in E(G)$. Then $v^{-} b_j \notin E(G)$ if $v^{-} \in x_P x_i \cup x_j \bar{P} y$ and $v^{+} b_j \notin E(G)$ if $v^{+} \in x_P x_i \cup x_j \bar{P} y$.

**Proof.** Suppose that $w \in x_P x_i$. If $w^{-} \in x_P x_i$ and $w^{-} v \in E(G)$, then the $(x, y)$-path $x_P w^{-} v \bar{Q} w u \bar{P} x_i x_P x_j \bar{P} y$ is Hamiltonian, and if $w^{+} \in x_P x_i$ and $w^{+} v \in E(G)$, then the $(x, y)$-path $x_P w u \bar{Q} v w^{+} \bar{P} x_i x_P x_j \bar{P} y$ is Hamiltonian, a contradiction. As for the case $w \in x_P \bar{P} y$, the proof is similar.

**Lemma 3.4.** Let $u, u^{+} \in V(P_i)$. If $u^{+} a_l \in E(G)$ for some $l \geq i + 1$, then $b_j u \notin E(G)$ for all $j \leq i$.

**Proof.** If $b_j u \in E(G)$ for some $j \leq i$, then the $(x, y)$-path $x_P b_j u \bar{P} x_j x_P x_i \bar{P} u^{+} a_l \bar{P} y$ is Hamiltonian, a contradiction.

**Lemma 3.5.** Let $z \in V(G) - N[x_P]$. If $|N(z) \cap A| \geq 2$, then $z^{-} z^{+} \notin E(G)$.

**Proof.** Let $a_l, a_m \in N(z)$ with $l < m$ and $z \in P_j$. If $z^{-} z^{+} \in E(G)$, then the $(x, y)$-path $x_P z^{-} z^{+} \bar{P} x_i x_P x_m \bar{P} a_l z a_m \bar{P} y$ is Hamiltonian if $j < l$, $x_P x_i x_P x_m \bar{P} z^{+} z^{-} \bar{P} a_l z a_m \bar{P} y$ is Hamiltonian if $l < j < m$, and $x_P x_i x_P x_m \bar{P} a_l z a_m \bar{P} z^{-} z^{+} \bar{P} y$ is Hamiltonian if $m \leq j$, and so on.
Lemma 3.6. Let $z, z^- \in P_i, w, w^- \in P_j$ with $i, j \geq 1$ and $k \geq 4$. If $|A - N(z)| \leq 1$ and $A \subseteq N(w)$, then $z^-w^- \notin E(G)$.

Proof. Suppose to the contrary $z^-w^- \in E(G)$. If $i = j$ and $w \in xPz$, then $a_iw \notin E(G)$ for otherwise $w$ is an $A$-vertex, which contradicts Lemma 3.1 since $A \subseteq N(w)$. Hence we have $A - \{a_i\} \subseteq N(z)$. Noting that $A \subseteq N(w)$ and $k \geq 4$, we have $w \neq z$ by Lemma 3.2. Thus, the $(x, y)$-path $xPw^-z^-Pwa_2Px_2Pw_2Pz_3P$ is hamiltonian if $i = 1$, $xPx_1Pz_1Pz_2Pw^-z^-Pwa_3P$ is hamiltonian if $i = 2$, and $xPx_1Pz_1Pz_2Pwa_2Pz^-w^-Pz_3P$ is hamiltonian if $i \geq 3$, a contradiction. If $i = j$ and $z \in xPw$, then since $a_iw \in E(G)$, $z$ is an $A$-vertex, which contradicts Lemma 3.1 since $|A - N(z)| \leq 1$. If $i \neq j$, then since $a_jw \in E(G)$, $w$ is an $A$-vertex. Since $z^-w^- \in E(G)$, by Lemma 3.1, $za_i \notin E(G)$. Thus, $xPz_1Pz_2Pz_3Pw^-z^-Pz_4P$ is a hamiltonian $(x, y)$-path if $i < j$, and $xPz_1Pz_2Pw_1Pz^-w^-Pz_4P$ is a hamiltonian $(x, y)$-path if $i > j$, also a contradiction. 

Lemma 3.7. Let $z, z^- \in P_i, w, w^- \in P_j$ with $i, j \geq 1$ and $k \geq 4$. If $|A \cup B - N(z)| \leq 1$ and $|A - N(w)| \leq 1$, then $w^-z^- \notin E(G)$.

Proof. We first show the following claim.

Claim 3.1. Let $u, v \in P_l, v^- \in P_m$ and $h \neq l, m$. If $u^-v^- \in E(G)$, then either $ua_h \notin E(G)$ or $vb_{h+1} \notin E(G)$.

Proof. Assume without loss of generality $v \in uP$. If $ua_h, vb_{h+1} \in E(G)$, then $u \neq v^-$ by Lemma 3.3. Thus the $(x, y)$-path $xPu^-v^-Pua_hPvb_{h+1}vP$ is hamiltonian if $h < l$, $xPu^-v^-Pua_hPvb_{h+1}vP$ is hamiltonian if $l < h < m$, and $xPu^-v^-Pua_hPvb_{h+1}vP$ is hamiltonian if $m < h$, a contradiction.

