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Analysis of Quantum Linear Systems' Response to Multi-photon States

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Abstract

The purpose of this paper is to present a mathematical framework for analyzing the response of quantum linear systems driven by multi-photon states. Both the factorizable (namely, no correlation among the photons in the channel) and unfactorizable multi-photon states are treated. Pulse information of a multi-photon input state is represented in terms of tensor, and the response of quantum linear systems to multi-photon input states is characterized by tensor operations. Analytic forms of output correlation functions and output states are derived. The proposed framework is applicable no matter whether the underlying quantum dynamic system is passive or active. Examples from the physics literature are used to illustrate the results presented.

Key words: quantum linear systems, multi-photon states, tensors.

1 Introduction

Analysis of system response to various types of input signals is fundamental to control systems engineering. Step response enables a control engineer to visualize system transient behavior such as rise time, overshoot and settling time; frequency response design methods are among the most powerful methods in classical control theory; response analysis of linear systems initialized in Gaussian states driven by Gaussian input signals is the basis of Kalman filtering and linear quadratic Gaussian (LQG) control.

Over the last two decades, there has been rapid advance in experimental demonstration and theoretical investigation of quantum (namely, non-classical) control systems due to their promising applications in a wide range of areas such as quantum communication, quantum computation, quantum metrology, laser-induced chemical reaction, and nano electronics (Gardiner & Zoller, 2000, Huang, Tarn & Clark, 1983, Doherty & Jacobs, 1999, Albertini & D'Alessandro, 2003, Yanagisawa & Kimura, 2003, Stockton, van Handel & Mabuchi, 2004, Altafini, 2007, Mirrahimi & van Handel, 2007, Gough & James, 2009, Li & Khaneja, 2009, Nurdin, James & Doherty, 2009, Yamamoto & Bouten, 2009, Bolognani & Ticozzi, 2010, Dong & Petersen, 2010, Wang & Schirmer, 2010, Zhang & James, 2011, Zhang, et al., 2012, Qi, 2013). Within this program quantum linear systems play a prominent role. Quantum linear systems are characterized by linear quantum stochastic differential equations (linear QSDEs). In quantum optics, linear systems are widely used because they are easy to manipulate and, more importantly, linear dynamics often serve well as good approximation of more general dynamics (Gardiner & Zoller, 2000, Wiseman & Milburn, 2010). Besides their broad applications in quantum optics, linear systems have also found applications in many other quantum-mechanical systems such as opto-mechanical systems Massel, et al., 2011, circuit quantum electrodynamics (circuit QED) systems Matyas, et al., 2011, and atomic ensembles Stockton, van Handel & Mabuchi, 2004. From a signals and systems point of view, quantum linear systems driven by Gaussian input states have been studied extensively, and results like quantum filtering and measurement-based feedback control have been well established (Wiseman & Milburn, 2010).

In addition to Gaussian states there are other types of non-classical states, for example single-photon states and multi-photon states. Such states describe electromagnetic fields with a definite number of photons. Due to their highly non-classical nature and recent hardware advance, there is rapidly growing interest in the generation

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and engineering (e.g., pulse shaping) of photon states, and it is generally perceived that these photon states hold promising applications in quantum communication, quantum computing, quantum metrology and quantum simulations (Gheri, Ellinger, Pellizzari & Zoller, 1998, Sanaka, Resch & Zeilinger, 2006, Cheung, Migdall & Rastello, 2009, Ou, 2007, Milburn, 2008, Bartley, et al., 2012, Gough, James & Nurdin, 2013). Thus, a new and important problem in the field of quantum control engineering is: How to characterize and engineer interaction between quantum linear systems and photon states? The interaction of quantum linear systems with continuous-mode photon states has recently been studied in the literature, primarily in the physics community. For example, interference phenomena of photons passing through beamsplitters have been studied, see, e.g., Sanaka, Resch & Zeilinger, 2006, Ou, 2007, Bartley, et al., 2012. Milburn discussed how to use an optical cavity to manipulate the pulse shape of a single-photon light field (Milburn, 2008). Quantum filtering for systems driven by single-photon fields has been investigated in Gough, James & Nurdin, 2013. Intensities of output fields of quantum systems driven by continuous-mode multi-photon light fields have been studied in Baragiola, Cook, Brańczyk & Combes, 2012. In Zhang & James, 2013 the response of quantum linear systems to single-photon states has been studied.

In the analysis of the response of quantum linear systems to single-photon states, matrix presentation is sufficient because two indices are adequate: one for input channels, and the other for output channels. However, this is not the case in the multi-photon setting. In addition to indices for input and output channels, we need another index to count photon numbers in channels. As a result, tensor representation and operation are essential in the multi-photon setting. To be specific, multi-photon state processing by quantum linear systems can be mathematically represented in terms of tensor processing by transfer functions. The key ingredient for such an operation is the following (for the passive case). Let $E(t) = (E^{jk}(t)) \in \mathbb{C}^{m\times m}$ be the transfer function of a quantum linear passive system with m input channels. For each $j = 1, \ldots, m$, let $\mathscr{V}_j(t_1, \ldots, t_{\ell_j})$ be an ℓ_j-way m-dimensional tensor function that encodes the pulse information of the j-th input channel containing ℓ_j photons. Denote the entries of $\mathscr{V}_j(t_1, \ldots, t_{\ell_j})$ by $\mathscr{V}_{j,k_1,\ldots,k_{\ell_j}}(t_1, \ldots, t_{\ell_j})$. For all given $1 \le r_1, \ldots, r_{\ell_j} \le m$, define an ℓ_j-way m-dimensional tensor $\mathscr{W}_j$ with entries given by the following multiple convolution

$$\begin{aligned}
&\mathscr{W}_{j,r_1,\ldots,r_{\ell_j}}(t_1, \ldots, t_{\ell_j}) \\
=& \sum_{k_1,\ldots,k_{\ell_j}=1}^{m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E^{r_1k_1}(t_1 - \iota_1)\times \\
&\cdots \times E^{r_{\ell_j}k_{\ell_j}}(t_{\ell_j} - \iota_{\ell_j})\mathscr{V}_{j,k_1,\ldots,k_{\ell_j}}(\iota_1, \ldots, \iota_{\ell_j})d\iota_1 \ldots d_{\ell_j}.
\end{aligned}$$

It turns out that the tensors $\mathscr{W}_j$ $(j = 1, \ldots, m)$ encode the pulse information of the output field. That is, an ℓ_j-way input tensor is mapped to an ℓ_j-way output tensor by the quantum linear passive system.

