Asymptotic Behavior of Underlying NT Paths in Interior Point Method for Monotone Semidefinite Linear Complementarity Problems

Chee-Khian Sim[1 2]

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[2]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. Email: macksim@inet.polyu.edu.hk

Abstract

An interior point method (IPM) defines a search direction at each interior point of the feasible region. These search directions form a direction field, which in turn gives rise to a system of ordinary differential equations (ODEs). Thus, it is natural to define the underlying paths of the IPM as solutions of the system of ODEs. In [32], these off-central paths are shown to be well-defined analytic curves and any of their accumulation points is a solution to the given monotone semidefinite linear complementarity problem (SDLCP). In [32]-[34], the asymptotic behavior of off-central paths corresponding to the HKM direction is studied. In particular, in [32], the authors study the asymptotic behavior of these paths for a simple example, while, in [33,34], the asymptotic behavior of these paths for a general SDLCP is studied. In this paper, we study off-central paths corresponding to another well-known direction, the Nesterov-Todd (NT) direction. Again, we give necessary and sufficient conditions for these off-central paths to be analytic w.r.t. $\sqrt{\mu}$ and then w.r.t. μ, at solutions of a general SDLCP. Also, as in [32], we present off-central path examples using the same SDP, whose first derivatives are likely to be unbounded as they approach the solution of the SDP. We work under the assumption that the given SDLCP satisfies a strict complementarity condition.

Keywords: Semidefinite linear complementarity problem; Interior point methods; NT direction; Local convergence; Ordinary differential equations

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1 Introduction.

The notion of a central path is introduced by Sonnevend [35] in 1985 to interior point methods (IPMs). Since then, researchers realize that an IPM is actually a homotopy method following underlying paths and that many remarkable properties of an IPM are attributed to the nice geometry of these paths. Readers who are interested to know more about basic geometry of these paths may refer to [2].

An important role the underlying paths play in the study of IPMs is to show its fast local convergence. The classical proof of local convergence of an iterative method, such as the Newton's method, for finding the solution of a system of equations relies on the nonsingularity of the Jacobian matrix. However, the Jacobian matrix of the equation system, defining the search direction in an IPM, may be singular at an optimal solution. Thus, the traditional approach of local convergence analysis does not work for IPM. Study of underlying paths, which mimic the behavior of iterates generated by an interior point algorithm, especially when the iterates are close together at a solution of the given problem, provides an alternative approach. Fast local convergence of IPMs has been successfully proved by relating it to the boundedness of derivatives of the underlying paths in [27,38,42,43]. See also [36,37]. Superlinear convergence of IPMs for linear complementarity problem (LCP) is proved in [18,19,20,14,6,24] by explicitly using the analyticity of off-central path. Also, [10,9,21,22,23] contain results about superlinear convergence of IPMs for SDP.

Study of fast local convergence is particularly important for the class of monotone semidefinite linear complementarity problems (SDLCPs), with the class of semidefinite programs (SDPs) as a special case, because, in contrast to a monotone linear complementarity problem (LCP), the exact solution of a SDLCP cannot be obtained from an approximate solution by determining a complementary basis.

The analysis for SDP, and hence for SDLCP, is considered to be more difficult than that for linear programming (LP). This arises mainly due to the difficulty in maintaining symmetry in the linearized complementarity [44]. Researchers working in the IPM area have proposed ways to overcome this drawback, which result in different symmetrized search directions, along which iterates generated by interior point algorithms move [1,7,13,28,29,30,31,41]. Among these search directions, the AHO, HKM and NT directions are more commonly used.

There are various ways in which the underlying paths, using these different search directions, for SDLCPs are defined in literature [3,15,17,26,32,40]. Paths arising from different search directions are likely to behave differently from each other. In [32], a new definition of the underlying paths of IPMs for SDLCPs, using ordinary differential equations (ODEs), is proposed. The motivation for defining paths in this way is to relate these paths to the vector field of search directions of an IPM. In [32]-[34], the behavior of these off-central paths, corresponding to the HKM direction, near solutions of a

SDLCP is investigated. It is found that, for off-central paths corresponding to the HKM direction, their asymptotic behavior [32] suggests that superlinear convergence of iterates, generated by a generic interior point algorithm using the HKM direction, may not be guaranteed. This contrasts with that for interior point algorithms using the AHO direction, where it has been shown that superlinear convergence of iterates, generated by a generic interior point algorithm (which does not, say, perform "narrowing" of the neighborhood), is possible [1,12,16,17,25,26]. It is interesting to investigate the asymptotic behavior of off-central paths corresponding to another well-known direction, the so-called NT direction [30,31], to see whether they behave like that corresponding to the AHO direction or that corresponding to the HKM direction.

In this paper, we use our definition of paths for SDLCPs [32] to study the asymptotic behavior of SDLCP paths, corresponding to the NT direction. To the author's knowledge, this study is the first to study off-central paths corresponding to the NT direction. In Section 2, through the same simple SDP example that we used in [32], we show the behavior of a few off-central paths as they approach the unique solution of the SDP. We show graphically that except for the central path, the first derivatives of all the other paths are likely to be unbounded as these paths approach the unique solution of the SDP. In Section 3, considering a general SDLCP, we give necessary and sufficient conditions for an off-central path, corresponding to the NT direction, to be analytic in the limit, w.r.t. $\sqrt{\mu}$ and then w.r.t. μ. Here, μ is related to the variable in the primal space and the variable in the dual space for a SDP (see Remark 2.1). Finally, we give some concluding remarks in Section 4.

1.1 Notations and Common Definitions.

The space of symmetric $n \times n$ matrices is denoted by S^n. Given matrices X and Y in $\Re^{p \times q}$, the standard inner product is defined by $X \bullet Y \equiv \mathrm{Tr}(X^T Y)$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. If $X \in S^n$ is positive semidefinite (resp., positive definite), we write $X \succeq 0$ (resp., $X \succ 0$). The cone of positive semidefinite (resp., positive definite) symmetric matrices is denoted by S^n_+ (resp., S^n_{++}). Either the identity matrix or operator will be denoted by I. $E_{ij} \in \Re^{n \times n}$ is a square matrix with 1 at its (i,j) entry and the rest of entries equal to zero. $\| \cdot \|$ for a vector in $\Re^n$ refers to its Euclidean norm, and for a matrix in $\Re^{p \times q}$, it refers to its Frobenius norm.

For a matrix $X \in \Re^{p \times q}$, we denote its component at the i^{th} row and j^{th} column by X_{ij}. Also, $X_{i\cdot}$ denotes the i^{th} row of X, and $X_{\cdot j}$ the j^{th} column of X. In case X is partitioned into blocks of submatrices, then X_{ij} refers to the submatrix in the corresponding (i,j) position.

Given square matrices $A_i \in \Re^{n_i \times n_i}$, $i = 1, \ldots, m$, $\mathrm{Diag}(A_1, \ldots, A_m)$ is a square matrix with A_i as its diagonal blocks arranged in accordance to the way they are lined up in $\mathrm{Diag}(A_1, \ldots, A_m)$. All the other entries in $\text{Diag}(A_1, \ldots, A_m)$ are taken to be zero.

Given functions $f : \Omega \longrightarrow E$ and $g : \Omega \longrightarrow \Re_{++}$, where Ω is an arbitrary set and E is a normed vector space, and a subset $\widetilde{\Omega} \subseteq \Omega$, we write $f(w) = O(g(w))$ for all $w \in \widetilde{\Omega}$ to mean that $\|f(w)\| \leq Mg(w)$ for all $w \in \widetilde{\Omega}$, where $M > 0$ is a positive constant. Suppose we have $E = S^n$. Then we write $f(w) = \Theta(g(w))$ if for all $w \in \widetilde{\Omega}$, $f(w) \in S^n_{++}$, $f(w) = \mathcal{O}(g(w))$ and $f(w)^{-1} = \mathcal{O}(1/g(w))$.

What the subset $\widetilde{\Omega}$ is should be clear from the context. Usually, $\widetilde{\Omega} = (0, \bar{w})$ for a small $\bar{w} > 0$.

A function $f = (f_1, \ldots, f_m)$ from an open subset $\mathcal{O}$ of $\Re^k$ to $\Re^m$ is analytic at a point $x = (x_1, \ldots, x_k) \in \mathcal{O}$ iff each f_i, $i = 1, \ldots, m$, can be written as a convergent power series expansion about x in an open neighborhood of x. Furthermore, if $x^0 \in \Re^k$ is on the boundary of $\mathcal{O}$, we say f is analytic at x^0 (or can be extended analytically to x^0), and we let $f(x^0) = \lim_{x \to x^0} f(x)$, iff there exists an analytic function g, which is analytic at x^0 and coincides with f wherever both are defined.

The above also applies if a component x_j of $x = (x_1, \ldots, x_k)$ is a symmetric matrix, instead of a number, in which case, we consider x_j to lie in an Euclidean space of appropriate dimension. If the range of f is in the space of (symmetric) matrices, we also view the space as an appropriate Euclidean space, when considering analyticity, so that the above applies.

2 Definition of an Off-central Path and an Investigation on NT Paths for an Example.

Let us consider the following SDLCP:

$$XY = 0, \tag{1}$$

$$\mathcal{A}(X) + \mathcal{B}(Y) = q, \tag{2}$$

$$X, Y \in \mathcal{S}^n_+, \tag{3}$$

where $\mathcal{A}, \mathcal{B} : \mathcal{S}^n \longrightarrow \Re^{\tilde{n}}$ are linear operators mapping $\mathcal{S}^n$ to the space $\Re^{\tilde{n}}$, where $\tilde{n} := n(n+1)/2$. Hence $\mathcal{A}$ and $\mathcal{B}$ have the form $\mathcal{A}(X) = (A_1 \bullet X, \ldots, A_{\tilde{n}} \bullet X)^T$, respectively, $\mathcal{B}(Y) = (B_1 \bullet Y, \ldots, B_{\tilde{n}} \bullet Y)^T$, where $A_i, B_i \in \mathcal{S}^n$ for all $i = 1, \ldots, \tilde{n}$. Also, $q \in \Re^{\tilde{n}}$.

We have the following assumptions on the SDLCP throughout the paper:

Assumption 2.1 *(a) SDLCP is monotone, i.e.,* $\mathcal{A}(X) + \mathcal{B}(Y) = 0$ *for* $X, Y \in \mathcal{S}^n \Rightarrow X \bullet Y \geq 0$.

(b) There exist $X^1, Y^1 \succ 0$*, such that* $\mathcal{A}(X^1) + \mathcal{B}(Y^1) = q$.

(c) $\{\mathcal{A}(X) + \mathcal{B}(Y) : X, Y \in S^n\} = \Re^{\tilde{n}}$.

Assumption 2.1(a)-(c) are basic assumptions used in the literature when SDLCP is studied in the context of IPM.

Let us now define the off-central path for SDLCP passing through a point (X^0, Y^0), $X^0, Y^0 \succ 0$, satisfying $A(X) + B(Y) = q$.