By Lemma 3.6, we may assume $B \subseteq N(z)$. If $w^-z^- \in E(G)$, then by Claim 3.1, $a_jw \notin E(G)$ for $l \neq i, j$. Noting $k \geq 4$ and $|A - N(w)| \leq 1$, we have $i \neq j$ and $wa_i, wa_j \in E(G)$. Since $wa_j \in E(G)$, $w$ is an $A$-vertex. If $za_i \in E(G)$, then $z$ is also an $A$-vertex which contradicts Lemma 3.1 since $i \neq j$ and $w^-z^- \in E(G)$. Hence, $za_i \notin E(G)$, which implies $za_j \in E(G)$ since $|A \cup B - N(z)| \leq 1$. If $j < k$, then $w^-P_{a_jw}w^-Pb_{j+1}v$ is a hamiltonian path in $G[V(P_j)]$, which contradicts Lemma 3.3 since $w^-z^-w^-z^-P_{a_iw}w^-Pb_{j+1}vP$ is a hamiltonian, a contradiction.

Lemma 3.8 (Chen et al. [4]). Let $z \in V(P) - X$ and $v \in A \cup B$. If $d(xP) = k \geq 4$ and $A \cup B - \{v\} \subseteq N(z)$, then $A \cup \{z^+\}$ is an independent set if $z^+ \in V(P)$ and $B \cup \{z^-\}$ is an independent set if $z^- \in V(P)$.

Lemma 3.9 (Chen et al. [5]). Let $u, v \notin V(P_i)$ and $\{u, v\} \supseteq V(P_i)$. If $ua_i, vb_{h+1} \in E(G)$, where $b_{k+1} = y$ if $i = k$, then there is some $w \in V(P_i)$ such that $uw, vw^+ \in E(G)$. 

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Let $z \in P_j$ and $[a_i, z] \rightarrow x_P$. We have the following five lemmas (3.10-3.14).

**Lemma 3.10.** If $2 \leq i \leq j$ and $z^+ \in V(P)$, then $A \cup \{x_P, z^+\}$ is an independent set.

**Proof.** Since $za_1 \in E(G)$, we have $a_1z^+ \notin E(G)$ for $2 \leq l \leq j$ by Lemma 3.2. If $a_1z^+ \in E(G)$ or $a_1z^+ \in E(G)$ for some $l \geq j + 1$, then by Lemmas 3.3 or 3.4 we have $b_2z \notin E(G)$ and hence $b_2a_i \in E(G)$. By Lemma 3.9, there is some $w \in P_l$ such that $wz, w^+ a_i \in E(G)$. Thus, the $(x, y)$-path $x \overrightarrow{P} x_1 x_P x_i \overrightarrow{P} w^+ a_i \overrightarrow{P} zw \overrightarrow{P} a_1 z^+ \overrightarrow{P} y$ is hamiltonian if $a_1z^+ \in E(G)$, and $x \overrightarrow{P} w z \overrightarrow{P} a_i w^+ \overrightarrow{P} x_i x_P x_1 \overrightarrow{P} z^+ a_i \overrightarrow{P} y$ is hamiltonian if $a_1z^+ \in E(G)$ for some $l \geq j + 1$, a contradiction. If $z \in B$, then $z = b_{j+1}$. By Lemma 3.1 we have $a_1 b_{j+1}, b_2 a_i \in E(G)$. By Lemma 3.9, there is some $w \in P_l$ such that $w b_{j+1}, w^+ a_i \in E(G)$, which contradicts Lemma 3.3. Thus, $z \notin B$ and hence $z^+ x_P \notin E(G)$, which implies $A \cup \{x_P, z^+\}$ is an independent set.

**Lemma 3.11.** If $2 \leq i \leq j$ and $|A| \geq 3$, then $B \cup \{z^-, x_P\}$ is an independent set.

**Proof.** Since $A - \{a_i\} \subseteq N(z)$ and $2 \leq i \leq j$, we have $b_l z^- \notin E(G)$ for $l \neq 1, j + 1$ by Lemma 3.3. If $b_l z^- \in E(G)$ or $z^- b_{j+1} \in E(G)$, then by Lemmas 3.2 or 3.1, we have $b_l \notin N(z)$. Since $[a_i, z] \rightarrow x_P$, we have $b_2 a_i \in E(G)$. By Lemma 3.9, there is some $u \in P_l$ such that $uz, u^+ a_i \in E(G)$. Thus, the $(x, y)$-path $x \overrightarrow{P} b_l z^- \overrightarrow{P} a_i u^+ \overrightarrow{P} x_i x_P x_1 \overrightarrow{P} uz \overrightarrow{P} y$ is hamiltonian if $b_l z^- \in E(G)$, and $x \overrightarrow{P} u z \overrightarrow{P} b_{j+1} z^- \overrightarrow{P} a_i u^+ \overrightarrow{P} x_i x_P x_{j+1} \overrightarrow{P} y$ is hamiltonian if $b_{j+1} z^- \in E(G)$, a contradiction. Since $|A| \geq 3$ and $[a_i, z] \rightarrow x_P$, by Lemma 3.1 we have $z \notin A$ which implies $z^- x_P \notin E(G)$. Thus, by Lemma 3.1 we can see that $B \cup \{z^-, x_P\}$ is an independent set.