The contributions of this paper are three-fold. First, the analytic form of the steady-state output state of a quantum linear system driven by a multi-photon input state is derived. When the quantum linear system is a beamsplitter (a static passive device), interesting multi-photon interference phenomena studied in (Sanaka, Resch & Zeilinger, 2006), (Ou, 2007), and (Bartley, et al., 2012) are re-produced by means of our approach, see Examples 1,2,3. Second, when the underlying quantum linear system is not passive (e.g., a degenerate parametric amplifier), the steady-state output state with respect to a multi-photon input state is not a multi-photon state. In terms of tensor representation, a more general class of states is defined. Such rigorous mathematical description paves the way for multi-photon state engineering. Third, both the factorizable and unfactorizable multi-photon states are treated in this paper. Here a factorizable multi-photon state is a state for which the photons in a given channel are not correlated, while for an unfactorizable multi-photon state there exists correlation among the photons. This difference cannot occur in the single-photon state case. Thus, the mathematical framework presented here is more general.

The rest of the paper is organized as follows. Preliminary results are presented in Section 2. Specifically, quantum linear systems are briefly reviewed in Subsection 2.1 with focus on stable inversion and covariance function transfer, in Subsection 2.2 several types of tensors and their associated operations are introduced. The multi-photon state processing when input states are factorizable in terms of pulse shapes is studied in Section 3. (Here the word "factorizable" means there is no correlation among photons in each specific channel.) Specifically, single-channel and multi-channel multi-photon states are presented in Subsections 3.1 and 3.2 respectively, covariance functions and intensities of output fields are studied in Subsection 3.3, while an analytic form of steady-state output states is derived in Subsection 3.4. The unfactorizable case is investigated in Section 4. Specifically, unfactorizable multi-channel multi-photon states are defined in Sebsection 4.1, the analytic form of the steady-state output state is presented in Subsection 4.2 where the underlying system is passive, the active case is studied in Subsection 4.3. Some concluding remarks are given in Section 5.

Notations. m is the number of input channels, and n is the number of degrees of freedom of a given quantum linear stochastic system. $|\phi\rangle$ denotes the initial state of the system which is always assumed to be vacuum, $|0\rangle$ denotes the vacuum state of free fields. Given a column vector of complex numbers or operators

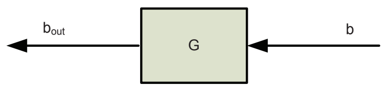

Fig. 1. Quantum linear system G with input b and output $b_{\rm out}$

$x = [\, x_1 \ \cdots \ x_k \,]^T$ where k is a positive integer, define $x^\# = [\, x_1^* \ \cdots \ x_k^* \,]^T$, where the asterisk $*$ indicates complex conjugation or Hilbert space adjoint. Denote $x^\dagger = (x^\#)^T$. Furthermore, define the doubled-up column vector to be $\breve{x} = [\, x^T \ (x^\#)^T \,]^T$. Let I_k be an identity matrix and 0_k a zero square matrix, both of dimension k. Define $J_k = \mathrm{diag}(I_k, -I_k)$ and $\Theta_k = [\, 0 \ I_k; \ -I_k \ 0 \,]$ (The subscript "k" is often omitted.) Then for a matrix $X \in \mathbb{C}^{2j\times 2k}$, define $X^\flat := J_k X^\dagger J_j$. $\otimes_c$ denotes the Kronecker product. Given a function $f(t)$ in the time domain, define its two-sided Laplace transform to be $F[s] = \mathscr{L}_b\{f(t)\}(s) := \int_{-\infty}^{\infty} e^{-st} f(t)dt$. Given two constant matrices $U, \ V \in \mathbb{C}^{r\times k}$, define $\Delta(U,V) = [U \ \ V; V^\# \ \ U^\#]$. Similarly, given time-domain matrix functions $E^-(t)$ and $E^+(t)$ of compatible dimensions, define $\Delta(E^-(t), E^+(t)) = [E^-(t) \ \ E^+(t); E^+(t)^\# \ \ E^-(t)^\#]$. Given two operators A and B, their commutator is defined to be $[A,B] := AB - BA$. For any integer $r > 1$, we write $\int_r$ for integration in the space $\mathbb{R}^r$. We also write $dt_{1\to r}$ for $dt_1 \cdots dt_r$. Finally, given a column vector a, we use a_j to denote its entries. Given a matrix A, we use A^{jk} to denote its entries. Given a 3-way tensor $\mathcal{A}$ (also called a tensor of order 3), we use $\mathcal{A}_{ijk}$ to denote its entries; we do the similar thing for higher order tensors.

2 Quantum linear systems and tensors

This section records preliminary results necessary for the development of the paper. Quantum linear systems are briefly discussed is Subsection 2.1. Tensors and their associated operations, the appropriate mathematical language for the describe the interaction of a quantum linear system with multi-photon channels, is introduced in Subsection 2.2.

2.1 Quantum linear systems

In this subsection we set up the model which is a quantum linear system driven by boson fields, Gardiner & Zoller, 2000,Wiseman & Milburn, 2010,Zhang & James, 2011.