Definition 2.1 *The solution $(X(\mu), Y(\mu))$, $X(\mu)$, $Y(\mu) \succ 0$, to*

$$H_P(XY' + X'Y) = \frac{1}{\mu} H_P(XY), \tag{4}$$

$$A(X') + B(Y') = 0, \tag{5}$$

where $\mu > 0$, with the initial condition $(X(1), Y(1)) = (X^0, Y^0)$, X^0, $Y^0 \succ 0$, is the **off-central path** *for SDLCP, corresponding to P, passing through (X^0, Y^0). Here $H_P(U) := \frac{1}{2}(PUP^{-1} + (PUP^{-1})^T)$, and $P \in \Re^{n\times n}$ is an invertible matrix.*

Assuming P be an analytic function of X, Y and $PXYP^{-1}$ be always symmetric (such P include the well-known directions like the HKM, and its dual, and NT directions), it is proved in [32] that the above is well-defined, and $(X(\mu), Y(\mu))$ is unique, analytic over $\mu \in (0, \infty)$. The motivation for defining an off-central path as in Definition 2.1 is also given in [32].

Remark 2.1 *The central path $(X_c(\mu), Y_c(\mu))$ for SDLCP, which satisfies $X_c(\mu)Y_c(\mu) = \mu I$, is a special, but important, example of an off-central path for SDLCP. When $\mu = 1$, it satisfies Tr$(X_c(1)Y_c(1)) = n$. Therefore, we also require the initial data (X^0, Y^0) when $\mu = 1$ in (4)-(5) to satisfy Tr$(X^0Y^0) = n$. In this case, using (4), it is easy to see that the parameter μ in the ODE system (4)-(5) actually represents the duality gap, $X(\mu) \bullet Y(\mu)$, at the point $(X(\mu), Y(\mu))$ on the path.*

Let us now consider a SDP example as follows:

$$(\mathcal{P}) \qquad \min \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet X$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \bullet X = 2, \quad \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \bullet X = 0, \ X \in S^2_+$$

and

$$(\mathcal{D}) \qquad \max \quad 2v_1$$

$$\text{subject to} \quad v_1 \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} + Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ Y \in S^2_+.$$

This example, which first appeared in [11], has all the nice properties (e.g. primal and dual nondegeneracy, strict complementarity, e.t.c.) for a SDP. The example is also used in [**?**] to analyze the asymptotic behavior of its off-central paths corresponding to the HKM direction.

It has an unique solution, $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$, which satisfies strict complementarity and nondegeneracy.

Written as a SDLCP, the above example can be expressed as

$$XY = 0$$
$$\mathcal{A}svec(X) + \mathcal{B}svec(Y) = q$$
$$X, Y \in \mathcal{S}^2_+,$$

where $\mathcal{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sqrt{2} & 2 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Note that $\mathcal{A}$ and $\mathcal{B}$ are the corresponding matrix representation of the linear operator A and B respectively in (2).

In [32], off-central paths for the example are analyzed. The off-central paths analyzed correspond to the dual HKM direction, which are obtained by setting $P = Y^{1/2}$ in (4). It is shown analytically in [32] that unless an off-central path satisfies certain algebraic condition, its first derivatives are unbounded in the limit as the path approaches the unique solution of the example. In this section, we consider off-central paths for the same SDP example, but corresponding to the NT direction, where P in (4) is such that $P^TP = W^{-1}$, W is a symmetric, positive definite matrix with $WYW = X$.

Although it is known that one can express W in terms of X and Y, it is still difficult to write down explicitly each entry of W in terms of that of X and Y for the example, even if these matrices are only of size two. We are therefore unable to further analyze the asymptotic behavior of these paths analytically, like the (dual) HKM case in [32]. As an alternative, we have written Matlab codes to study the behavior of the first derivatives of various off-central paths corresponding to the NT direction, for the example.

Using $\mathrm{Tr}(X^0Y^0) = 2$ and (5), we have the off-central path for the example, $(X(\mu), Y(\mu))$, passing through $(X^0, Y^0) = (X(1), Y(1))$, is of the form

$$X(\mu) = \begin{pmatrix} 1 & 2\mu - y_1(\mu) \\ 2\mu - y_1(\mu) & 2\mu - y_1(\mu) \end{pmatrix} \text{ and } \mathrm{Y}(\mu) = \begin{pmatrix} y_1(\mu) & y_2(\mu) \\ y_2(\mu) & 1 - 2y_2(\mu) \end{pmatrix}.$$

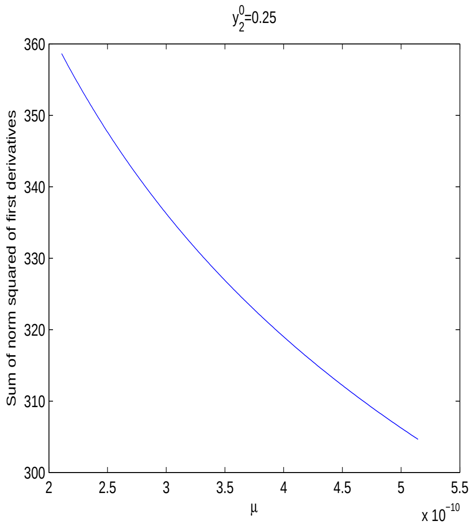
The central path for the example is given by

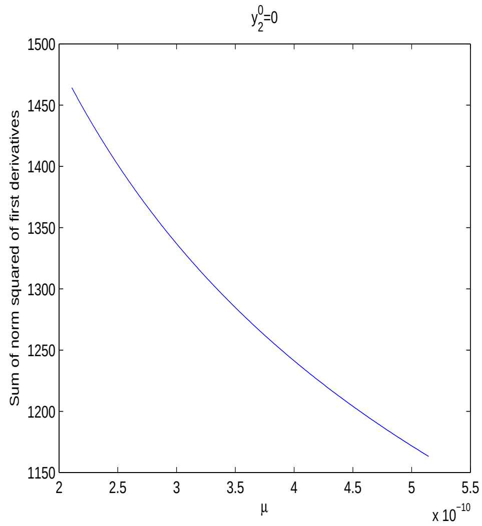
$$X_c(\mu) = \begin{pmatrix} 1 & 2\mu - y_1^c(\mu) \\ 2\mu - y_1^c(\mu) & 2\mu - y_1^c(\mu) \end{pmatrix} \text{ and } \mathrm{Y_c}(\mu) = \begin{pmatrix} y_1^c(\mu) & y_2^c(\mu) \\ y_2^c(\mu) & 1 - 2y_2^c(\mu) \end{pmatrix},$$

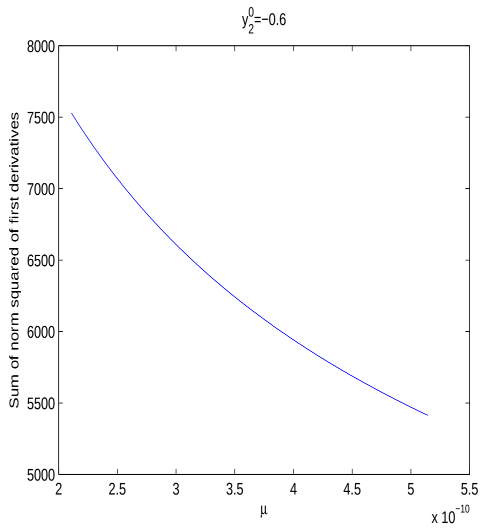
where $y_2^c(\mu) = -y_1^c(\mu)$ and

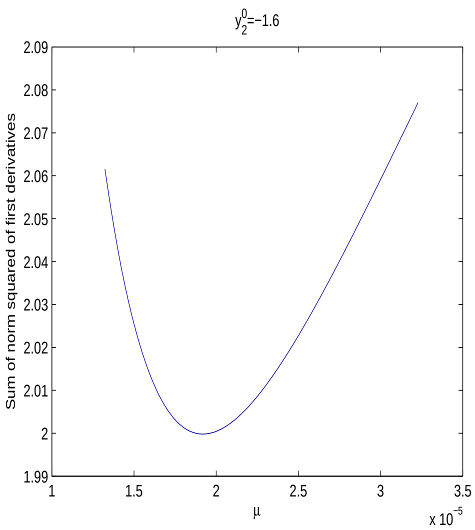
$$y_1^c(\mu) = \frac{2\mu}{\sqrt{1 + 4\mu^2} + 1 - 2\mu}. \tag{6}$$

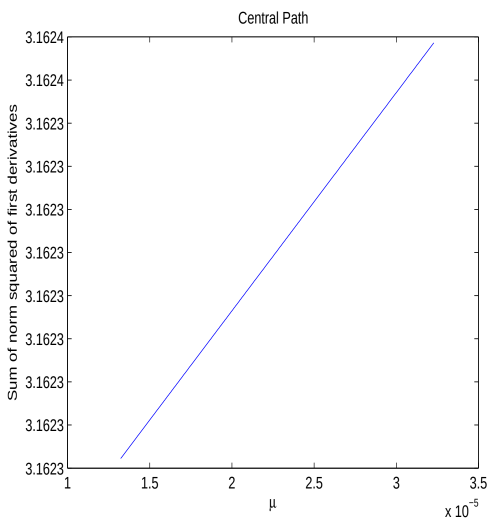
In the below plots, X^0 are all set to be equal to $X_c(1)$, while Y^0 are varied by varying $y_2^0 = y_2(1)$, keeping $y_1^0 = y_1(1)$ equal to $y_1^c(1) = \frac{2}{\sqrt{5}-1}$.

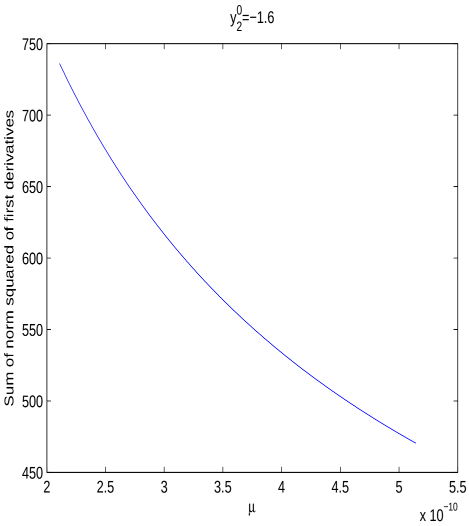


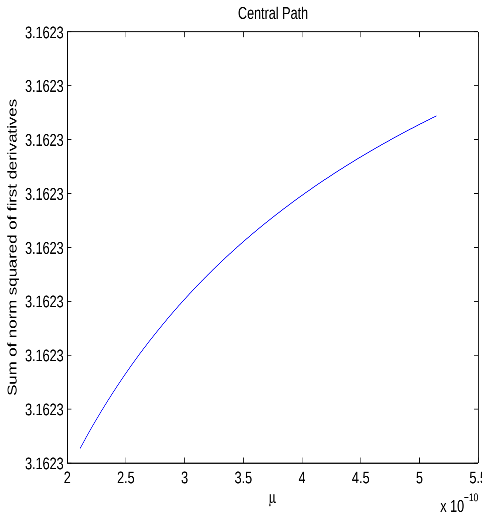












As observed from these plots, we see that except for the case when the off-central path is actually the central path, the sums of norm squared of first derivatives[1] of the given paths get larger as μ approaches zero. The first three plots are for paths that are initiated relatively far from the central path, while the fourth and sixth plots are for the same off-cental path that is initiated near to the central path. Comparison between the fourth and fifth plot shows that for this off-central path with $y_2^0 = -1.6$, for the same range of μ, the sum of norm squared of its derivatives starts to increase while that of the central path remains stable. For the case of the central path, one can observe that its sum of first derivatives' norm squared is approaching $3.162\ldots$ as μ approaches zero (seventh plot), which should be the case, as can be verified using $y_2^c(\mu) = -y_1^c(\mu)$ and (6).