**Lemma 3.12.** If $j + 1 < i$, then $A \cup \{z^+, x_P\}$ is an independent set.

**Proof.** Since $a_{j+1} z \in E(G)$, by Lemma 3.2 we have $a_1 z^+ \notin E(G)$ for all $l$ with $l \neq j + 1$. If $a_{j+1} z^+ \in E(G)$, then by Lemma 3.3 we have $b_{j+2} z \notin E(G)$ and hence $a_{j+2} \in E(G)$. By Lemma 3.9, there is some $u \in P_{j+1}$ such that $uz, u^+ a_i \in E(G)$. Thus, the $(x, y)$-path $x \overrightarrow{P} u z \overrightarrow{P} a_{j+1} z^+ \overrightarrow{P} x_{j+1} x_P x_i \overrightarrow{P} u^+ a_i \overrightarrow{P} y$ is hamiltonian, a contradiction. If $z \in B$, then $z = b_{j+1}$. Since $[a_i, z] \rightarrow x_P$ and $j + 1 < i$, there is some $u \in P_{j+1}$ such that $uz, u^+ a_i \in E(G)$, which contradicts Lemma 3.4. Hence $z \notin B$ which implies $z^+ x_P \notin E(G)$. Thus, $A \cup \{z^+, x_P\}$ is an independent set by Lemma 3.1.

**Lemma 3.13.** Let $|A| \geq 3$. If $j + 1 < i$ and $z^- \in V(P)$, then $B \cup \{z^-, x_P\}$ is an independent set.

**Proof.** Since $a_{j+1} z \in E(G)$, we have $b_l z^- \notin E(G)$ for $l \neq j + 1$ by Lemmas 3.3 and 3.4. If $b_{j+1} z^- \in E(G)$, then $z$ is a $B$-vertex. By Lemma 3.1 we have $z b_{j+2} \notin E(G)$, which implies $a_i b_{j+2} \in E(G)$. By Lemma 3.9, there is some $w \in P_{j+1}$ such that $zw, w^+ a_i \in E(G)$. Thus, the $(x, y)$-path $x \overrightarrow{P} z^- b_{j+1} \overrightarrow{P} w \overrightarrow{P} x_{j+1} x_P x_i \overrightarrow{P} w^+ a_i \overrightarrow{P} y$ is hamiltonian, a contradiction. Since $|A| \geq 3$ and $[a_i, z] \rightarrow x_P$, we have $z \notin A$ by Lemma 3.1 and hence $z^- x_P \notin E(G)$. Thus, $B \cup \{z^-, x_P\}$ is an independent set.
The following two lemmas can be extracted from [5]: Lemma 3.14 is extracted from the Case 2 of Lemma 2.8(2) and Lemma 3.15 from Lemma 2.9 in [5].

Lemma 3.14 (Chen et al. [5]). If \( j = i - 1 \geq 1 \), \( d(x_P) = k \geq 4 \) and \( \{x, y\} \subseteq N(x_Q) \) for any longest \((x, y)\)-path \( Q \), then \( B \cup \{z^-, x_P\} \) is an independent set.

Lemma 3.15 (Chen et al. [5]). Suppose that \( P \) is a longest \((x, y)\)-path such that \(|X \cap \{x, y\}| \) is as small as possible and that for this path, \( d(x_P) = k \geq 4 \). If \( G \) is 3-critical, then there exists an independent set \( I \) such that either \( \{x_P\} \cup A \subseteq I \) or \( \{x_P\} \cup B \subseteq I \) and \(|I| \geq k + 1 \).

4. Proof of Theorem 4

Let \( G \) be a 3-connected 3-critical graph with \( \alpha(G) = \delta(G) + 1 \geq 5 \). If \( G \) is not Hamilton-connected, then by Theorem 5, there are two vertices \( x, y \in V(G) \) such that \( p(x, y) = n - 2 \). Among all the longest \((x, y)\)-paths, we choose \( P \) such that \(|\{x, y\} \cap N(x_P)| \) is as small as possible. Choose an orientation of \( P \) such that \(|A| \geq |B| \). Assume without loss of generality that the orientation is from \( x \) to \( y \). We still use the notations given in Section 3.

Since \( \alpha(G) = \delta(G) + 1 \geq 5 \), by the choice of \( P \) and Lemma 3.15, \( d(x_P) = k = \delta \geq 4 \). We first show the following claims.

Claim 4.1. Let \( z \in P_j \) and \([a_i, z] \rightarrow x_P\). If \(|A| = k \) and \( j = i - 1 \geq 1 \), then \( B \cup \{z^-, x_P\} \) is an independent set.

Proof. Let \( U = N[x_P] \cup A \). By Lemmas 2.1 and 2.2, we may assume that \([a_i, x_j] \rightarrow a_{i+1} \) for \( 1 \leq l \leq k - 1 \). Thus, noting that \(|A| = k \), we have

\[
d_U(x_l) \geq \delta \text{ for any } x_l \in N(x_P).
\]  

(4-1)

Assume \( b_i \in B \) and \( b_iz^- \in E(G) \). Since \( A - \{a_i\} \subseteq N(z) \), by Lemma 3.3, \( l \in \{1, j + 1, i + 1\} \). If \( j = 1 \), then \( i = 2 \). Since \( a_3z \in E(G) \), by Lemma 3.4, \( l \neq 1 \) and hence \( l \in \{2, 3\} \). If \( l = 2 \) or \( 3 \), then by Lemma 3.2 we have \( b_iz \notin E(G) \) and hence \( a_2b_1 \in E(G) \). Since \( za_3, a_2b_4 \in E(G) \), by Lemma 3.1 we have \(|P_1| \geq 2 \) and \(|P_2| \geq 2 \), which implies \( b_2, b_3 \notin U \). Thus we have \( d(x_2) \geq \delta + 1 \) and \( d(x_3) \geq \delta + 1 \) by (4-1).