2.1.1 Fields and systems

The triple (S, L, H) provides a compact way for the description of open quantum systems (Gough & James, 2009). Here the self-adjoint operator H is the initial system Hamiltonian, S is a unitary scattering operator, and L is a coupling operator that describes how the system is coupled to its environment. The environment is an m-channel electromagnetic field in free space, represented by a column vector of annihilation operators $b(t) = [b_1(t), \cdots, b_m(t)]^T$. Let t_0 be the initial time, namely, the time when the quantum system starts interacting with its environment. Define a gauge process $\Lambda(t)$ by $\Lambda(t) = \int_{t_0}^t b^\#(\tau) b^T(\tau) d\tau = \left(\Lambda^{jk}(t)\right)_{j,k=1,\dots,m}$ with operator entries $\Lambda^{jk}(t)$ on the Fock space $\mathfrak{F}$ for the free field. In this paper it is assumed that these quantum stochastic processes are *canonical*, that is, they have the following non-zero Ito products

$$dB_j(t)dB_k^*(t) = \delta_{jk}dt, \ d\Lambda^{jk}dB_l^*(t) = \delta_{kl}dB_j^*(t),$$
$$dB_j(t)d\Lambda^{kl}(t) = \delta_{jk}dB_l(t), \ d\Lambda^{jk}(t)d\Lambda^{lm}(t) = \delta_{kl}d\Lambda^{jm}(t),$$

where $B(t) = [B_1(t), \cdots, B_m(t)]^T$ is a column vector of the integrated field operators defined via $B(t) := \int_{t_0}^t b(r)dr$. In the *interaction picture* the stochastic Schrodinger's equation for the open quantum system driven by the free field $b(t)$ is, in Ito form (Gardiner & Zoller, 2000, Chapter 11),

$$dU(t,t_0) = \Big\{ \mathrm{Tr}[(S - I_m)d\Lambda(t)^T] + dB^\dagger(t)L - L^\dagger S dB(t) - (\frac{1}{2}L^\dagger L + iH)dt \Big\} U(t,t_0), \quad t \ge t_0, \qquad (1)$$

with $U(t,t_0) = I$ being an identity operator for all $t \le t_0$.

Specific to the linear case, the open quantum linear system G shown in Fig. 1 represents a collection of n interacting quantum harmonic oscillators $a(t) = [a_1(t), \dots, a_n(t)]^T$ (defined on a Hilbert space $\mathfrak{H}_G$) coupled to m boson fields $b(t)$ described above. Here, a_j $(j = 1, \dots, n)$ is the annihilation operator of the jth oscillator satisfying the canonical commutation relations $[a_j, a_k^*] = \delta_{jk}$. Denote $\breve{a}(t_0) = \breve{a}$. The vector operator $L \in \mathfrak{H}_G$ is defined as $L = C_- a + C_+ a^\#$ with $C_-, C_+ \in \mathbb{C}^{m\times n}$. The initial Hamiltonian $H \in \mathfrak{H}_G$ is $H = \frac{1}{2}\breve{a}^\dagger \Delta(\Omega_-, \Omega_+)\breve{a}$ with $\Omega_-, \Omega_+ \in \mathbb{C}^{n\times n}$ satisfying $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^T$. By (1), the dynamic model for the system G is

$$\dot{\breve{a}}(t) = A\breve{a}(t) + B\breve{b}(t), \quad \breve{a}(t_0) = \breve{a}, \qquad (2)$$
$$\breve{b}_{\rm out}(t) = C\breve{a}(t) + D\breve{b}(t), \qquad (3)$$

in which system matrices are given in terms of the physical parameters S, L, H, specifically,

$$D = \Delta(S, 0), \quad C = \Delta(C_-, C_+),$$
$$B = -C^\flat \Delta(S, 0), \quad A = -\frac{1}{2}C^\flat C - iJ_n\Delta(\Omega_-, \Omega_+).$$

The *transfer function* for the system G is

$$g_G(t) := \begin{cases} \delta(t)D + Ce^{At}B, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (4)$$

This, together with (2) and (3), yields

$$\breve{b}_{\rm out}(t) = Ce^{A(t-t_0)}\breve{a} + \int_{t_0}^{t} g_G(t-r)\breve{b}(r)dr. \quad (5)$$

The system G is said to be *passive* if both $C^+ = 0$ and $\Omega^+ = 0$. The system G is said to be *asymptotically stable* if the matrix A is Hurwitz.

Define matrix functions

$$g_{G^-}(t) := \begin{cases} \delta(t)S - [\, C_- \;\; C_+ \,]e^{At} \begin{bmatrix} C_-^\dagger \\ -C_+^\dagger \end{bmatrix} S, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$g_{G^+}(t) := \begin{cases} -[\, C_- \;\; C_+ \,]e^{At} \begin{bmatrix} -C_+^T \\ C_-^T \end{bmatrix} S^\#, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

(Note that when G is passive, $g_{G^+}(t) \equiv 0$.) With these functions, the transfer function in (4) can be re-written as $g_G(t) = \Delta\left(g_{G^-}(t), g_{G^+}(t)\right)$. Finally, assume that the system (2)-(3) is asymptotically stable. Letting $t_0 \to -\infty$, equation (5) becomes

$$\breve{b}_{\rm out}(t) = \int_{-\infty}^{\infty} g_G(t-r)\breve{b}(r)dr, \quad (6)$$

which characterizes the steady-state relation between the input and the output.

2.1.2 Stable inversion

For the transfer function $g_G(t)$ defined in (4), let $\Xi_G[s]$ be its two-sided Laplace transform. Define $g_{G^{-1}}(t) := \mathscr{L}_b^{-1}\{\Xi_G[s]^{-1}\}(t)$. The following results are proved in Zhang & James, 2013.

Lemma 1 *Assume that the system G is asymptotically stable. Then $g_{G^{-1}}(t) = \Delta\left(g_{G^-}(-t)^\dagger, -g_{G^+}(-t)^T\right)$.*

Lemma 2 *Assume the quantum linear system G is asymptotically stable. Define an operator*

$$\breve{b}^-(t,t_0) := U(t,t_0)\breve{b}(t)U(t,t_0)^*, \quad t \geq t_0.$$

Then $\breve{b}^-(t,-\infty) = \int_{-\infty}^{\infty} g_{G^{-1}}(t-r)\breve{b}(r)dr$.