Hence, from these graphical examples, they indicate the possibility that unless the off-central paths corresponding to the NT direction, for the example, satisfy certain condition, their first derivatives are likely to be unbounded as μ tends to zero.

3 Asymptotic Analyticity Behavior of NT Paths for a general SDLCP.

Based on what is observed of the first derivatives of off-central paths for the SDP example in the above section, it suggests that off-central paths corresponding to the NT direction are in general not analytic at solutions of a SDLCP. In this section, we study the conditions that are required to ensure analyticity of paths corresponding to the NT direction, for a general SDLCP. We investigate the asymptotic analyticity of an off-central path, corresponding to the NT direction, first w.r.t. $\sqrt{\mu}$ and then w.r.t.

[1]Sum of norm squared of first derivatives here stands for $\sqrt{\|X'(\mu)\|^2 + \|Y'(\mu)\|^2}$.

μ. By studying the conditions which ensure asymptotic analyticity of paths, one can hope to ensure that iterates generated by interior point algorithms are guaranteed to converge superlinearly as they approach a solution of a given SDLCP.

We require an additional assumption, besides Assumption 2.1, to carry out the analysis.

Assumption 3.1 *There exists a strictly complementary solution,* (X^*, Y^*)*, to SDLCP (1)-(3).*

The analysis of the asymptotic behavior of an off-central path for a general SDLCP is considered to be difficult without this assumption (Assumption 3.1). However, we note that there have been some work done in this area for special classes of SDLCP without the assumption. See for example [5].

Let (X^*, Y^*) be a strictly complementary solution to SDLCP (1)-(3), which exists by Assumption 3.1.

Since X^* and Y^* commute, they are jointly diagonalizable by some orthogonal matrix. So, using a suitable orthogonal similarity transformation of the matrices in SDLCP (1)-(3), we may assume, without loss of generality, that

$$X^* = \begin{pmatrix} \Lambda_{11}^* & 0 \\ 0 & 0 \end{pmatrix}, Y^* = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{22}^* \end{pmatrix},$$

where $\Lambda_{11}^* = \mathrm{Diag}(\lambda_1^*, \ldots, \lambda_m^*) \succ 0$ and $\Lambda_{22}^* = \mathrm{Diag}(\lambda_{m+1}^*, \ldots, \lambda_n^*) \succ 0$. Here $\lambda_1^*, \ldots, \lambda_n^*$ are real numbers greater than zero.

Hereafter, whenever we partition a matrix $S \in S^n$, we do it in a similar way, i.e., S is always partitioned as $\begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix}$, where $S_{11} \in S^m, S_{22} \in S^{n-m}$ and $S_{12} \in \Re^{m \times (n-m)}$. The same holds for $S \in \Re^{n \times n}$, which may not necessarily be symmetric.

3.1 Analyticity w.r.t. $\sqrt{\mu}$.

Let us now analyze the analyticity of off-central paths, corresponding to the NT direction, w.r.t. $\sqrt{\mu}$.

From now onwards, we may occasionally suppress the dependence of a variable on another variable. For example, instead of writing $X(\mu)$ for the primal part of an off-central path, we simply write X. We do this only when the dependency is clear from the context, and also for the sake of not cluttering an expression with too many symbols.

Written in matrix-vector form, the ODE system (4)-(5) can be expressed as:

$$\begin{pmatrix} \mathrm{svec}(A_1)^T & \mathrm{svec}(B_1)^T \\ \vdots & \vdots \\ \mathrm{svec}(A_{\tilde{n}})^T & \mathrm{svec}(B_{\tilde{n}})^T \\ P \otimes_s (P^{-T}Y) & (PX) \otimes_s P^{-T} \end{pmatrix} \begin{pmatrix} \mathrm{svec}(X') \\ \mathrm{svec}(Y') \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} 0 \\ \mathrm{svec}(H_P(XY)) \end{pmatrix}, \quad (7)$$

where $\tilde{n} = n(n+1)/2$.

Here, the operation $\otimes_s$ and the map svec (with inverse smat) are used, whose properties are given on pp. 775 – 776 and the appendix of [39].

Considering the NT direction, which corresponds to P, such that $P^TP = W^{-1}$, W a symmetric, positive definite matrix with $WYW = X$, (7) becomes

$$\begin{pmatrix} \text{svec}(A_1)^T & \text{svec}(B_1)^T \\ \vdots & \vdots \\ \text{svec}(A_{\tilde{n}})^T & \text{svec}(B_{\tilde{n}})^T \\ I & W \otimes_s W \end{pmatrix} \begin{pmatrix} \text{svec}(X') \\ \text{svec}(Y') \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} 0 \\ \text{svec}(X) \end{pmatrix}, \tag{8}$$

by taking $P = W^{-1/2}$ in (7).

Remark 3.1 *Note that W can be explicitly expressed in terms of X, Y by*

$$W = X^{1/2}(X^{1/2}YX^{1/2})^{-1/2}X^{1/2} = Y^{-1/2}(Y^{1/2}XY^{1/2})^{1/2}Y^{-1/2},$$

using $WYW = X$, X, Y, W symmetric, positive definite matrices.

Following the approach in [33], we analyze the asymptotic analyticity behavior of an off-central path, $(X(\mu), Y(\mu))$, w.r.t. $\sqrt{\mu}$, by performing a transformation on (8) to an equivalent ODE system so that analysis is possible.

Let $X_1(t) := X(t^2)$, $Y_1(t) := Y(t^2)$. Define $\widetilde{X}_1(t)$, $\widetilde{Y}_1(t)$ by

$$X_1(t) = \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1(t) \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \tag{9}$$

and

$$Y_1(t) = \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}_1(t) \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}. \tag{10}$$

Now $W(\mu)$ for an off-central path, $(X(\mu), Y(\mu))$, corresponding to the NT direction, is such that $WYW = X$. From this, we have

$$W \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}_1 \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} W = \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1 \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}$$

using (9) and (10). This implies that

$$\frac{1}{t} \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} W \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}_1 \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} W \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \frac{1}{t} = \widetilde{X}_1.$$

Define

$$\widetilde{W}_1(t) := \frac{1}{t}\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} W(t^2) \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}. \tag{11}$$

Remark 3.2 *$\widetilde{X}_1(t)$, $\widetilde{Y}_1(t)$ and $\widetilde{W}_1(t)$ (of an off-central path) given above in (9)-(11) satisfy $\widetilde{W}_1\widetilde{Y}_1\widetilde{W}_1 = \widetilde{X}_1$. Hence, we can write*

$$\widetilde{W}_1 = \widetilde{X}_1^{1/2}(\widetilde{X}_1^{1/2}\widetilde{Y}_1\widetilde{X}_1^{1/2})^{-1/2}\widetilde{X}_1^{1/2} \tag{12}$$

by Remark 3.1. Therefore, $\widetilde{W}_1$ is an analytic function of $\widetilde{X}_1, \widetilde{Y}_1$, and for $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ corresponding to the off-central path $(X(\mu), Y(\mu))$, $\widetilde{W}_1(t)$ is an analytic function of t. Also, by (9), and that

$$X_1(t) = \begin{pmatrix} \Theta(1) & \mathcal{O}(t) \\ \mathcal{O}(t) & \Theta(t^2) \end{pmatrix},$$

we have $\widetilde{X}_1(t)$ is symmetric, positive definite in the limit as t approaches zero. Similar property holds for $\widetilde{Y}_1(t)$. Using (12), $\widetilde{W}_1(t)$ is hence symmetric, positive definite in the limit as t approaches zero.

Let us now reformulate (8) in terms of $\widetilde{X}_1(t), \widetilde{Y}_1(t), \widetilde{W}_1(t)$, their derivatives and t, in order to analyze the asymptotic analyticity behavior of $(X_1(t), Y_1(t)) = (X(t^2), Y(t^2))$, as t approaches zero.

From (8), we have

$$\text{svec}(X') + (W \otimes_s W)\text{svec}(Y') = \frac{1}{\mu}\text{svec}(X).$$

Expressing the above in terms of $\widetilde{X}_1(t), \widetilde{Y}_1(t), \widetilde{W}_1(t)$, their derivatives and $t(= \sqrt{\mu})$, we obtain

$$\begin{aligned} &\text{svec}(\widetilde{X}_1') + (\widetilde{W}_1 \otimes_s \widetilde{W}_1)\text{svec}(\widetilde{Y}_1') \\ = \ & \frac{1}{t}\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right)\text{svec}(\widetilde{X}_1) - \frac{1}{t}(\widetilde{W}_1 \otimes_s \widetilde{W}_1)\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right)\text{svec}(\widetilde{Y}_1). \end{aligned} \tag{13}$$

Following [33],

$$\begin{pmatrix} \text{svec}(A_1)^T \\ \vdots \\ \text{svec}(A_{\tilde{n}})^T \end{pmatrix}\text{svec}(X') + \begin{pmatrix} \text{svec}(B_1)^T \\ \vdots \\ \text{svec}(B_{\tilde{n}})^T \end{pmatrix}\text{svec}(Y') = 0$$

from (8) can be expressed as

$$\mathcal{A}(t)\text{svec}(\widetilde{X}_1') + \mathcal{B}(t)\text{svec}(\widetilde{Y}_1') = -\mathcal{G}(t)\text{svec}(\widetilde{X}_1) - \mathcal{H}(t)\text{svec}(\widetilde{Y}_1), \tag{14}$$

where

$$\mathcal{A}(t)_{k\cdot} := \begin{cases} \left(\operatorname{svec}\begin{pmatrix} (A_k)_{11} & t(A_k)_{12} \\ t(A_k)_{12}^T & t^2(A_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \leq k \leq i_1 \\ \left(\operatorname{svec}\begin{pmatrix} 0 & (A_k)_{12} \\ (A_k)_{12}^T & t(A_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1+1 \leq k \leq i_1+i_2 \\ \left(\operatorname{svec}\begin{pmatrix} 0 & 0 \\ 0 & (A_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1+i_2+1 \leq k \leq \tilde{n} \end{cases}, \tag{15}$$

$$\mathcal{B}(t)_{k\cdot} := \begin{cases} \left(\operatorname{svec}\begin{pmatrix} t^2(B_k)_{11} & t(B_k)_{12} \\ t(B_k)_{12}^T & (B_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \leq k \leq i_1 \\ \left(\operatorname{svec}\begin{pmatrix} t(B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & 0 \end{pmatrix}\right)^T & \text{for } i_1+1 \leq k \leq i_1+i_2 \\ \left(\operatorname{svec}\begin{pmatrix} (B_k)_{11} & 0 \\ 0 & 0 \end{pmatrix}\right)^T & \text{for } i_1+i_2+1 \leq k \leq \tilde{n} \end{cases}, \tag{16}$$

$$\mathcal{G}(t)_{k\cdot} := \begin{cases} \left(\operatorname{svec}\begin{pmatrix} 0 & (A_k)_{12} \\ (A_k)_{12}^T & 2t(A_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \leq k \leq i_1 \\ \left(\operatorname{svec}\begin{pmatrix} 0 & 0 \\ 0 & (A_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1+1 \leq k \leq i_1+i_2 \\ 0 & \text{for } i_1+i_2+1 \leq k \leq \tilde{n} \end{cases}, \tag{17}$$

and

$$\mathcal{H}(t)_{k\cdot} := \begin{cases} \left(\operatorname{svec}\begin{pmatrix} 2t(B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & 0 \end{pmatrix}\right)^T & \text{for } 1 \leq k \leq i_1 \\ \left(\operatorname{svec}\begin{pmatrix} (B_k)_{11} & 0 \\ 0 & 0 \end{pmatrix}\right)^T & \text{for } i_1+1 \leq k \leq i_1+i_2 \\ 0 & \text{for } i_1+i_2+1 \leq k \leq \tilde{n} \end{cases}, \tag{18}$$

for some fixed i_1, i_2.