If \( l = 2 \), then \( Q = xPz^-b_2Pza_3Pb_4a_2P_{x_3x_4P}y \) is an \((x, y)\)-path of length \( n - 2 \) with \( d(x_Q) = d(x_2) \geq \delta + 1 \) and if \( l = 3 \), then \( R = xPz^-b_3Pb_2a_3P_{x_2x_3x_4P}y \) is an \((x, y)\)-path of length \( n - 2 \) with \( d(x_R) = d(x_3) \geq \delta + 1 \). Since \( \alpha(G) = \delta(G) + 1 \), by Lemma 3.1 we have \( y \in N(x_2) \) if \( l = 2 \) and \( y \in N(x_3) \) if \( l = 3 \). If \( y \neq a_k \), then \( d(x_2) \geq \delta + 2 \) if \( l = 2 \) and \( d(x_3) \geq \delta + 2 \) if \( l = 3 \), which implies \( \alpha(G) \geq \delta(G) + 2 \) by Lemma 3.1, a contradiction. Hence \( y = a_k \). Thus, \( xPz^-b_2P_{x_3x_4P}x_3a_3P_{x_2a_k} \) is a hamiltonian \((x, y)\)-path if \( l = 2 \) and \( xPz^-b_3P_{x_2a_3x_3a_k} \) is a hamiltonian \((x, y)\)-path if \( l = 3 \), a contradiction. Hence we have \( j \geq 2 \). Since \( l \in \{1, j + 1, i + 1\} \), we have
contains a vertex $b_2z \notin E(G)$ by Lemma 3.2 and hence $b_2a_i \in E(G)$. If $l = 1$, then since $[a_i, z] \rightarrow x_P$, we have $zz_1 \in E(G)$ or $a_ix_1 \in E(G)$. Thus, $xPb_1z\overline{P}x_2px_3\overline{P}x_1\overline{P}b_2a_i\overline{P}y$ is a hamiltonian $(x, y)$-path if $zz_1 \in E(G)$ and $xPb_1z\overline{P}a_iz\overline{P}x_3px_1a_i\overline{P}y$ is a hamiltonian $(x, y)$-path if $a_ix_1 \in E(G)$. If $j + 1$, then $Q = xPb_1x_2x_3z\overline{P}a_iz\overline{P}b_2a_i\overline{P}y$ is a hamiltonian $(x, y)$-path of length $n - 2$ with $x_Q = x_{j + 1}$. Since $|P_j| \geq 2$, $b_{j + 1} \notin U$ which implies $d(x_{j + 1}) \geq 3 + 1$ by (4.1). Since $a(G) = \delta(G) + 1$, by Lemma 3.1 we have $xx_{j + 1} \in E(G)$ and $x = x_1$. In this case, $xx_{j + 1}xp_{x_2}z\overline{P}a_iz\overline{P}b_2a_i\overline{P}y$ is a hamiltonian $(x, y)$-path. If $l = i + 1$, then since $[a_i, z] \rightarrow x_P$, we have $zz_{i + 1} \in E(G)$ or $a_ix_{i + 1} \in E(G)$. Thus, $xPb_2a_i\overline{P}b_{i + 1}z\overline{P}x_2px_1z\overline{P}a_{i + 1}\overline{P}y$ in the former case and $xPb_1x_2p_{x_1}z\overline{P}a_iz\overline{P}b_2a_i\overline{P}y$ in the latter case, is a hamiltonian $(x, y)$-path, a contradiction. Therefore, $B \cup \{z\}$ is an independent set. On the other hand, since $k \geq 4$ and $[a_i, z] \rightarrow x_P$, by Lemma 3.1, we have $z \notin A$ and hence $z^- \notin E(G)$. Thus by Lemma 3.1, $B \cup \{z, x_P\}$ is an independent set.

Claim 4.2. Let $I = \{x_P\} \cup W$ with $|I| = k + 1 \geq 5$ be an independent set. If $W = A$ or $I$ is obtained by one of the Lemmas 3.8 and 3.10-3.15, then $[x_P, x_1] \rightarrow w$ is impossible for any $x_1 \in X$ and $w \in W$.

Proof. If $[x_P, x_1] \rightarrow w$ for some $w \in W$ and $x_1 \in X$, then by Lemmas 2.5 and 2.8, $W$ contains a vertex $w'$ such that $V(G) \subseteq N[w']$. If $W = A$, then by Lemma 3.1, $G^*$ contains a hamiltonian $(x, y)$-path and hence $p(x, y) = n - 1$ by Theorem 7, a contradiction. If $I$ is obtained by one of the Lemmas 3.8 and 3.10-3.15, then by the proofs of these lemmas, we can see that $G^*$ contains a hamiltonian $(x, y)$-path, which implies $p(x, y) = n - 1$ by Theorem 7, also a contradiction.