2.1.3 Steady-state covariance transfer

Assume the quantum linear system G is in the vacuum state $|\phi\rangle$. Assume further that the input field is in a zero-mean state $\rho_{\rm f}$. (specific types of $\rho_{\rm f}$ will be studied in the sequel.) Denote the covariance functions of the input filed $b(t)$ and the output field $b_{\rm out}(t)$ by $R(t,r)$ and $R_{\rm out}(t,r)$ respectively, that is,

$$\begin{aligned} R(t,r) &= {\rm Tr}[\rho_{\rm f}\breve{b}(t)\breve{b}^\dagger(r)], \quad (7) \\ R_{\rm out}(t,r) &= {\rm Tr}[|\phi\rangle\langle\phi| \otimes \rho_{\rm f}\breve{b}_{\rm out}(t)\breve{b}_{\rm out}^\dagger(r)]. \end{aligned}$$

According to (6) and (7) we have

Lemma 3 *Assume that the system (2)-(3) is asymptotically stable. Let the input field have covariance $R(t,r)$ defined in (7). The steady-state (namely $t_0 \to -\infty$) output covariance function $R_{\rm out}(t,r)$ is*

$$R_{\rm out}(t,r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_G(t-\tau_1)R(\tau_1,\tau_2)g_G(r-\tau_2)^\dagger d\tau_1 d\tau_2. \quad (8)$$

In the frequency domain, we have

Theorem 4 *Assume that the system (2)-(3) is asymptotically stable. If the input field is stationary with spectral density matrix $R[i\omega]$ (namely, the Fourier transform of $R(t,t)$), the output spectral density matrix is given by*

$$R_{\rm out}[i\omega] = \Xi_G[i\omega]R[i\omega]\Xi_G[i\omega]^\dagger. \quad (9)$$

In what follows we focus on the Gaussian input field states. Denote the initial joint system-field Gaussian state by $\rho_{0{\rm g}} = |\phi\rangle\langle\phi| \otimes \rho_{\rm f}$ where $\rho_{\rm f}$ is a Gaussian field state,Zhang & James, 2013. Define the steady-state joint state

$$\rho_{\infty{\rm g}} := \lim_{\substack{t_0\to-\infty \\ t\to\infty}} U(t,t_0)\rho_{0{\rm g}}U(t,t_0)^*, \quad (10)$$

and the steady-state output field state $\rho_{\rm field,g} := {\rm Tr}_{\rm sys}[\rho_{\infty{\rm g}}]$ where the subscript "sys" indicates that the trace operation is performed with respect to the system. According to Theorem 4, if the input state $\rho_{\rm f}$ is stationary with spectral density $R[i\omega]$, then $\rho_{\rm field,g}$ is the steady-state output field density with covariance function $R_{\rm out}[i\omega]$ given in (9).

2.2 Tensors

In this subsection several types of tensors and their associated operations are introduced. Because different channels may have different numbers of photons, fibers of the tensors may thus have different lengths, see e.g., (11). Nonetheless, with slight abuse of notation, we still call these objects tensors.

Given positive integers m and $\ell_1, \dots, \ell_m$, let $\mathbb{C}^{m\times(\ell_1,\dots,\ell_m)}$ be a space of matrix-like objects, whose element ξ is of the form

$$\xi = \begin{bmatrix} \xi^{11} & \cdots & \xi^{1\ell_1} \\ \vdots & \ddots & \vdots \\ \xi^{m1} & \cdots & \xi^{m\ell_m} \end{bmatrix}.$$

In this paper ξ is used to represent m-channel multi-photon input states with ℓ_j denoting the photon number in the j-th channel, $j = 1, \dots, m$. Because channels may have different numbers of photons, $\ell_1, \dots, \ell_m$ may not equal each other. Nonetheless in the paper we still call ξ a matrix. Next we define a tensor space $\mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$, whose elements $\mathscr{S}$ are defined in the following way: For each $i, j = 1, \dots, m$, the model-3 fiber is

$$\mathscr{S}_{ij:} = \begin{bmatrix} \mathscr{S}_{ij1} \\ \vdots \\ \mathscr{S}_{ij\ell_j} \end{bmatrix} \in \mathbb{C}^{\ell_j}. \tag{11}$$

That is, when the first two indices i, j are fixed, we have a vector of dimension ℓ_j. $\mathscr{S}$ looks like a 3-way tensor (Kolda & Bader, 2009), but its mode-3 fibers may have different dimensions for different j. Nevertheless, in this paper we still call $\mathscr{S}$ a 3-way tensor and $\mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$ a space of 3-way tensors over the field of complex numbers. Given a matrix $\xi \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)}$, we may represent it as a 3-way tensor $\xi^\uparrow \in \mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$, by defining horizontal slices to be

$$\xi^\uparrow_{i::} = \begin{bmatrix} \left.\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}\right\}_{i-1} \\ \xi^{i1} \quad \cdots \quad \xi^{i\ell_i} \\ \left.\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}\right\}_{m-i} \end{bmatrix} \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)}, \ \forall i = 1, \dots, m. \tag{12}$$

This update turns out to be very useful because the output state of a quantum passive linear system driven by an m-channel multi-photon state encoded by a matrix $\xi \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)}$ can be characterized by a tensor in $\mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$, see Sec. 3.4.

We adopt notations in Kolda & Bader, 2009. For each $j = 1, \dots, m$ and $k = 1, \dots, \ell_j$,

$$\mathscr{S}_{:jk} = \begin{bmatrix} \mathscr{S}_{1jk} \\ \vdots \\ \mathscr{S}_{mjk} \end{bmatrix} \in \mathbb{C}^m$$

is mode-1 (column) fiber. $\mathscr{S}_{i::}$ and $\mathscr{S}_{:j:}$ are respectively horizontal and lateral slices (matrices) of the form

$$\mathscr{S}_{i::} = \begin{bmatrix} \mathscr{S}^T_{i1:} \\ \vdots \\ \mathscr{S}^T_{im:} \end{bmatrix}, \quad \mathscr{S}_{:j:} = \begin{bmatrix} \mathscr{S}_{:j1} & \cdots & \mathscr{S}_{:j\ell_j} \end{bmatrix}.$$

Finally, let $\mathscr{C} \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)\times(\ell_1,\dots,\ell_m)}$ be a 3-way tensor. We say that $\mathscr{C}$ is *partially Hermitian* in modes 2 and 3 if all the horizontal slices are Hermitian matrices. That is, for all $i = 1, \dots, m$, the horizontal slices $\mathscr{C}_{i::} \in \mathbb{C}^{\ell_i\times\ell_i}$ satisfy $\mathscr{C}^\dagger_{i::} = \mathscr{C}_{i::}$. This is a natural extension of the concept *partially symmetric* discussed in (Kolda & Bader, 2009) to the complex domain.