Details on how $\mathcal{A}(t), \mathcal{B}(t), \mathcal{G}(t)$ and $\mathcal{H}(t)$ are derived can be obtained from [33].

Combining (13) and (14), we have that $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$, corresponding to the off-central path

$(X(\mu), Y(\mu))$, satisfies

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{W}_1 \otimes_s \widetilde{W}_1 \end{pmatrix} \begin{pmatrix} \text{svec}(\widetilde{X}_1') \\ \text{svec}(\widetilde{Y}_1') \end{pmatrix} =$$

$$\begin{pmatrix} -\mathcal{G}(t) & -\mathcal{H}(t) \\ \frac{1}{t}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I & -\frac{1}{t}(\widetilde{W}_1 \otimes_s \widetilde{W}_1)\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right) \end{pmatrix} \begin{pmatrix} \text{svec}(\widetilde{X}_1) \\ \text{svec}(\widetilde{Y}_1) \end{pmatrix}. \quad (19)$$

We have the following proposition:

Proposition 3.1

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{W}_1 \otimes_s \widetilde{W}_1 \end{pmatrix}$$

which appears on the left hand side of (19) is analytic and invertible for all $t \geq 0$ *and* $\widetilde{W}_1$ *positive definite.*

Proof: The analyticity of

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{W}_1 \otimes_s \widetilde{W}_1 \end{pmatrix}$$

is clear.

To show that it is invertible for all $t \geq 0$ and $\widetilde{W}_1$ positive definite, it suffices to show that

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{W}_1 \otimes_s \widetilde{W}_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad \Rightarrow \quad u = v = 0.$$

Now

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{W}_1 \otimes_s \widetilde{W}_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

implies that

$$\begin{aligned} \mathcal{A}(t)u + \mathcal{B}(t)v &= 0 \\ u + (\widetilde{W}_1 \otimes_s \widetilde{W}_1)v &= 0. \end{aligned} \quad (20)$$

$\mathcal{A}(t)u + \mathcal{B}(t)v = 0 \Rightarrow u^T v \geq 0$, a proof of which can be found from the proof of Proposition 2.3 in [33].

By positive definite of $\widetilde{W}_1$, and (20), we then have $u = v = 0$. **QED**

Consider

$$\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right) \text{svec}(\widetilde{X}_1) - (\widetilde{W}_1 \otimes_s \widetilde{W}_1)\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right) \text{svec}(\widetilde{Y}_1)$$

in (19).

Expanding the above expression and in matrix form, it is equal to

$$\begin{pmatrix} (\widetilde{X}_1)_{11} & 0 \\ 0 & -(\widetilde{X}_1)_{22} \end{pmatrix} - \widetilde{W}_1 \begin{pmatrix} (\widetilde{Y}_1)_{11} & 0 \\ 0 & -(\widetilde{Y}_1)_{22} \end{pmatrix} \widetilde{W}_1. \tag{21}$$

Using (9)-(11), (21) and Proposition 3.1 on the ODE system (19), we have the following theorem:

Theorem 3.1 *Let $(X(\mu), Y(\mu))$, $\mu > 0$, be an off-central path for SDLCP (1)-(3) under Assumptions 2.1 and 3.1. Then $X(\mu), Y(\mu)$ are analytic as functions of $t = \sqrt{\mu}$ at $t = 0$ if and only if*

$$\frac{1}{t}\begin{pmatrix} I & 0 \\ 0 & \frac{1}{t}I \end{pmatrix}\left[\begin{pmatrix} X_{11}(\mu) & 0 \\ 0 & -X_{22}(\mu) \end{pmatrix} - W(\mu)\begin{pmatrix} Y_{11}(\mu) & 0 \\ 0 & -Y_{22}(\mu) \end{pmatrix} W(\mu)\right]\begin{pmatrix} I & 0 \\ 0 & \frac{1}{t}I \end{pmatrix}$$

is analytic as a function of $t = \sqrt{\mu}$ at $t = 0$.

A sufficient condition for analyticity of $(X(\mu), Y(\mu))$ as a function of $t = \sqrt{\mu}$ is given in the following:

Theorem 3.2 *Let $(X(\mu), Y(\mu))$, $\mu > 0$, be an off-central path for SDLCP (1)-(3) under Assumptions 2.1 and 3.1. If $W_{12}(\mu)$ converges to zero as μ tends to zero, and is analytic as a function of $t = \sqrt{\mu}$ at $t = 0$, then $X(\mu), Y(\mu)$ are analytic as functions of $t = \sqrt{\mu}$ at $t = 0$.*

Proof. Suppose that $W_{12}(\mu)$ converges to zero as μ tends to zero, and is analytic as a function of $t = \sqrt{\mu}$ at $t = 0$. From (11), this is equivalent to $(\widetilde{W}_1)_{12}(t)$ converges to zero as $t \to 0$, and is analytic at $t = 0$, since $(\widetilde{W}_1)_{12}(t) = W_{12}(t^2)$.

Given $\widetilde{X}_1, \widetilde{Y}_1, \widetilde{W}_1$, symmetric, positive definite, with $\widetilde{W}_1\widetilde{Y}_1\widetilde{W}_1 = \widetilde{X}_1$. It is easy to show that each block component of the matrix in (21) can be expressed in terms of $(\widetilde{W}_1)_{12}$ (or its transpose), other block entries of $\widetilde{W}_1$ and block entries of $\widetilde{X}_1, \widetilde{Y}_1$.

Now, $(\widetilde{W}_1)_{12}(t) = tV(t)$, where $V(t)$ is analytic at $t = 0$. Hence, the matrix in (21) can be expressed as a product of t and an analytic matrix function of $(t, \widetilde{X}_1, \widetilde{Y}_1)$.

Therefore, $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ of the given off-central path satisfies an ODE system, in standard form ($y' = f(t, y)$), derived from (19), whose right hand side is analytic at any accumulation points of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$, as $t \to 0$. Hence, by Theorem 4.1 of [4], pp. 15 and Theorem 2.1 of [32], we have $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ can be analytically extended to $t = 0$, which implies that $X(\mu), Y(\mu)$ are analytic as functions of $t = \sqrt{\mu}$ at $t = 0$. **QED**

Theorem 3.2 is similar to Theorem 3.2 in [33], but in the latter, the sufficient condition is also necessary for analyticity. One may ask whether the same holds here.

If $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ is analytic at $t = 0$, then clearly, $(\widetilde{W}_1)_{12}(t)$ is analytic at $t = 0$. What about the condition "$(\widetilde{W}_1)_{12}(t)$ tends to zero as $t \to 0$"? If this condition holds, then the sufficient condition for asymptotic analyticity of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ in the above Theorem 3.2, which is equivalent to "$(\widetilde{W}_1)_{12}(t)$ converges to zero as $t \to 0$ and is analytic at $t = 0$", is also necessary.

Suppose $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ derived from an off-central path, $(X(\mu), Y(\mu))$, corresponding to the NT direction is analytic at $t = 0$. These lead to the following conditions which must be satisfied by $\widetilde{X}_1^* = \lim_{t\to 0} \widetilde{X}_1(t)$, $\widetilde{Y}_1^* = \lim_{t\to 0} \widetilde{Y}_1(t)$, $\widetilde{W}_1^* = \lim_{t\to 0} \widetilde{W}_1(t)$:

$$\widetilde{W}_1^* \widetilde{Y}_1^* \widetilde{W}_1^* \quad = \quad \widetilde{X}_1^*, \tag{22}$$

$$\widetilde{W}_1^* \begin{pmatrix} (\widetilde{Y}_1^*)_{11} & 0 \\ 0 & -(\widetilde{Y}_1^*)_{22} \end{pmatrix} \widetilde{W}_1^* \quad = \quad \begin{pmatrix} (\widetilde{X}_1^*)_{11} & 0 \\ 0 & -(\widetilde{X}_1^*)_{22} \end{pmatrix}. \tag{23}$$

Here, $\widetilde{X}_1^*, \widetilde{Y}_1^*, \widetilde{W}_1^*$ are symmetric, positive definite matrices.

Condition (22) is derived from the fact that $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ is derived from an off-central path corresponding to the NT direction, while Condition (23) is necessary for $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ to be analytic at $t = 0$.

Also, from the monotonicity assumption of SDLCP (1)-(3), Assumption 2.1(a), we have $\widetilde{X}_1^*, \widetilde{Y}_1^*$ must also satisfy

$$(\widetilde{X}_1^*)_{12} \bullet (\widetilde{Y}_1^*)_{12} \geq 0. \tag{24}$$

Does Conditions (22)-(24) suffice to conclude that $(\widetilde{W}_1)_{12}(t)$ converges to zero as $t \to 0$, that is, $(\widetilde{W}_1^*)_{12} = 0$?

It turns out that in general, using Conditions (22)-(24), we cannot conclude that $(\widetilde{W}_1^*)_{12} = 0$. Here is a counterexample:

Example 3.1 *Let*

$$(\widetilde{W}_1^*)_{12} = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, (\widetilde{Y}_1^*)_{12} = \begin{pmatrix} 0 & \beta_1 \\ -\beta & 0 \end{pmatrix},$$

$$(\widetilde{Y}_1^*)_{11} = (\widetilde{Y}_1^*)_{22} = (\widetilde{W}_1^*)_{11} = (\widetilde{W}_1^*)_{22} = I_{2\times 2}.$$

Choose β be a small value between 0 and 1, and

$$\beta_1^2 = -\beta^4 + \beta^2.$$

We have $\widetilde{X}_1^*, \widetilde{Y}_1^*$ and $\widetilde{W}_1^*$ defined this way, where $\widetilde{X}_1^* = \widetilde{W}_1^* \widetilde{Y}_1^* \widetilde{W}_1^*$, are symmetric, positive definite matrices satisfying (22)-(24) with (24) being satisfied with equality, while $(\widetilde{W}_1^*)_{12} \neq 0$.

However, under a further restriction on the matrices involved, the conclusion "$(\widetilde{W}_1^*)_{12} = 0$" holds, as shown in the following proposition:

Proposition 3.2 *If the upper left hand block of every matrix involved is one dimensional, Conditions (22)-(24) implies that* $(\widetilde{W}_1^*)_{12} = 0$.

Proof. By pre-multiplying and post-multiplying $\widetilde{Y}_1^*$ by

$$\begin{pmatrix} (\widetilde{Y}_1^*)_{11}^{-1/2} & 0 \\ 0 & (\widetilde{Y}_1^*)_{22}^{-1/2} \end{pmatrix}, \tag{25}$$

we can assume the diagonal blocks of $\widetilde{Y}_1^*$ to be identity matrices. Also, in order for Conditions (22)-(24) to hold for the new $\widetilde{Y}_1^*$, we have to take the new $\widetilde{W}_1^*$ to be $\widetilde{W}_1^*$ pre- and post-multiplied by the inverse of the above matrix (25).