If $N(x_P) \cap \{x, y\} = \emptyset$, then $|A| = |B| = k$. By Lemmas 2.1 and 2.2, we may assume $[a_i, x_j] \rightarrow a_{i + 1}$ for $1 \leq l \leq k - 1$. Since $k \geq 4$, by Lemma 2.5 there is some $a_i$ with $i \geq 2$ and a vertex $z \in V(G) - N[x_P]$ such that $[x_P, z] \rightarrow a_i$ or $[a_i, z] \rightarrow x_P$. If $[x_P, z] \rightarrow a_i$, then $\gamma \geq \delta + 2$ by Lemma 3.8 and if $[a_i, z] \rightarrow x_P$, then $\gamma \geq \delta + 2$ by Lemmas 3.10-3.14 and Claim 4.1, a contradiction. Thus, $|N(x_P) \cap \{x, y\}| \geq 1$. By the choice of the orientation of $P$, we have $x = x_1$.

Claim 4.3. For any $a_i \in A$ and any $z \in V(G) - N[x_P]$, $[x_P, z] \rightarrow a_i$ is impossible.

Proof. Suppose to the contrary there is some $z \in V(G) - N[x_P]$ such that $[x_P, z] \rightarrow a_i$. Since $x = x_1$, by Lemma 3.8, $B \cup \{x_P, z^-\}$ is an independent set, and if $|A| = k - 1$, then $A \cup \{x_P, z^+\}$ is also an independent set. Noting that $A \cup \{x_P\}$ or $A \cup \{x_P, z^+\}$ is a maximum independent set and $k \geq 4$, by Claim 4.2, there are some $a_j \in A$ with $j \neq 1, i$ and $w \in V(G) - N[x_P]$ such that $[x_P, w] \rightarrow a_j$ or $[a_j, w] \rightarrow x_P$. In both cases, we have $w \neq z$ and $|A - N(w)| \leq 1$. By Lemma 3.8 or Lemmas 3.11, 3.13, 3.14 and Claim 4.1, $B \cup \{x_P, w^-\}$ is an independent set. By Lemma 3.7, $w^-z^- \notin E(G)$. Thus, $B \cup \{x_P, z^-, w^-\}$ is an independent set of order $k + 2$, a contradiction.
If $|A| = k - 1$, then Lemma 3.15 and the symmetry of $A$ and $B$, we may assume that $G$ contains an independent set $I$ such that $A \cup \{x_P\} \subseteq I$ and $|I| = k + 1$. If $|A| = k$, then $A \cup \{x_P\}$ is a maximum independent set. Thus, by Claim 4.2, $[x_P, x_l] \rightarrow a$ is impossible for any $a \in A$ and $x_l \in X$. Since $A \cup \{x_P\}$ is an independent set by Lemma 3.1 and $G$ is 3-critical, by Claim 4.3 we may assume in the following proof that $[a_i, z_i] \rightarrow x_P$ for all $a_i \in A$.

We now consider the following two cases separately.

**Case 1.** $|N(x_P) \cap \{x, y\}| = 1$

Let $w \in P_l$ and $wa_i \in E(G)$. If $a_i \overline{P}w \not\subset N[a_i]$, say, $v \in a_i \overline{P}w$ is the last vertex that is not adjacent to $a_i$ along $a_i \overline{P}w$, then since $wa_i \in E(G)$, $v$ is an $A$-vertex. Thus, $A \cup \{x_P, v\}$ is an independent set of order $k + 2$ by Lemma 3.1 and hence we have

$$a_i \overline{P}w \subseteq N[a_i] \text{ if } w \in P_l \text{ and } wa_i \in E(G). \quad (4-2)$$

Since $\alpha = \delta + 1$, by Lemmas 3.10-3.14 and Claim 4.1, we have $z_i \in P_{l-1}$ or $z_i = y$ for $2 \leq i \leq k$. If there are two vertices $z_i$ and $z_j$ such that $z_i \in P_{l-1}$ and $z_j \in P_{j-1}$, then both $B \cup \{x_P, z_i^+\}$ and $B \cup \{x_P, z_j^+\}$ are independent sets by Claim 4.1. Since $a_{i-1}z_i, a_{j-1}z_j \in E(G)$, $z_i^+$ and $z_j^+$ are $A$-vertices and hence $z_i^+z_j^+ \notin E(G)$ by Lemma 3.1, which implies $B \cup \{x_P, z_i^+, z_j^+\}$ is an independent set of order $k + 2$, a contradiction. Thus, noting that $k \geq 4$, there exist at least two vertices $z_i, z_j$ with $i, j \neq k$ such that $z_i = z_j = y$, which implies $A \subseteq N(y)$ and $B \cup \{y^-\}$ is an independent set by Lemma 3.11. If there is some $z_i$ with $i \geq 2$ such that $z_i \neq y$, then $z_i y^- \notin E(G)$ by Lemma 3.6 and hence $B \cup \{x_P, z_i^-, y^-\}$ is an independent set of order $k + 2$, a contradiction. Thus, we have $z_i = y$ for $2 \leq i \leq k$. By (4-2), $P_k \subseteq N[a_k]$, which implies each vertex of $P_k - \{y\}$ is an $A$-vertex. Let $z_1 \in P_j$. If $z_1 \neq y$, then $j \leq k - 1$. Since $a_{j+1}z_1 \in E(G)$, we have $b_jz_1^- \notin E(G)$ for $l \neq j + 1$ by Lemmas 3.3 and 3.4. Since $z_1 \notin N(y)$ and $[a_1, z_1] \rightarrow x_P$, by Lemma 3.10 there is some vertex $w \in P_k$ such that $wz_1, w^+a_1 \in E(G)$, which implies $z_1^{-}b_jz_{j+1} \in E(G)$ by Lemma 3.3. By Lemma 3.6, $z_1^+y^- \notin E(G)$ and hence $B \cup \{x_P, z_1^-, y^-\}$ is an independent set of order $k + 2$, a contradiction. Thus, $z_i = y$ and hence we have