In what follows we define operations associated to these tensors. Given 3-way tensors $\mathscr{S}(t), \mathscr{T}(r) \in \mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$ and partially Hermitian tensor $\mathscr{C} \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)\times(\ell_1,\dots,\ell_m)}$, we define a matrix $\mathscr{S}(t) \circledast \mathscr{T}(r) \in \mathbb{C}^{m\times m}$ whose (i,k)-th entry is

$$(\mathscr{S}(t) \circledast \mathscr{T}(r))_{ik} := \sum_{j=1}^m \frac{1}{N_{\ell_j}} \sum_{\alpha,\beta=1}^{\ell_j} \mathscr{C}_{j\alpha\beta} \mathscr{S}_{ij\alpha}(t) \mathscr{T}_{kj\beta}(r), \tag{13}$$

where N_{ℓ_j} $(j = 1, \dots, m)$ are positive scalars. (The physical interpretation of N_{ℓ_j} will be given in Sec. 3.) It can be verified that

$$(\mathscr{S}(t) \circledast \mathscr{T}(r))^\dagger = \mathscr{T}(r)^\# \circledast \mathscr{S}(t)^\#. \tag{14}$$

In this paper, we call $\mathscr{C}$ a "core tensor" for the operation $\circledast$. According to (13) and $\xi^\uparrow$ in (12), we have

$$\mathrm{diag}_{j=1,\dots,m}\Big(\frac{1}{N_{\ell_j}} \sum_{i,k=1}^{\ell_j} \mathscr{C}_{jik} \xi^{ji}(r)^* \xi^{jk}(t)\Big) = \xi^\uparrow(r)^\# \circledast \xi^\uparrow(t).$$

Given a matrix function $E(t) \in \mathbb{C}^{m\times m}$ and a 3-way tensor $\mathscr{S}(t) \in \mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$, define $\mathscr{T} \in \mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$ whose (i, j, k)-th element is

$$\mathscr{T}_{ijk}(t) := \sum_{r=1}^m \int_{-\infty}^{\infty} E^{ir}(t-\tau) \mathscr{S}_{rjk}(\tau) d\tau.$$

In compact form we write $\mathscr{T} = \mathscr{S} \times_1 E$, where $\times_1$ is the so-called *1-mode matrix product* (Kolda & Bader, 2009, Sec. 2.5).

Given two matrices $E(t), F(t) \in \mathbb{C}^{m\times m}$ and two tensors $\mathscr{S}(t), \mathscr{T}(t) \in \mathbb{C}^{m\times m\times(l_1,\dots,\ell_m)}$, define

$$\begin{aligned} &\Delta(\mathscr{S}, \mathscr{T}) \times_1 \Delta(E, F) \\ &:= \Delta(\mathscr{S} \times_1 E + \mathscr{T}^\# \times_1 F, \mathscr{T} \times_1 E + \mathscr{S}^\# \times_1 F). \end{aligned} \tag{15}$$

That is, the operation $\times_1$ is performed block-wise. This operation is useful in studying the output state of a quantum linear system driven by a multi-channel multi-photon input state.

Finally, we define another type of operations between matrices and tensors. Let $E(t) = (E^{jk}(t)) \in \mathbb{C}^{m\times m}$ be the transfer function of the underlying quantum linear system with m input channels. For each $j = 1, \dots, m$, let $\mathscr{V}_j(t_1, \dots, t_{\ell_j})$ be an ℓ_j-way m-dimensional tensor function that encodes the pulse information of the jth input field containing ℓ_j photons. Denote the entries of $\mathscr{V}_j(t_1, \dots, t_{\ell_j})$ by $\mathscr{V}_{j,k_1,\dots,k_{\ell_j}}(t_1, \dots, t_{\ell_j})$. Define an ℓ_j-way m-dimensional tensor $\mathscr{W}_j$ with entries given by the following multiple convolution

$$\mathscr{W}_{j,r_1,\dots,r_{\ell_j}}(t_1, \dots, t_{\ell_j}) = \sum_{k_1,\dots,k_{\ell_j}=1}^{m} \int_{\ell_j} E^{r_1k_1}(t_1 - \iota_1)\times$$

$$\cdots \times E^{r_{\ell_j}k_{\ell_j}}(t_{\ell_j} - \iota_{\ell_j})\mathscr{V}_{j,k_1,\dots,k_{\ell_j}}(\iota_1, \dots, \iota_{\ell_j})d\iota_1 \dots d_{\ell_j}$$

for all $1 \le r_1, \dots, r_{\ell_j} \le m$. In compact form we write

$$\mathscr{W}_j = \mathscr{V}_j \times_1 E \times_2 \cdots \times_{\ell_j} E, \quad \forall j = 1, \dots, m, \qquad (16)$$

cf. Kolda & Bader, 2009, Sec. 2.5.

3 The factorizable case

In this section we study how a quantum linear system responds to a factorizable multi-photon input state. Here the word "factorizable" means that photons in each input channel are not statistically correlated.

3.1 Single-channel multi-photon states

In this subsection single-channel ℓ-photon states are defined and their statistical properties are discussed.

For any given positive integer ℓ and real numbers $t_1, \dots, t_\ell$, let $P(t_1, \dots, t_\ell)$ be a permutation of the numbers $t_1, \dots, t_\ell$. Denote the set of all such permutations by S_ℓ. For arbitrary functions $\xi_1(t), \dots, \xi_\ell(t)$ defined on the real line, define

$$N_\ell \qquad (17)$$
$$:= \sum_{P\in S_\ell} \int_\ell \xi_\ell(t_\ell)^* \cdots \xi_1(t_1)^* \xi_1(P(t_1)) \cdots \xi_\ell(P(t_\ell)) dt_{1\to\ell},$$

provided the above multiple integral converges (this is always assumed in the paper). The subscript "ℓ" in N_ℓ indicates the number of photons. It can be shown that $N_\ell > 0$. A single-channel *continuous-mode* ℓ-photon state $|\psi_\ell\rangle$ is defined via

$$|\psi_\ell\rangle := \frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} B^*(\xi_k)|0\rangle, \qquad (18)$$

where $B^*(\xi_k) := \int_{-\infty}^{\infty} b^*(t)\xi_k(t)dt$, $(k = 1, \dots, m.)$ Because $|\psi_\ell\rangle$ is a product of single integrals, there is no correlation among the photons. This type of multi-photon states is therefore called *factorizable* photon states. It can be shown that $\langle\psi_\ell|\psi_\ell\rangle = 1$.