With these transformations, if the upper left hand block of every matrix involved is one dimensional, we have

$$\widetilde{Y}_1^* = \begin{pmatrix} 1 & y^T \\ y & I \end{pmatrix}, \widetilde{W}_1^* = \begin{pmatrix} w_{11} & w^T \\ w & (\widetilde{W}_1^*)_{22} \end{pmatrix},$$

where w_{11} is a scalar and y, w are column vectors.

We are done if we can show that $w = 0$.

From Conditions (22) and (23), we have

$$w_{11}w^Ty + \|w\|^2 = 0, \tag{26}$$

$$2ww^T + (\widetilde{W}_1^*)_{22}yw^T + wy^T(\widetilde{W}_1^*)_{22} = 0, \tag{27}$$

$$w_{11}w = (\widetilde{W}_1^*)_{22}w. \tag{28}$$

Post-multiplying the equation in (27) by w, we obtain

$$2\|w\|^2w + \|w\|^2(\widetilde{W}_1^*)_{22}y + wy^T(\widetilde{W}_1^*)_{22}w = 0.$$

Using (28) and then (26) on the above equation, we obtain $y = -(\widetilde{W}_1^*)_{22}^{-1}w$, if we assume $w \neq 0$.

Using Condition (24), we have

$$w_{11}w^Ty + (w^Ty)^2 + w_{11}y^T(\widetilde{W}_1^*)_{22}y + w^T(\widetilde{W}_1^*)_{22}y \geq 0.$$

Substituting $y = -(\widetilde{W}_1^*)_{22}^{-1} w$ derived earlier into the above inequality and using (28), we obtain

$$\|w\|^2 \left(\frac{\|w\|^2}{w_{11}^2} - 1 \right) \geq 0.$$

Since we assume $w \neq 0$, we then have $\|w\| \geq w_{11}$. But this is a contradiction to the positive definiteness of $\widetilde{W}_1^*$. Hence we are done. **QED**

From Proposition 3.2, we have the following corollary:

Corollary 3.1 *Let $m = 1$ or $m = n-1$, that is, a primal optimal solution X^* to SDLCP (1)-(3) has rank 1 or $n-1$.*
Suppose $(X(\mu), Y(\mu))$, $\mu > 0$, is an off-central path for SDLCP (1)-(3) under Assumptions 2.1 and 3.1. Then $X(\mu), Y(\mu)$ are analytic as functions of $t = \sqrt{\mu}$ at $t = 0$ if and only if $W_{12}(\mu)$ converges to zero as μ tends to zero, and is analytic as a function of $t = \sqrt{\mu}$ at $t = 0$.

Remark 3.3 *From Example 3.1, we see that for $2 \leq m \leq n-2$, the sufficient condition in Theorem 3.2 is unlikely to be also necessary for analyticity of an off-central path corresponding to the NT direction, w.r.t. $\sqrt{\mu}$, in general.*

3.2 Analyticity w.r.t. μ.

Now, let us turn our attention to the analyticity of off-central paths, corresponding to the NT direction, w.r.t. μ.

To analyze the analyticity of an off-central path, $(X(\mu), Y(\mu))$, with respect to μ, a necessary condition is that $X_{12}(\mu) = \mathcal{O}(\mu)$ and $Y_{12}(\mu) = \mathcal{O}(\mu)$. Hence, we take $X_{12}(\mu) = \mathcal{O}(\mu), Y_{12}(\mu) = \mathcal{O}(\mu)$ from now onwards. Also, a necessary condition for $X(\mu), Y(\mu)$ to be analytic at $\mu = 0$ is $W_{12}(\mu) = O(\sqrt{\mu})$. To see this, we observe that $X_{12}(\mu) = O(\mu), Y_{12}(\mu) = O(\mu)$ imply that $W_{12}(\mu) \to 0$ as $\mu \to 0$. Also, $X(\mu), Y(\mu)$ analytic at $\mu = 0$ implies that they are analytic as a function of $t = \sqrt{\mu}$ at $t = 0$. That is, $\widetilde{X}_1(t), \widetilde{Y}_1(t)$ are analytic at $t = 0$. Hence, $\widetilde{W}_1(t)$, in particular, $(\widetilde{W}_1)_{12}(t) = W_{12}(t^2)$ is analytic at $t = 0$. Therefore, with $W_{12}(\mu) \to 0$ as $\mu \to 0$, we have $W_{12}(\mu) = O(\sqrt{\mu})$.

Analyzing analyticity w.r.t. μ involves different transformation on $X(\mu)$ and on $Y(\mu)$, as compared to the case, ((9), (10)), of analyticity w.r.t. $\sqrt{\mu}$, [34].

Following [34], let us define $\widetilde{X}(\mu)$ and $\widetilde{Y}(\mu)$ which are related to $X(\mu), Y(\mu)$ by

$$X(\mu) = \widetilde{X}(\mu) \begin{pmatrix} I & 0 \\ 0 & \mu I \end{pmatrix} \tag{29}$$

and

$$Y(\mu) = \begin{pmatrix} \mu I & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}(\mu). \tag{30}$$

Since $X_{11}(\mu) = \Theta(1), X_{12}(\mu) = \mathcal{O}(\mu), X_{22}(\mu) = \Theta(\mu)$, we have $\widetilde{X}(\mu) = \mathcal{O}(1)$. Similarly, we have $\widetilde{Y}(\mu) = \mathcal{O}(1)$.

From $WYW = X$ which must be satisfied by $(X(\mu), Y(\mu))$ - an off-central path corresponding to the NT direction, we have

$$W \begin{pmatrix} \mu I & 0 \\ 0 & I \end{pmatrix} \widetilde{Y} W = \widetilde{X} \begin{pmatrix} I & 0 \\ 0 & \mu I \end{pmatrix}$$

which implies that

$$W \begin{pmatrix} \sqrt{\mu} I & 0 \\ 0 & \frac{1}{\sqrt{\mu}} I \end{pmatrix} \widetilde{Y} W \begin{pmatrix} \sqrt{\mu} I & 0 \\ 0 & \frac{1}{\sqrt{\mu}} I \end{pmatrix} = \widetilde{X}.$$

Define

$$\widetilde{W}(\mu) := W(\mu) \begin{pmatrix} \sqrt{\mu} I & 0 \\ 0 & \frac{1}{\sqrt{\mu}} I \end{pmatrix}. \tag{31}$$

Therefore, we have $\widetilde{W}\widetilde{Y}\widetilde{W} = \widetilde{X}$. Also, $\widetilde{X}(\mu), \widetilde{Y}(\mu), \widetilde{W}(\mu)$ are invertible even in the limit as $\mu \to 0$. Note that accumulation points of $\widetilde{W}(\mu)$ exist as $\mu \to 0$ since $\widetilde{W}(\mu) = O(1)$. The latter follows from $W_{11}(\mu) = O\left(\frac{1}{\sqrt{\mu}}\right)$, $W_{22} = O(\sqrt{\mu})$ and $W_{12}(\mu) = O(\sqrt{\mu})$.

Since the "tilde" matrices are in general nonsymmetric, instead of using the operation $\otimes_s$ and the map svec, we consider the operation $\otimes$ and the map vec (with inverse mat) from now on. Properties of these can be found for example in Chapter 4 of [8] and the appendix of [39].

From (4) (with P, such that $P^T P = W^{-1}$, W a symmetric, positive definite matrix and $WYW = X$), we have

$$X' + WY'W = \frac{1}{\mu} X.$$

Expressing the latter in terms of $\widetilde{X}, \widetilde{Y}, \widetilde{W}$, their derivatives and μ, we obtain

$$\mathrm{vec}(\widetilde{X}') + (\widetilde{W}^T \otimes \widetilde{W}) \mathrm{vec}(\widetilde{Y}') = \frac{1}{\mu} \mathrm{vec} \left(\widetilde{X} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \widetilde{Y} \widetilde{W} \right). \tag{32}$$

Now, in the wider context when we treat $X, Y \in \Re^{n \times n}$ instead of in S^n, let us introduce the following:

$$(E_{ij} - E_{ji}) \cdot X = 0, \quad 1 \leq i < j \leq n, \tag{33}$$

$$(E_{ij} - E_{ji}) \cdot Y = 0, \quad 1 \leq i < j \leq n, \tag{34}$$

to (2) to form

$$(\mathcal{A}_1 \;\; \mathcal{B}_1) \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}, \tag{35}$$

where

$$
\begin{aligned}
(\mathcal{A}_1 \;\; \mathcal{B}_1) &= \begin{pmatrix} \mathrm{vec}((A_1)_1) & \mathrm{vec}((B_1)_1) \\ \vdots & \vdots \\ \mathrm{vec}((A_1)_{n^2+n(n-1)/2}) & \mathrm{vec}((B_1)_{n^2+n(n-1)/2}) \end{pmatrix} \\
&:= \begin{pmatrix} \mathrm{vec}(A_1)^T & \mathrm{vec}(B_1)^T \\ \vdots & \vdots \\ \mathrm{vec}(A_{\widetilde{n}})^T & \mathrm{vec}(B_{\widetilde{n}})^T \\ \mathrm{vec}(E_{ij}-E_{ji})^T_{1\le i<j\le n} & 0 \\ 0 & \mathrm{vec}(E_{ij}-E_{ji})^T_{1\le i<j\le n} \end{pmatrix} \in \Re^{(n^2+n(n-1)/2)\times 2n^2}.
\end{aligned}
$$

Conditions (33), (34) are introduced to ensure that although X, Y belong to $\Re^{n\times n}$, they are necessarily symmetric.

From (35), together with (32) (see [34]), we have the following ODE system:

$$
\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\widetilde{X}') \\ \mathrm{vec}(\widetilde{Y}') \end{pmatrix} =
\frac{1}{\mu}\begin{pmatrix} -\mathcal{A}_1(\mu)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes I\right)\mathrm{vec}(\widetilde{X}) - \mathcal{B}_1(\mu)\left(I \otimes \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}\left(\widetilde{X}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Y}\widetilde{W}\right) \end{pmatrix}, \tag{36}
$$

where

$$
\mathcal{A}_1(\mu)_{k\cdot} := \begin{cases} \left(\mathrm{vec}\begin{pmatrix} ((A_1)_k)_{11} & \mu((A_1)_k)_{12} \\ ((A_1)_k)_{21} & \mu((A_1)_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \le k \le i_1' \\ \left(\mathrm{vec}\begin{pmatrix} 0 & ((A_1)_k)_{12} \\ 0 & ((A_1)_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1'+1 \le k \le n^2+n(n-1)/2 \end{cases} \tag{37}
$$

and

$$
\mathcal{B}_1(\mu)_{k\cdot} := \begin{cases} \left(\mathrm{vec}\begin{pmatrix} \mu((B_1)_k)_{11} & \mu((B_1)_k)_{12} \\ ((B_1)_k)_{21} & ((B_1)_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \le k \le i_1' \\ \left(\mathrm{vec}\begin{pmatrix} ((B_1)_k)_{11} & ((B_1)_k)_{12} \\ 0 & 0 \end{pmatrix}\right)^T & \text{for } i_1'+1 \le k \le n^2+n(n-1)/2 \end{cases}, \tag{38}
$$

for some fixed i_1'.