$$z_i = y \text{ for } 1 \leq i \leq k. \quad (4-3)$$

Since $A \subseteq N(y)$, by Lemma 3.1, we have $y \neq a_k$ and hence $y^-x_P \notin E(G)$. If there is some $z \in V(G) \setminus N[x_P]$ such that $[x_P, z] \rightarrow y^-$, then $z \neq y$. By Lemma 3.8, $A \cup \{x_P, z^+\}$ is an independent set of order $k + 2$, a contradiction. Since $B \cup \{y^-, x_P\}$ is a maximum independent set, by Claim 4.2, there is no vertex $x_l \in X$ such that $[x_P, x_l] \rightarrow y^-$. Thus, there is some vertex $z \in P_l$ such that $[y^-, z] \rightarrow x_P$. If $z \neq y$, then since $a_k y \in E(G)$, all vertices of $a_k \overline{P}y^-$ are $A$-vertices by (4-2), which implies $z \notin P_k$ since otherwise $\{y^-, z\} \neq A - \{a_k\}$ by Lemma 3.1. Since $y^-$ is an $A$-vertex, we have $A - \{a_k\} \subseteq N(z)$, which implies $b_lz^- \notin E(G)$ for $l \neq i + 1$. If $z^-b_{i+1} \in E(G), \quad 14$
then $z$ is a $B$-vertex. Thus, noting that $B \cup \{y^\ast\}$ is an independent set, we can see $\{y^\ast, z\} \not\subseteq B - \{b_{i+1}\}$, a contradiction. Thus we have $z - b_i \not\subseteq E(G)$ for $2 \leq l \leq k$. Since $y^\ast$ is an $A$-vertex, $k \geq 4$ and $[y^\ast, z] \rightarrow x_P$, we have $z \not\subseteq A$ and hence $z - x_P \not\subseteq E(G)$.

By Lemma 3.6, $y^\ast - z \not\subseteq E(G)$. Thus, $B \cup \{x_P, y^\ast, z\}$ is an independent set of order $k + 2$, also a contradiction. Thus we have $z = y$, that is,

$$[y, y^\ast] \rightarrow x_P. \quad (4-4)$$

By Lemma 3.1, (4-2) and (4-3), $P_k \subseteq N[y]$. By Lemma 3.11, (4-3) and (4-4), $A \cup B \subseteq N[y]$. For $1 \leq i \leq k - 1$, if there is some $u \in P_i$ such that $uy \not\subseteq E(G)$, then $u^+, u^- \in P_i$ since $A \cup B \subseteq N(y)$. By (4-3), $A \subseteq N(u)$. By Lemma 3.5, we have $u^\ast - u^+ \not\subseteq E(G)$. By Lemma 3.6, $u^\ast - y^\ast \not\subseteq E(G)$. If $u^\ast + y^\ast \not\subseteq E(G)$, then the $(x, y)$-path $xP_ixPxkPu^+y^\astP_a\bar{k}u\bar{P}a_iy$ is hamiltonian and hence $u^\ast + y^\ast \not\subseteq E(G)$. By Lemma 3.3, $u^\ast - b_i, u^\ast + b_i \not\subseteq E(G)$ for $l \neq i + 1$, which implies $B \cup \{x_P, u^\ast, u^+, y^\ast\} - \{b_{i+1}\}$ is an independent set of order $k + 2$, a contradiction. Thus, we have $P_i \subseteq N[y]$ for $1 \leq i \leq k - 1$ and hence $\{x_P, y\} \not\subseteq V(G)$, a contradiction.

**Case 2.** $|N(x_P) \cap \{x, y\}| = 2$

In this case, we let $z_2 \in P_i$.

Suppose $i = 1$, $l \geq 3$ and $z_l \in P_j$. Assume $z_l \neq z_2$. If $j \neq 1$, then $z_2^\ast z_l^\ast \not\subseteq E(G)$ for otherwise the $(x, y)$-path $xP_ixP\bar{x}Pz_2Pz_2^\astPz_l^\ast\bar{P}z_lP\bar{y}$ is hamiltonian. If $j = 1$ and $z_2^\ast z_l^\ast \subseteq E(G)$, then $z_l$ is an $A$-vertex if $z_l \in xPz_2$ and $z_2$ an $A$-vertex if $z_2 \in xPz_l$. By Lemma 3.1, $z_2a_2, z_2a_l \not\subseteq E(G)$, which is impossible since $[a_2, z_2] \rightarrow x_P$ and $[a_l, z_l] \rightarrow x_P$.