Define a matrix $\mathcal{C} \in \mathbb{C}^{\ell\times\ell}$ whose entries are

$$\mathcal{C}^{ik} = \langle 0| \prod_{\alpha=1,\alpha\neq i}^{\ell} B(\xi_\alpha) \prod_{\beta=1,\beta\neq k}^{\ell} B^*(\xi_\beta) |0\rangle, \ \ell \ge 2. \qquad (19)$$

Clearly, $\mathcal{C} = \mathcal{C}^\dagger$.

In what follows we study statistical properties of the ℓ-photon state $|\psi_\ell\rangle$. It is easy to show that for all $t \ge t_0$, $\langle\psi_\ell|b(t)|\psi_\ell\rangle = 0$. That is, the field has zero average field amplitude. The following result summarizes the second-order statistical information of the ℓ-photon state $|\psi_\ell\rangle$.

Lemma 5 *Let $\bar{n}(t)$ denote the field intensity with respect to the state $|\psi_\ell\rangle$, namely, $\bar{n}(t) = \langle\psi_\ell|b^*(t)b(t)|\psi_\ell\rangle$. (In quantum optics, the second-order moment $\bar{n}(t)$ is the count rate (Gardiner & Zoller, 2000).) Moreover, let the field covariance function be $\langle\psi_\ell|\breve{b}(t)\breve{b}^\dagger(r)|\psi_\ell\rangle$ as given by (7). Then we have*

$$R(t,r) = \delta(t-r)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{N_\ell}\sum_{i=1}^{\ell}\sum_{k=1}^{\ell} \Delta\left(\mathcal{C}^{ik}\xi_k(t)\xi_i(r)^*, 0\right), \qquad (20)$$

$$\bar{n}(t) = \frac{1}{N_\ell}\sum_{i=1}^{\ell}\sum_{k=1}^{\ell} \mathcal{C}^{ik}\xi_i(t)^*\xi_k(t). \qquad (21)$$

Due to page limitation, the proof of Lemma 5 is omitted.

Remark 1. According to Lemma 5, the ℓ-photon state $|\psi_\ell\rangle$ is not Gaussian; it may not be stationary either. So, its first and second order moments cannot provide all statistical information of the input field.

3.2 Multi-channel multi-photon states

Let there be m input field channels. Similar to (18), define the j-th channel ℓ_j-photon state by

$$|\Psi_j\rangle := \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} B_j^*(\xi^{jk})|0\rangle, \qquad (22)$$

where the subscript "j" indicates the j-th channel, and ℓ_j indicates that there are ℓ_j photons in this channel. For each $j = 1, \dots, m$, the normalization coefficient N_{ℓ_j} is defined in the similar way as N_ℓ in (17). We define an m-channel multi-photon state as

$$|\Psi\rangle := |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_m\rangle. \tag{23}$$

In particular, for each $j = 1, \dots, m$, if $\xi^{j1}(t) \equiv \cdots \equiv \xi^{j\ell_j}(t)$, then (23) defines a multi-channel continuous-mode Fock state, see eg., Baragiola, Cook, Brańczyk & Combes, 2012, (D1).

3.3 Output covariance functions and intensities

3.3.1 Steady-state output covariance function

In this part we derive an explicit expression of $R_{\rm out}(t,r)$ when the input is in the multi-channel multi-photon state $|\Psi\rangle$ defined in (23).

We first introduce some notation. Define a 3-way tensor $\mathscr{C} \in \mathbb{C}^{m\times(\ell_1,\dots,\ell_m)\times(\ell_1,\dots,\ell_m)}$, whose elements are

$$\mathscr{C}_{jik} := \langle 0| \prod_{\alpha=1,\alpha\neq i}^{\ell_j} B_j(\xi^{j\alpha}) \prod_{\beta=1,\beta\neq k}^{\ell_j} B_j^*(\xi^{j\beta})\, |0\rangle\,. \tag{24}$$

Clearly, $\mathscr{C}$ is partially Hermitian, that is, $\mathscr{C}_{j::} = \mathscr{C}_{j::}^\dagger \in \mathbb{C}^{\ell_j\times\ell_j}$, $(j = 1, \dots, m)$. The input covariance function is thus of the form

$$\begin{aligned} R(t,r) = &\,\delta(t-r)\begin{bmatrix} I_m & \\ & 0_m \end{bmatrix} \\ &+ \begin{bmatrix} \xi^{\uparrow}(r)^{\#} \circledast \xi^{\uparrow}(t) & \\ & \left(\xi^{\uparrow}(r)^{\#} \circledast \xi^{\uparrow}(t)\right)^{\dagger} \end{bmatrix}. \end{aligned} \tag{25}$$

For state $|\Psi\rangle$ defined in (23), let $\xi^\uparrow$ be the 3-way tensor defined via (12). Then we can define 3-way tensors $\eta^-, \eta^+ \in \mathbb{C}^{m\times m\times(\ell_1,\dots,\ell_m)}$ by

$$\Delta\left(\eta^-, \eta^+\right) := \Delta\left(\xi^\uparrow, \mathbf{0}\right) \times_1 g_G, \tag{26}$$

where the operation $\times_1$ has been defined in (15).

Theorem 6 *Assume that the quantum linear system G is asymptotically stable. Let the input field have covariance $R(t,r)$ be that given in (25). Then the steady-state output covariance function is*

$$\begin{aligned} R_{\rm out}(t,r) = &\int_{-\infty}^{\infty} g_G(t-\tau)\begin{bmatrix} I_m & \\ & 0_m \end{bmatrix} g_G(r-\tau)^\dagger d\tau \\ &+ \Delta\left(\left(\eta^-(r)^{\#} \circledast \eta^-(t)\right), \left(\eta^+(r) \circledast \eta^-(t)\right)\right)^T \\ &+ \Delta\left(\eta^+(t) \circledast \eta^+(r)^{\#}, \eta^+(t) \circledast \eta^-(r)\right), \end{aligned} \tag{27}$$

where $\eta^-(t)$ and $\eta^+(t)$ are given by (26), and the core tensor for the operation $\circledast$ is the tensor $\mathscr{C}$ given in (24).