Details on how $\mathcal{A}_1(\mu), \mathcal{B}_1(\mu)$ are derived and their properties can be obtained from [34].

The analyticity of the matrix

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix},$$

which appears on the left hand side of (36), in an open neighborhood of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, and its accumulation points, is given in the following proposition. Analyticity of this matrix is required to analyze the asymptotic analyticity behavior of paths corresponding to the NT direction, using ODE system (36) (or (42)).

Before we state and prove the proposition, let us first state a well-known result which will be used in the proof of the proposition.

Result 3.1 *Let $A \in \Re^{p\times p}$ and $B \in \Re^{q\times q}$. The equation $AX + XB = C$ has an unique solution $X \in \Re^{p\times q}$ for each $C \in \Re^{p\times q}$ if and only if $\sigma(A) \cap \sigma(-B) = \emptyset$.*
Here, $\sigma(A)$ refers to the set of all eigenvalues of the square matrix A.

The above result can be found for example in Chapter 4 of [8].

Proposition 3.3 *There exists an open neighborhood $\mathcal{U} \subset \Re \times \Re^{n\times n} \times \Re^{n\times n}$ of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, and $(0, \widetilde{X}^*, \widetilde{Y}^*)$ corresponding to any accumulation point of the off-central path, such that for $(\mu, \widetilde{X}, \widetilde{Y}) \in \mathcal{U}$, the matrix*

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix}$$

is analytic at $(\mu, \widetilde{X}, \widetilde{Y})$, where $\widetilde{W}$ exists, is an analytic function of $(\widetilde{X}, \widetilde{Y})$, and satisfies $\widetilde{W}\widetilde{Y}\widetilde{W} = \widetilde{X}$.

Proof. Consider

$$f : \Re^{n\times n} \times \Re^{n\times n} \times \Re^{n\times n} \longrightarrow \Re^{n\times n}$$

defined by

$$f(\widetilde{X}, \widetilde{Y}, \widetilde{W}) = \widetilde{W}\widetilde{Y}\widetilde{W} - \widetilde{X}.$$

We have

$$\nabla_{\widetilde{W}} f(\widetilde{X}, \widetilde{Y}, \widetilde{W})U = \widetilde{W}\widetilde{Y}U + U\widetilde{Y}\widetilde{W}, \quad U \in \Re^{n\times n}.$$

We need only show that $\nabla_{\widetilde{W}} f(\widetilde{X}, \widetilde{Y}, \widetilde{W})$ is invertible for $(\widetilde{X}, \widetilde{Y}, \widetilde{W})$ with $\widetilde{W}\widetilde{Y}\widetilde{W} = \widetilde{X}$ and $(\widetilde{X}, \widetilde{Y})$ derived from the set of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, or any of its accumulation points. The proposition then follows from the Implicit Function Theorem.

First, we consider $(\widetilde{X}, \widetilde{Y})$ to be derived from the set $\{(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu)); \mu > 0\}$.

From (30) and (31), we have

$$\widetilde{W}\widetilde{Y} = \frac{1}{\sqrt{\mu}} WY$$

and

$$\widetilde{Y}\widetilde{W} = \frac{1}{\sqrt{\mu}} \begin{pmatrix} \frac{1}{\sqrt{\mu}} I & 0 \\ 0 & \sqrt{\mu} I \end{pmatrix} YW \begin{pmatrix} \sqrt{\mu} I & 0 \\ 0 & \frac{1}{\sqrt{\mu}} I \end{pmatrix}.$$

Since W and Y are symmetric and positive definite, we see from the above relations that $\sigma(\widetilde{W}\widetilde{Y}) \cap \sigma(-\widetilde{Y}\widetilde{W}) = \emptyset$. Hence, by Result 3.1, $\nabla_{\widetilde{W}} f(\widetilde{X}, \widetilde{Y}, \widetilde{W})$ is invertible.

Let $(\widetilde{X}, \widetilde{Y})$ be derived from one of the accumulation points of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$ as μ tends to zero, with corresponding $\widetilde{W}$. Let us denote them by $\widetilde{X}^*, \widetilde{Y}^*$ and $\widetilde{W}^*$ respectively.

Let $(\widetilde{X}_k, \widetilde{Y}_k, \widetilde{W}_k)$ be derived from an off-central path corresponding to $\mu_k > 0$, with $\lim_{k\to\infty} \mu_k = 0$, $\lim_{k\to\infty} \widetilde{X}_k = \widetilde{X}^*$, $\lim_{k\to\infty} \widetilde{Y}_k = \widetilde{Y}^*$ and $\lim_{k\to\infty} \widetilde{W}_k = \widetilde{W}^*$. (Actually, we should have written $\widetilde{X}_k = \widetilde{X}(\mu_k), \widetilde{Y}_k = \widetilde{Y}(\mu_k), \widetilde{W}_k = \widetilde{W}(\mu_k)$. They are written this way for the sake of simplification of notations.)

We have $\widetilde{W}_k\widetilde{Y}_k = \frac{1}{\sqrt{\mu_k}} W_k Y_k$ by (30) and (31), where again W_k and Y_k stand for $W(\mu_k)$ and $Y(\mu_k)$ respectively.

Now,

$$W_k = \sqrt{\mu_k} \begin{pmatrix} \frac{1}{\sqrt{\mu_k}} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} (\check{W}_k)_{11} & \sqrt{\mu_k}(\check{W}_k)_{12} \\ \sqrt{\mu_k}(\check{W}_k)_{12}^T & (\check{W}_k)_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\mu_k}} I & 0 \\ 0 & I \end{pmatrix},$$

where $(\check{W}_k)_{11}, (\check{W}_k)_{22}$ are symmetric, positive definite matrices even in the limit as $k \to \infty$, $(\check{W}_k)_{11} = \Theta(1), (\check{W}_k)_{22} = \Theta(1)$ and $(\check{W}_k)_{12} = \mathcal{O}(1)$, and

$$Y_k = \begin{pmatrix} \sqrt{\mu_k} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} (\check{Y}_k)_{11} & \sqrt{\mu_k}(\check{Y}_k)_{12} \\ \sqrt{\mu_k}(\check{Y}_k)_{12}^T & (\check{Y}_k)_{22} \end{pmatrix} \begin{pmatrix} \sqrt{\mu_k} I & 0 \\ 0 & I \end{pmatrix},$$

where $(\check{Y}_k)_{11}, (\check{Y}_k)_{22}$ are symmetric, positive definite matrices even in the limit as $k \to \infty$, $(\check{Y}_k)_{11} = \Theta(1), (\check{Y}_k)_{22} = \Theta(1)$ and $(\check{Y}_k)_{12} = \mathcal{O}(1)$.

The above properties on $(\check{W}_k)_{ij}, (\check{Y}_k)_{ij}$ hold due to $(W_k)_{11} = \Theta\left(\frac{1}{\sqrt{\mu_k}}\right), (W_k)_{22} = \Theta(\sqrt{\mu_k}), (W_k)_{12} = \mathcal{O}(\sqrt{\mu_k})$, and $(Y_k)_{11} = \Theta(\mu_k), (Y_k)_{22} = \Theta(1), (Y_k)_{12} = \mathcal{O}(\mu_k)$.

Note that the limit of each "check" submatrix exists as k tends to ∞. (If not, then they exist by taking a suitable subsequence.)

Therefore,

$$\widetilde{W}_k\widetilde{Y}_k = \begin{pmatrix} (\check{W}_k)_{11}(\check{Y}_k)_{11} + \mu_k(\check{W}_k)_{12}(\check{Y}_k)_{12}^T & (\check{W}_k)_{11}(\check{Y}_k)_{12} + (\check{W}_k)_{12}(\check{Y}_k)_{22} \\ \mu_k(\check{W}_k)_{12}^T(\check{Y}_k)_{11} + \mu_k(\check{W}_k)_{22}(\check{Y}_k)_{12}^T & \mu_k(\check{W}_k)_{12}^T(\check{Y}_k)_{12} + (\check{W}_k)_{22}(\check{Y}_k)_{22} \end{pmatrix}.$$

Letting $\mu_k \to 0$, we have

$$\widetilde{W}^*\widetilde{Y}^* = \begin{pmatrix} \check{W}_{11}^*\check{Y}_{11}^* & \check{W}_{11}^*\check{Y}_{12}^* + \check{W}_{12}^*\check{Y}_{22}^* \\ 0 & \check{W}_{22}^*\check{Y}_{22}^* \end{pmatrix}.$$

Similarly,

$$\widetilde{Y}^*\widetilde{W}^* = \begin{pmatrix} \check{Y}_{11}^*\check{W}_{11}^* & \check{Y}_{11}^*\check{W}_{12}^* + \check{Y}_{12}^*\check{W}_{22}^* \\ 0 & \check{Y}_{22}^*\check{W}_{22}^* \end{pmatrix}.$$

Therefore, $\sigma(\widetilde{W}^*\widetilde{Y}^*) \cap \sigma(-\widetilde{Y}^*\widetilde{W}^*) = \emptyset$. And $\nabla_{\widetilde{W}} f(\widetilde{X}^*, \widetilde{Y}^*, \widetilde{W}^*)$ is invertible, by Result 3.1 again.

Hence, we are done. **QED**

Note that the matrix

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix}$$

is not square. However, we show in the following proposition that it is one-to-one in an open neighborhood of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, and any of its accumulation points.

Proposition 3.4 *There exists an open neighborhood $\mathcal{U} \subset \Re \times \Re^{n\times n} \times \Re^{n\times n}$ of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, and $(0, \widetilde{X}^*, \widetilde{Y}^*)$ corresponding to any accumulation point of the off-central path, such that the matrix*

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix}$$

is one-to-one in $\mathcal{U}$, where $\widetilde{W}$ exists, is unique, and is such that $\widetilde{W}\widetilde{Y}\widetilde{W} = \widetilde{X}$, for $(\mu, \widetilde{X}, \widetilde{Y}) \in \mathcal{U}$.

Proof. For $\mu > 0$, we need only show that for $\widetilde{X}_1 = \widetilde{X}_1(\mu)$ and $\widetilde{Y}_1 = \widetilde{Y}_1(\mu)$, where $(\widetilde{X}_1(\mu), \widetilde{Y}_1(\mu))$ is derived from an off-central path, if

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ I & \widetilde{W}^T \otimes \widetilde{W} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(U) \\ \mathrm{vec}(V) \end{pmatrix} = 0, \quad \text{for } U, V \in \Re^{n\times n}, \tag{39}$$

then $U = V = 0$.

From $\mathcal{A}_1(\mu)\mathrm{vec}(U) + \mathcal{B}_1(\mu)\mathrm{vec}(V) = 0$ in (39), using monotonicity, we have $\mathrm{vec}(V^T)^T\mathrm{vec}(U) \geq 0$. Therefore, using $\mathrm{vec}(U) + (\widetilde{W}^T \otimes \widetilde{W})\mathrm{vec}(V) = 0$ in (39), we have $\mathrm{vec}(V^T)^T(\widetilde{W}^T \otimes \widetilde{W})\mathrm{vec}(V) \leq 0$.