Thus, $z_2^\ast z_l^\ast \not\subseteq E(G)$ and hence $B \cup \{x_P, z_2^\ast, z_l^\ast\}$ is an independent set of order $k + 2$ by Lemmas 3.11, 3.13 and 3.14. Therefore, we have

$$z_l = z_2 \text{ for } 3 \leq l \leq k - 1 \text{ if } i = 1. \quad (4-5)$$

If $i \geq 2$, then $A \cup \{x_P, z_2^\ast\}$ is an independent set by Lemma 3.10. If $i = 1$, then by (4-5) and Lemma 3.12, $A \cup \{x_P, z_2^\ast\}$ is an independent set. By Lemmas 3.11 and 3.14, $B \cup \{x_P, z_2^\ast\}$ is an independent set. Thus, both $B \cup \{x_P, z_2^-\}$ and $A \cup \{x_P, z_2^\ast\}$ are independent sets.

If there is some $w \in V(G) - N[x_P]$ such that $[x_P, w] \rightarrow z_2^\ast$ ([x_P, w] \rightarrow z_2^\ast$, respectively), then $w \neq z_2$. By Lemma 3.8, $B \cup \{x_P, w^\ast\}$ is an independent set. By Lemma 3.7 we have $z_2^\ast w^\ast \not\subseteq E(G)$ and hence $B \cup \{x_P, w^\ast, z_2^\ast\}$ is an independent set of order $k + 2$, a contradiction. Thus, noting that both $B \cup \{x_P, z_2^-\}$ and $A \cup \{x_P, z_2^\ast\}$ are maximum independent sets, by Claim 4.2, we may assume $[z_2^\ast, w_1] \rightarrow x_P$ and $[z_2^\ast, w_2] \rightarrow x_P$.

Let $w_1 \in P_j$. If $w_1 \neq z_2$, then since $k \geq 4$, $A \cup \{z_2^-\}$ is an independent set and $[z_2^\ast, w_1] \rightarrow x_P$, we have $w_1 \not\subseteq A$, which implies $w_1^\ast x_P \not\subseteq E(G)$, and $A \subseteq N(w_1)$, which implies $w_1^\ast b_l \not\subseteq E(G)$ for $l \neq j + 1$ by Lemma 3.3. If $w_1^\ast b_{j+1} \in E(G)$, then $w_1$ is a $B$-vertex. Thus by Lemma 3.1 we have $B - \{b_{j+1}\} \subseteq N(z_2^-)$. If $j = 2$, then since $k \geq 4$,
there is some $l$ with $l \neq 2, i$ such that $z_2a_l \in E(G)$, which implies $z_2^+b_{l+1} \notin E(G)$ by Lemma 3.3, a contradiction. If $j \neq 2$, then by Lemma 3.5 we have $z_2^+z_2^j \notin E(G)$, which implies $w_1z_2^j \in E(G)$. Since $a_jz_2 \in E(G)$, by Lemma 3.3 we have $i = j$. Thus, since $k \geq 4$, there is some $l$ with $l \neq j$ such that $z_2a_l \in E(G)$, which implies $z_2^+b_{l+1} \notin E(G)$ by Lemma 3.3, also a contradiction. Hence, $B \cup \{x_P, w_1^-\}$ is an independent set. By Lemma 3.6, $z_2^-w_1^- \notin E(G)$. Thus by Lemma 3.1, $B \cup \{x_P, z_2^-, w_1^-\}$ is an independent set of order $k + 2$, a contradiction. Hence we have $w_1 = z_2$, that is,

$$[z_2^+, z_2] \to x_P. \quad (4-6)$$

If $w_2 \neq z_2$, then since $B \cup \{z_2^-, x_P\}$ is an independent set, we have $B \subseteq N(w_2)$. By (4-6), we have $A \subseteq N(z_2) \in E(G)$, which implies $z_2^-$ is an $A$-vertex. Thus, $A = \{a_1\} \subseteq N(w_2)$, which implies $|A \cup B - N(w_2)| \leq 1$. By Lemmas 3.7 and 3.8, we can see that $B \cup \{x_P, z_2^-, w_2\}$ is an independent set of order $k + 2$, a contradiction. Hence we have $w_2 = z_2$, that is,

$$[z_2^-, z_2] \to x_P. \quad (4-7)$$

By (4-6) and (4-7), $A \cup B \subseteq N(z_2)$. If there is some vertex $v \in a_1\overline{P}z_2$ such that $va_i \notin E(G)$ and $v^+a_i \in E(G)$, then $v$ is an $A$-vertex. If $vz_2^+ \in E(G)$, then $z_2$ is an $A$-vertex, which contradicts Lemma 3.1. Thus, $A \cup \{x_P, v, z_2^+\}$ is an independent set of order $k + 2$, a contradiction. Noting that $z_2 \in N(a_i)$, we have $a_i\overline{P}z_2 \subseteq N[a_i]$. By symmetry, we have $z_2\overline{P}b_{i+1} \subseteq N[b_{i+1}]$. If $N(z_2^+) \cap a_i\overline{P}z_2^- \neq \emptyset$, then since $a_i\overline{P}z_2 \subseteq N[a_i], z_2$ is an $A$-vertex and if $N(z_2^-) \cap z_2^+\overline{P}b_{i+1} \neq \emptyset$, then since $z_2\overline{P}b_{i+1} \subseteq N[b_{i+1}], z_2$ is a $B$-vertex, which contradicts Lemma 3.1 since $A \cup B \subseteq N(z_2)$. Thus, we have

$$N(z_2^+) \cap a_i\overline{P}z_2^- = \emptyset \text{ and } N(z_2^-) \cap z_2^+\overline{P}b_{i+1} = \emptyset. \quad (4-8)$$