Proof. (27) can be derived by substituting (25) into (8) and with the aid of (14)-(15) and (26).

3.3.2 Steady-state output intensity

For the multi-channel multi-photon input state $|\Psi\rangle$ defined in (23), the steady-state ($t_0 \to -\infty$) intensity of the output field is

$$\bar{n}_{\rm out}(t) = \left\langle \phi\Psi | b_{\rm out}^{\#}(t) b_{\rm out}^{T}(t) | \phi\Psi \right\rangle. \tag{28}$$

As $\bar{n}_{\rm out}(t)$ is the 2-by-2 block of $R_{\rm out}(t,t)$, the following result is an immediate consequence of Theorem 6.

Corollary 7 *Assume the quantum linear system G is asymptotically stable. The steady-state ($t_0 \to -\infty$) intensity $\bar{n}_{\rm out}(t)$ of the output field defined in (28), of the system G driven by the m-channel multi-photon input field $|\Psi\rangle$ defined in (23), is given by*

$$\begin{aligned} \bar{n}_{\rm out}(t) = &\int_{-\infty}^{\infty} g_{G^+}(t)^{\#} g_{G^+}(t)^T dt \\ &+ \left(\eta^+(t) \circledast \eta^+(t)^{\#}\right)^T + \eta^-(t)^{\#} \circledast \eta^-(t), \end{aligned} \tag{29}$$

where $\eta^-(t)$ and $\eta^+(t)$ are given by (26), and the core tensor for the operation $\circledast$ is given in (24).

3.4 Steady-state output state

The preceding subsections studied the first and second order moments of output fields of quantum linear systems driven by multi-photon states. Since the output states are not Gaussian, these moments cannot provide the complete information of output fields. In this subsection we derive the analytic form of output states.

A multi-channel continuous-mode multi-photon state $|\Psi\rangle$ defined in (23) is parameterized by the functions $\xi^{jk}(t)$, each of which has two indices j and k. The index j (from 1 to m) indicates the j-th input channel, while the index k (from 1 to ℓ_j for each given j) is used to count the number of photons in each channel. On the other hand, according to (26), the steady-state output covariance function (in (27)) and intensity $\bar{n}_{\rm out}(t)$ (in (29)) of the linear quantum system G driven by $|\Psi\rangle$ are parameterized by tensor functions $\eta_{ijk}^-(t)$ and $\eta_{ijk}^+(t)$, each of which has three indices i, j, k. Formally, the index i indicates that each output channel is a linear combination of the input channels. Interestingly, let $\xi^- = \xi^\uparrow$ (c.f. (12)), that is,

$$\xi_{ijk}^-(t) := \begin{cases} 0, & i \neq j, \\ \xi^{jk}(t), & i = j. \end{cases}$$

Define further $\xi^+ \in \mathscr{C}^{m\times m\times(\ell_1,\ldots,\ell_m)}$ to be a zero tensor. Then the m-channel multi-photon state $|\Psi\rangle$ can be re-written as

$$|\Psi\rangle = \prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+))|0^{\otimes m}\rangle. \tag{30}$$

Accordingly, (26) can be re-written as

$$\Delta(\eta^-, \eta^+) = \Delta(\xi^-, \xi^+) \times_1 g_G. \tag{31}$$

In light of the above discussion, we derive the steady-state output state of the quantum linear system G driven by an input state of the form

$$\begin{aligned}
&\rho_{\xi,R}\\
=&\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+))\rho_R\\
&\times \left(\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right)^*,
\end{aligned}$$

where

$$\xi(t) = \Delta(\xi^-(t), \xi^+(t)), \tag{32}$$

with $\xi^-, \xi^+ \in \mathbb{C}^{m\times m\times(\ell_1,\ldots,\ell_m)}$ and ρ_R is a stationary zero-mean Gaussian state with covariance function $R[i\omega]$.

In order for $\rho_{\xi,R}$ to be a valid state, ξ^-, ξ^+ and ρ_R have to satisfy certain conditions. We first introduce some notation. Given a tensor $\varphi \in \mathbb{C}^{2m\times m\times(\ell_1,\ldots,\ell_m)}$, define tensor products consisting of ℓ_j vectors, each of dimension $2m$:

$$\begin{aligned}
&M_{\varphi_{:j}}(t_{1\to\ell_j}) := \varphi_{:j1}(t_1) \otimes_c \cdots \otimes_c \varphi_{:j\ell_j}(t_{\ell_j}),\\
&M_{\varphi_{:j}}^+(t_{1\to\ell_j}) := \varphi_{:j\ell_j}(t_1) \otimes_c \cdots \otimes_c \varphi_{:j1}(t_j),\ j=1,\ldots,m,
\end{aligned}$$

where $\otimes_c$ is the Kronecker product as introduced in the *Notations* part. Then define tensor products of the form

$$\begin{aligned}
&M_\varphi(t_{1\to\ell_1+\cdots+\ell_m})\\
:=&\frac{1}{\sqrt{N_{\ell_1}}} M_{\varphi_{:1}}(t_{1\to\ell_1}) \otimes_c \cdots\\
&\otimes_c \frac{1}{\sqrt{N_{\ell_m}}} M_{\varphi_{:m}}(t_{\ell_1+\cdots+\ell_{m-1}+1\to\ell_1+\cdots+\ell_m}),\\
&M_\varphi^+(t_{1\to\ell_1+\cdots+\ell_m})\\
:=&\frac{1}{\sqrt{N_{\ell_m}}} M_{\varphi_{:m}}^+(t_{1\to\ell_m}) \otimes_c \cdots\\
&\otimes_c \frac{1}{\sqrt{N_{\ell_1}}} M_{\varphi_{:1}}^+(t_{\ell_2+\cdots+\ell_m+1\to\ell_1+\cdots+\ell_m}).
\end{aligned}$$

Similarly, for the operators $\breve{b}(t)$, define

$$M_{\breve{b}}(t_{1\to k}) := \breve{b}(t_1) \otimes_c \cdots \otimes_c \breve{b}(t_k), \quad \forall k \geq 1.$$

Finally for a matrix A, let $A^{\otimes_c^k} := A \otimes_c \cdots \otimes_c A$ be an k-way Kronecker tensor product.