Let us now look at the expression $\text{vec}(V^T)^T(\widetilde{W}^T \otimes \widetilde{W})\text{vec}(V)$. We have

$$\begin{aligned}
& \text{vec}(V^T)^T(\widetilde{W}^T \otimes \widetilde{W})\text{vec}(V) \\
= \; & \text{Tr}(V\widetilde{W}V\widetilde{W}) \\
= \; & \frac{1}{\mu}\text{Tr}\left(\left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V\widetilde{W} \left(\begin{array}{cc} I & 0 \\ 0 & \mu I \end{array}\right) \left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V\widetilde{W} \left(\begin{array}{cc} \frac{1}{\mu} & 0 \\ 0 & I \end{array}\right)\right) \\
= \; & \frac{1}{\mu}\text{Tr}\left(\left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V\widetilde{W} \left(\begin{array}{cc} \frac{1}{\sqrt{\mu}} I & 0 \\ 0 & \sqrt{\mu} I \end{array}\right) \left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V\widetilde{W} \left(\begin{array}{cc} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} I \end{array}\right)\right) \\
= \; & \frac{1}{\mu}\left\| \left(\widetilde{W}\left(\begin{array}{cc} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} I \end{array}\right)\right)^{1/2} \left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V \left(\widetilde{W}\left(\begin{array}{cc} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} I \end{array}\right)\right)^{1/2} \right\|^2 \geq 0,
\end{aligned}$$

since $\widetilde{W}\left(\begin{array}{cc} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} I \end{array}\right)$ is symmetric, positive definite, and $\left(\begin{array}{cc} \mu I & 0 \\ 0 & I \end{array}\right) V$ is symmetric.

Hence, we conclude that $U = V = 0$.

Now, suppose $(\widetilde{X}^*, \widetilde{Y}^*)$ is an accumulation point of $(\widetilde{X}(\mu), \widetilde{Y}(\mu))$ as $\mu \to 0$. We have $\widetilde{W}^*$ is such that $\widetilde{W}^*\widetilde{Y}^*\widetilde{W}^* = \widetilde{X}^*$. Let us consider

$$\left(\begin{array}{cc} \mathcal{A}_1(0) & \mathcal{B}_1(0) \\ I & (\widetilde{W}^*)^T \otimes \widetilde{W}^* \end{array}\right) \left(\begin{array}{c} \text{vec}(U_0) \\ \text{vec}(V_0) \end{array}\right) = 0, \quad \text{for } U_0, V_0 \in \Re^{n\times n}. \tag{40}$$

Again, we need only show that $U_0 = V_0 = 0$.

By monotonicity, we have using $\mathcal{A}_1(0)\text{vec}(U_0) + \mathcal{B}_1(0)\text{vec}(V_0) = 0$ in (40) that $\text{vec}(V_0^T)^T\text{vec}(U_0) \geq 0$. Also, we can show that $(V_0)_{11} = (V_0)_{11}^T, (V_0)_{22} = (V_0)_{22}^T$ and $(V_0)_{21} = 0$. (See proof of Proposition 2.2 in [34] for details.)

Note that $\widetilde{W}_{21}^* = 0$, and $\widetilde{W}_{11}^*, \widetilde{W}_{22}^*$ are symmetric, positive definite. Together with the above properties of V_0, we have

$$\begin{aligned}
& \text{vec}(V_0^T)^T((\widetilde{W}^*)^T \otimes \widetilde{W}^*)\text{vec}(V_0) \\
= \; & \text{Tr}(V_0\widetilde{W}^*V_0\widetilde{W}^*) \\
= \; & \text{Tr}\left(\left(\begin{array}{cc} (V_0)_{11} & (V_0)_{12} \\ 0 & (V_0)_{22} \end{array}\right) \left(\begin{array}{cc} \widetilde{W}_{11}^* & \widetilde{W}_{12}^* \\ 0 & \widetilde{W}_{22}^* \end{array}\right) \times \right. \\
& \left. \left(\begin{array}{cc} (V_0)_{11} & (V_0)_{12} \\ 0 & (V_0)_{22} \end{array}\right) \left(\begin{array}{cc} \widetilde{W}_{11}^* & \widetilde{W}_{12}^* \\ 0 & \widetilde{W}_{22}^* \end{array}\right)\right) \\
= \; & \text{Tr}((V_0)_{11}\widetilde{W}_{11}^*(V_0)_{11}\widetilde{W}_{11}^*) + \text{Tr}((V_0)_{22}\widetilde{W}_{22}^*(V_0)_{22}\widetilde{W}_{22}^*) \\
= \; & \|(\widetilde{W}_{11}^*)^{1/2}(V_0)_{11}(\widetilde{W}_{11}^*)^{1/2}\| + \|(\widetilde{W}_{22}^*)^{1/2}(V_0)_{22}(\widetilde{W}_{22}^*)^{1/2}\| \geq 0.
\end{aligned}$$

Since $\mathrm{vec}(V_0^T)^T\mathrm{vec}(U_0) \geq 0$, we also have $\|(\widetilde{W}_{11}^*)^{1/2}(V_0)_{11}(\widetilde{W}_{11}^*)^{1/2}\| + \|(\widetilde{W}_{22}^*)^{1/2}(V_0)_{22}(\widetilde{W}_{22}^*)^{1/2}\| \leq 0$. Hence, $(V_0)_{11}, (V_0)_{22} = 0$.

Using $\mathrm{vec}(U_0) + ((\widetilde{W}^*)^T \otimes \widetilde{W}^*)\mathrm{vec}(V_0) = 0$ in (40), we deduce from what is known of V_0 and $\widetilde{W}^*$, that $(U_0)_{11} = 0, (U_0)_{22} = 0$ and $(U_0)_{21} = 0$. Also,

$$(U_0)_{12} + \widetilde{W}_{11}^*(V_0)_{12}\widetilde{W}_{22}^* = 0. \tag{41}$$

By monotonicity arguments, we have $\mathrm{Tr}((V_0)_{12}^T(U_0)_{12}) \geq 0$ (see proof of Proposition 2.2 in [34]). Hence, using (41), we conclude that $(U_0)_{12} = (V_0)_{12} = 0$. **QED**

As in [34], observe that we can further reduce the number of rows in the above ODE system (36) to form the following ODE system:

$$\begin{pmatrix} \mathcal{A}_1(\mu) & \mathcal{B}_1(\mu) \\ \widehat{I} & \widehat{\widetilde{W}^T \otimes \widetilde{W}} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\widetilde{X}') \\ \mathrm{vec}(\widetilde{Y}') \end{pmatrix} =$$

$$\frac{1}{\mu}\begin{pmatrix} -\mathcal{A}_1(\mu)\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes I\right)\mathrm{vec}(\widetilde{X}) - \mathcal{B}_1(\mu)\left(I \otimes \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\mathrm{vec}(\widetilde{Y})\right. \\ \widehat{\mathrm{vec}}\left(\widetilde{X}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Y}\widetilde{W}\right) \end{pmatrix}, \tag{42}$$

where the "hat" operation is defined as follows:

Definition 3.1 *Given $D \in \Re^{n^2 \times n^2}$, define $\widehat{D} \in \Re^{(n^2 - m(n-m)) \times n^2}$ such that for all $u \in \Re^{n^2}$, $\widehat{D}u$ consists of entries of V_{11}, V_{12}, V_{22} in the order in which they appear in the image of Du, where $Du = vec\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$.*
Also, define the map $\widehat{vec}$ from $\Re^{n \times n}$ to $\Re^{n^2 - m(n-m)}$ such that, given $V \in \Re^{n \times n}$, $\widehat{vec}(V)$ is the vector in $\Re^{n^2 - m(n-m)}$ obtained from $vec(V)$ by removing entries of V_{21}.

Remark 3.4 *The matrix on the left hand side of the above ODE system (42) is again one-to-one and analytic in an open neighborhood of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, $\mu > 0$, of an off-central path, and any of its accumulation points.*

Using the ODE system (42) (or (36)), we have the following theorem, which is similar to Theorem 2.1 in [34]:

Theorem 3.3 *Let $(X(\mu), Y(\mu))$, $\mu > 0$, be an off-central path for SDLCP (1)-(3) under Assumptions 2.1 and 3.1. Then $X(\mu), Y(\mu)$ are analytic at $\mu = 0$ if and only if $X_{12}(\mu) = \mathcal{O}(\mu)$, $Y_{12}(\mu) = \mathcal{O}(\mu)$, $W_{12}(\mu) = \mathcal{O}(\sqrt{\mu})$, and $\frac{1}{\mu}(W_{11}Y_{11}W_{12} + W_{11}Y_{12}W_{22})(\mu) \to 0$ as $\mu \to 0$ and is analytic at $\mu = 0$.*

Proof. First, observe that "$\frac{1}{\mu}(W_{11}Y_{11}W_{12} + W_{11}Y_{12}W_{22})(\mu) \to 0$ as $\mu \to 0$ and is analytic at $\mu = 0$" is equivalent to "$(\widetilde{W}_{11}\widetilde{Y}_{11}\widetilde{W}_{12} + \widetilde{W}_{11}\widetilde{Y}_{12}\widetilde{W}_{22})(\mu) \to 0$ as $\mu \to 0$ and is analytic at $\mu = 0$". Also, $X_{12}(\mu) = \mathcal{O}(\mu) \Leftrightarrow \widetilde{X}_{12}(\mu) = \mathcal{O}(1)$, $Y_{12}(\mu) = \mathcal{O}(\mu) \Leftrightarrow \widetilde{Y}_{12}(\mu) = \mathcal{O}(1)$ and $W_{12}(\mu) = \mathcal{O}(\sqrt{\mu}) \Leftrightarrow \widetilde{W}_{12}(\mu) = \mathcal{O}(1)$.

($\Rightarrow$) Suppose $X(\mu), Y(\mu)$ are analytic at $\mu = 0$.
Then it is clear that $X_{12}(\mu) = \mathcal{O}(\mu)$ and $Y_{12}(\mu) = \mathcal{O}(\mu)$, and $W_{12}(\mu) = \mathcal{O}(\sqrt{\mu})$.
Also, since the left hand side of the ODE system (42) is analytic at $\mu = 0$, and that

$$\widetilde{W}_{11}\widetilde{Y}_{11}\widetilde{W}_{12} + \widetilde{W}_{11}\widetilde{Y}_{12}\widetilde{W}_{22} = \left(\widetilde{X}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Y}\widetilde{W}\right)_{12},$$

we have the required result.