Assume $z_1 \in P_3$ and $z_1 \neq z_2$. Since $[a_1, z_1] \to x_P$ and $k \geq 4$, by Lemma 3.1 we have $z_1 \notin A$, which implies $z_1^-x_P \notin E(G)$. If $j \neq k - 1$, then since $z_1a_{j+1} \in E(G)$, we have $b_1z_1^- \notin E(G)$ for $l \neq j + 1$ by Lemmas 3.3 and 3.4. If $b_{j+1}z_1^- \in E(G)$, then $z_1$ is a $B$-vertex. Thus, by Lemmas 3.1 and 3.9, there is some vertex $w \in P_{k-1}$ such that $w^+a_1, z_1w \in E(G)$, which contradicts Lemma 3.3. Hence, $B \cup \{x_P, z_1^-\}$ is an independent set. If $j = k - 1$, then $i \neq k - 1$ for otherwise $\{a_1, z_1\} \neq z_2^j$ if $z_1 \in a_{k-1}\overline{P}z_2^j$ by Lemma 3.10 and (4-8), and $\{a_1, z_1\} \neq z_2^j$ if $z_1 \in z_2^+\overline{P}b_k$ by (4-8) and Lemma 3.1 since $z_2^j$ is an $A$-vertex. Since $a_2z_1 \in E(G)$, we have $b_1z_1^- \notin E(G)$ for $l \neq 2, k$ by Lemma 3.3. If $b_2z_1^- \in E(G)$, then $b_3z_1 \notin E(G)$ by Lemma 3.2 which implies $a_1b_3 \in E(G)$. Since $[a_1, z_1] \to x_P$, we can see that either $a_1x_3 \in E(G)$ or $z_1x_3 \in E(G)$. Thus, the $(x, y)$-path $x_Px_2^+\overline{P}a_1^+b_3 \overline{P}a_3z_1 \overline{P}y$ is hamiltonian in the former case, and $xx_Px_2^+\overline{P}a_1^+b_3 \overline{P}a_3z_1 \overline{P}y$ is hamiltonian in the latter case, a contradiction. If $z_1^-b_k \in E(G)$, then $z_1$ is a $B$-vertex. By (4-8), $z_2^j$ is a $B$-vertex, which implies $z_2^+z_1 \notin E(G)$ by Lemma 3.1 and hence $\{a_1, z_1\} \neq z_2^j$, a contradiction. Thus, $B \cup \{x_P, z_1^-\}$ is an independent set. By (4-6) and (4-7), we have $A \cup B \subseteq N(z_2)$,
which implies $z_l^{-}z_2^{-} \notin E(G)$ by Lemma 3.7. Thus, $B \cup \{x_P, z_l^{-}, z_2^{-}\}$ is an independent set of order $k + 2$ and hence we have $z_1 = z_2$. By (4-5), we have $z_l = z_2$ for $l \geq 3$ if $i = 1$. If $i \geq 2$ and there is some $z_l$ with $l \geq 3$ such that $z_l \neq z_2$, then $B \cup \{x_P, z_l^{-}\}$ is an independent set by Lemmas 3.11, 3.13 and 3.14. By (4-6), $A \subseteq N(z_2)$ and hence $z_2^{-}z_l^{-} \notin E(G)$ by Lemma 3.6. Thus, $B \cup \{x_P, z_2^{-}, z_l^{-}\}$ is an independent set of order $k + 2$, a contradiction. Thus we have

$$z_l = z_2 \text{ for } l \neq 2.$$ \hspace{1cm} (4-9)

By (4-6), (4-7) and (4-8), we have $P_l \subseteq N[z_2]$ and $A \cup B \subseteq N(z_2)$. Let $l \neq i$. If there is some $u \in P_l$ such that $uz_2 \notin E(G)$, then $u^+, u^- \notin N(x_P)$ and $A \subseteq N(u)$ by (4-9). By Lemma 3.3, $b_mu^+, b_mu^- \notin E(G)$ for $m \neq l + 1$. By Lemma 3.5, $u^+u^- \notin E(G)$. By Lemma 3.7, $u^-z_2^- \notin E(G)$. If $u^+z_2^- \in E(G)$, then the $(x, y)$-path $xP_1xp_1Px_1Pz_2^-P_2a_1PuP_2a_1P_1y$ is hamiltonian if $l < i$ and if $l > i$, then $xP_1xp_1Pz_2^-P_2a_1PuP_2a_1P_1Pz_2^-u^+P_1y$ is hamiltonian, a contradiction. Thus, we have $u^+z_2^- \notin E(G)$, which implies $B \cup \{x_P, u^+, u^-, z_2^-\} - \{b_{l+1}\}$ is an independent set of order $k + 2$, a contradiction. Therefore, we have $P_l \subseteq N[z_2]$ for $l \neq i$, which implies $\{x_P, z_2\} \succ V(G)$, a contradiction.

The proof of Theorem 4 is complete. \hfill \blacksquare

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References