($\Leftarrow$) Suppose $\widetilde{X}_{12}(\mu) = \mathcal{O}(1)$, $\widetilde{Y}_{12}(\mu) = \mathcal{O}(1)$, $\widetilde{W}_{12}(\mu) = \mathcal{O}(1)$, and $(\widetilde{W}_{11}\widetilde{Y}_{11}\widetilde{W}_{12} + \widetilde{W}_{11}\widetilde{Y}_{12}\widetilde{W}_{22})(\mu) \to 0$ as $\mu \to 0$ and is analytic at $\mu = 0$.
We have $(\widetilde{X}(\mu), \widetilde{Y}(\mu), \widetilde{W}(\mu))$ satisfies $\widetilde{W}\widetilde{Y}\widetilde{W} = \widetilde{X}, \widetilde{X}_{21} = \mu\widetilde{X}_{12}^T, \widetilde{Y}_{21} = \mu\widetilde{Y}_{12}^T, \widetilde{W}_{21} = \mu\widetilde{W}_{12}^T$ and

$$\widetilde{X}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Y}\widetilde{W}$$
$$= \begin{pmatrix} \widetilde{X}_{11} - \widetilde{W}_{11}\widetilde{Y}_{11}\widetilde{W}_{11} - \widetilde{W}_{11}\widetilde{Y}_{12}\widetilde{W}_{21} & -\widetilde{W}_{11}\widetilde{Y}_{11}\widetilde{W}_{12} - \widetilde{W}_{11}\widetilde{Y}_{12}\widetilde{W}_{22} \\ \widetilde{X}_{21} - \widetilde{W}_{21}\widetilde{Y}_{11}\widetilde{W}_{11} - \widetilde{W}_{21}\widetilde{Y}_{12}\widetilde{W}_{21} & -\widetilde{W}_{21}\widetilde{Y}_{11}\widetilde{W}_{12} - \widetilde{W}_{21}\widetilde{Y}_{12}\widetilde{W}_{22} \end{pmatrix}.$$

These imply that for $\widetilde{X} = \widetilde{X}(\mu), \widetilde{Y} = \widetilde{Y}(\mu)$,

$$\widehat{\text{vec}}\left(\widetilde{X}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Y}\widetilde{W}\right)$$

in (42) can be written as a product of μ, and an analytic function of $\mu, \widetilde{X}, \widetilde{Y}$ at any accumulation points of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$, derived from the given off-central path, as μ tends to zero.

Also, for $\widetilde{X} = \widetilde{X}(\mu), \widetilde{Y} = \widetilde{Y}(\mu)$,

$$-\mathcal{A}_1(\mu)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes I\right)\text{vec}(\widetilde{X}) - \mathcal{B}_1(\mu)\left(I \otimes \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\text{vec}(\widetilde{Y})$$

in (42) can be written as a product of μ and an analytic function of $\widetilde{X}, \widetilde{Y}$ (see [34]).

Together with Remark 3.4, we see that $(\widetilde{X}(\mu), \widetilde{Y}(\mu))$ of the given off-central path satisfies an ODE system in standard form ($y' = f(t, y)$), derived from (42), whose right hand side is analytic at any accumulation points of $(\mu, \widetilde{X}(\mu), \widetilde{Y}(\mu))$. Hence, by Theorem 4.1 of [4], pp. 15 and Theorem 2.1 of [32], $(\widetilde{X}(\mu), \widetilde{Y}(\mu))$ can be analytically extended to $\mu = 0$, which implies that the same holds for $(X(\mu), Y(\mu))$, due to (29), (30). And we are done. **QED**

4 Conclusion.

In this paper, we provide a few off-central path examples, corresponding to the NT direction, for a particularly simple and nice SDP, whose behavior near the unique solution of this SDP is not well-behaved. We also provide in Section 3 necessary and sufficient conditions for when an off-central path, corresponding to the NT direction, for a general SDLCP is analytic, first w.r.t. $\sqrt{\mu}$ and then w.r.t. μ, at $\mu = 0$. In the analysis in Section 3, we follow the approach in [33,34]. Similar analysis like in [32] for the NT case for the simple SDP example in Section 2 proves to be too difficult for this paper.

References

1. Alizadeh, F., Haeberly, J.A., Overton, M.: Primal-dual interior-point methods for semidefinite programming: Convergence Rates, Stability and Numerical Results. SIAM J. Optim. **8**, 746-768 (1998).

2. Bayerm, D.A., Lagarias, J.C.: The nonlinear geometry of linear programming, I, II, III. Trans. Amer. Math. Soc. **314**, 499-526 (1989), 527-581 (1989), Vol. 320, 193-225 (1990).

3. Chua, C.B.: Analyticity of weighted central paths and error bounds for semidefinite programming. Math. Program. Series A. **115**, 239-271 (2008).

4. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. International Series in Pure and Applied Mathematics (1955).

5. da Cruz Neto, J. X., Ferreira, O.P., Monteiro, R.D.C.: Asymptotic behavior of the central path for a special class of degenerate SDP problems. Math. Program. Series A. **103**, 487-514 (2005).

6. Gurtuna, F., Petra, C., Potra, F.A., Shevchenko, O., Vancea, A.: Corrector-predictor infeasible interior point methods for sufficient linear complementarity problems. Comput. Optim. Appl. Online First, DOI 10.1007/s10589-009-9263-4 (2009).

7. Helmberg, C., Rendl, F., Vanderbei, R.J., Wolkowicz, H.: An interior-point method for semidefinite programming. SIAM J. Optim. **6**, 342-361 (1996).

8. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, New York (1991).

9. Ji, J., Potra, F.A., Sheng, R.: On a general class of interior-point algorithms for semidefinite programming with polynomial complexity and superlinear convergence. Methods Appl. Anal. **6** (1999).

10. Ji, J., Potra, F.A., Sheng, R.: On the local convergence of a predictor-corrector method for semidefinite programming. SIAM J. Optim. **10**, 195-210 (1999).

11. Kojima, M., Shida, M., Shindoh, S.: Local convergence of predictor-corrector infeasible-interior-point algorithms for SDPs and SDLCPs. Math. Program. Series A. **80**, 129-160 (1998).

12. Kojima, M., Shida, M., Shindoh, S.: A predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem using the Alizadeh-Haeberly-Overton search direction. SIAM J. Optim. **9**, 444-465 (1999).

13. Kojima, M., Shindoh, S., Hara, S.: Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. SIAM J. Optim. **7**, 86-125 (1997).

14. Liu, X., Potra, F.A.: Corrector-predictor methods for sufficient linear complementarity problems in a wide neighborhood of the central path. SIAM J. Optim. **17**, 871-890 (2006).

15. Lu, Z., Monteiro, R.D.C.: Error bounds and limiting behavior of weighted paths associated with the SDP map $X^{1/2}SX^{1/2}$. SIAM J. Optim. **15**, 348-374 (2004).

16. Lu, Z., Monteiro, R.D.C.: A note on the local convergence of a predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem based on the Alizadeh-Haeberly-Overton search direction. SIAM J. Optim. **15**, 1147-1154 (2005).

17. Lu, Z., Monteiro, R.D.C.: Limiting behavior of the Alizadeh-Haeberly-Overton weighted paths in semidefinite programming. Optim. Methods Softw. **22**, 849-870 (2007).

18. Potra, F.A.: A superlinearly convergent predictor-corrector method for degenerate LCP in a wide neighborhood of the central path with $O(\sqrt{n}L)$-iteration complexity. Math. Program. Series A. **100**, 317-337 (2004).

19. Potra, F.A.: Corrector-predictor methods for monotone linear complementarity problems in a wide neighborhood of the central path. Math. Program. Series B. **111**, 243-272 (2008).

20. Potra, F.A., Liu, X.: Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path. Optim. Methods Softw. **20**, 145-168 (2005).

21. Potra, F.A., Sheng, R.: Superlinear convergence of interior-point algorithms for semidefinite programming. J. Optim. Theory Appl. **99**, 103-119 (1998).

22. Potra, F.A., Sheng, R.: A superlinearly convergent primal-dual infeasible-interior-point algorithm for semidefinite programming. SIAM J. Optim. **8**, 1007-1028 (1998).

23. Potra, F.A., Sheng, R.: Superlinear convergence of a predictor-corrector method for semidefinite programming without shrinking central path neighborhood. Bulletin Mathématique de la Societé des Sciences Mathématiques de Roumanie, Vol. 43, 106-123 (2000).

24. Potra, F.A., Stoer, J.: On a class of superlinearly convergent polynomial time interior point methods for sufficient LCP. SIAM J. Optim. **20**, 1333-1363 (2009).

25. Preiß, M., Stoer, J.: Analysis of infeasible-interior-point paths and high-order methods for solving SDLCPs. Parametric Optimization and Related Topics, VII. Aportaciones Matemáticas: Investigación, 18, 235-251 (2004).

26. Preiß, M., Stoer, J.: Analysis of infeasible-interior-point paths arising with semidefinite linear complementarity problems. Math. Program. Series A. **99**, 499-520 (2004).

27. Mehrotra, S.: Quadratic convergence in a primal-dual method. Math. Oper. Res. **18**, 741-751 (1993).

28. Monteiro, R.D.C.: Primal-dual path following algorithms for semidefinite programming. SIAM J. Optim. **7**, 663-678 (1997).

29. Monteiro, R.D.C., Tsuchiya, T.: Polynomial convergence of a new family of primal-dual algorithms for semidefinite programming. SIAM J. Optim. **9**, 551-577 (1999).

30. Nesterov, Y., Todd, M.: Self-scaled barriers and interior-point methods for convex programming. Math. Oper. Res. **22**, 1-42 (1997).

31. Nesterov, Y., Todd, M.: Primal-dual interior-point methods for self-scaled cones. SIAM J. Optim. **8**, 324-364 (1998).

32. Sim, C.-K., Zhao, G.: Underlying paths in interior point methods for the monotone semidefinite linear complementarity problem. Math. Program. Series A. **110**, 475-499 (2007).

33. Sim, C.-K., Zhao, G.: Asymptotic behavior of Helmberg-Kojima-Monteiro (HKM) paths in interior point methods for monotone semidefinite linear complementarity problems: general theory. J. Optim. Theory Appl. **137**, 11-25 (2008).

34. Sim, C.-K.: On the analyticity of underlying HKM paths for monotone semidefinite linear complementarity problems. J. Optim. Theory Appl. **141**, 193-215 (2009).

35. Sonnevend, G.: An analytic center for polyhedrons and new classes for linear programming. In: A. Prekopa (ed.): System Modelling and Optimization, Lecture Notes in Control and Information Sciences, Vol. 84, pp. 866-876, Springer-Verlag, Berlin (1985).

36. Stoer, J., Wechs, M.: On the analyticity properties of infeasible-interior-point paths for monotone linear complementarity problems. Numer. Math. **81**, 631-645 (1999).

37. Stoer, J., Wechs, M., Mizuno, S.: High order infeasible-interior-point methods for solving sufficient linear complementarity problems. Math. Oper. Res. **23**, 832-862 (1998).

38. Sturm, J.F.: Superlinear convergence of an algorithm for monotone linear complementarity problems, when no strictly complementary solution exists. Math. Oper. Res. **24**, 72-94 (1999).

39. Todd, M.J., Toh, K.C., Tütüncü, R.H.: On the Nesterov-Todd direction in semidefinite programming. SIAM J. Optim. **8**, 769-796 (1998).

40. Halická, M., Trnovská, M.: Limiting behavior and analyticity of weighted central paths in semidefinite programming. Optim. Methods Softw. **25**, 247-262 (2010).

41. Tseng, P.: Search directions and convergence analysis of some infeasible path-following methods for the monotone semi-definite LCP. Optim. Methods Softw. **9**, 245-268 (1998).

42. Ye, Y., Anstreicher, K.: On quadratic and $O(\sqrt{n}L)$ convergence of a predictor-corrector algorithm for LCP. Math. Program. Series A. **62**, 537-551 (1993).

43. Ye, Y., Güler, O., Tapia, R.A., Zhang, Y.: A quadratically convergence $O(\sqrt{n}L)$-iteration algorithm for linear programming. Math. Program. Series A. **59**, 151-162 (1993).

44. Zhang, Y.: On extending some primal-dual interior-point algorithms from linear programming to semidefinite programming. SIAM J. Optim. **8**, 365-386 (1998).