The following equation will be used in Definition 8.

$$\begin{aligned}
&\int_{2\sum_{j=1}^m \ell_j} (M_\xi^+(t_{1\to\ell_1+\cdots+\ell_m})^\#\\
&\quad \otimes_c M_\xi(t_{\ell_1+\cdots+\ell_m+1\to 2(\ell_1+\cdots+\ell_m)}))^T J^{\otimes_c^{\ell_1+\cdots+\ell_m}}\\
&\quad \otimes_c \Theta^{\otimes_c^{\ell_1+\cdots+\ell_m}} \mathrm{Tr}[\rho_R M_{\breve{b}}(t_{1\to 2(\ell_1+\cdots+\ell_m)})]\\
&\quad dt_{1\to 2(\ell_1+\cdots+\ell_m)}\\
=&1,
\end{aligned} \tag{33}$$

where ξ is given in (32) and $\Theta = [\,0\ \ I;\ -I\ \ 0\,]$.

Definition 8 *A state $\rho_{\xi,R}$ is said to be a* photon-Gaussian *state if it belongs to the set*

$$\begin{aligned}
\mathcal{F}_0 := \Bigg\{ &\rho_{\xi,R} = \prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+))\rho_R\\
&\times \left(\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right)^*\\
&: \xi \text{ and } \rho_{\mathrm{R}} \text{ satisfy (33)} \Bigg\}.
\end{aligned} \tag{34}$$

Remark 2. Clearly, the m-channel multi-photon state $|\Psi\rangle$ defined in (30) belongs to $\mathcal{F}_0$.

Proposition 9 *The photon-Gaussian states $\rho_{\xi,R} \in \mathcal{F}_0$ are normalized, that is* $\mathrm{Tr}[\rho_{\xi,R}] = 1$.

Due to page limitation, the proof of Proposition 9 is omitted.

The following result is the main result of this subsection.

Theorem 10 *Suppose that the linear quantum system G is asymptotically stable. Then the steady-state output state of G driven by a state $\rho_{\xi,R} \in \mathcal{F}_0$ is*

$$\begin{aligned}
\rho_{\eta,R_{\mathrm{out}}} = &\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+))\rho_{\mathrm{field,g}}\\
&\left(\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^{m} (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \right)^*,
\end{aligned} \tag{35}$$

where the 3-way tensors η^- and η^+ are given by (31), and $\rho_{\rm field,g}$ is a stationary zero-mean Gaussian field whose covariance function is $R_{\rm out}[i\omega] = G[i\omega]R[i\omega]G[i\omega]^\dagger$ given by the Gaussian transfer (9) in Theorem 4. Clearly, $\rho_{\eta,R_{\rm out}} \in \mathcal{F}_0$.

Proof. Let $\rho(t,t_0)$ be the density operator of the composite system. Then

$$\rho(t,t_0) = U(t,t_0)|\phi\rangle\langle\phi| \otimes \rho_{\xi,R} U(t,t_0)^*.$$

We study the steady-state behavior of the state, that is, we assume that the interaction starts in the distant past ($t_0 \to -\infty$), and also let $t \to \infty$.

$$\begin{aligned} &\rho_\infty \\ &:= \lim_{\substack{t_0\to-\infty\\ t\to\infty}} \rho(t,t_0) \\ &= \lim_{\substack{t_0\to-\infty\\ t\to\infty}} U(t,t_0)|\phi\rangle\langle\phi| \otimes \rho_{\xi,R} U(t,t_0)^* \\ &= \lim_{\substack{t_0\to-\infty\\ t\to\infty}} \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t,t_0) \\ &\quad \left(I_{sys} \otimes (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+))\right) U(t,t_0)^* \rho_{\infty,\rm g} \\ &\quad \times \lim_{\substack{t_0\to-\infty\\ t\to\infty}} \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t,t_0) \right. \\ &\quad \left. \times \left(I_{sys} \otimes (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+))\right) U(t,t_0)^* \right)^*, \end{aligned} \tag{36}$$

where $\rho_{\infty,\rm g}$ is given in (10). By means of Lemmas 1 and 2, we have

$$\begin{aligned} &\lim_{\substack{t_0\to-\infty\\ t\to\infty}} \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t,t_0) \\ &\qquad \times \left(I_{sys} \otimes \left(B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)\right)\right) U(t,t_0)^* \\ &= \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m \left(B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)\right), \end{aligned} \tag{37}$$

where 3-way tensors η^- and η^+ are given by (31). Substituting (37) into (36) we have

$$\begin{aligned} \rho_\infty = &\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \rho_{\infty,\rm g} \\ &\times \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \right)^*. \end{aligned}$$

Tracing out the system part, (35) is obtained. The proof is completed.

Specific to the passive case, the steady-state output state is a multi-photon state, as given by the following result.

Corollary 11 *Assume that the quantum linear system G is asymptotically stable and passive. The steady-state output state of G driven by the multi-photon state $|\Psi\rangle$ is a pure state*

$$|\Psi_{\rm out}\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m B_i^*(\eta_{ijk}^-)|0^{\otimes m}\rangle.$$

Example 1: Beamsplitter. Consider a beamsplitter with parameters $L = 0$, $H = 0$, and

$$S = \begin{bmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ -\sqrt{1-\eta} & \sqrt{\eta} \end{bmatrix}, \quad (0 < \eta < 1).$$

Let each input channel have two photons. According to Corollary 11,

$$\begin{aligned} &|\Psi_{\rm out}\rangle \\ =&\frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11}) B_1^*(\xi^{12}) B_1^*(\xi^{21}) B_1^*(\xi^{22}) |0^{\otimes 4}\rangle \\ &+ \frac{\eta\sqrt{\eta(1-\eta)}}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11}) B_1^*(\xi^{12}) B_1^*(\xi^{22}) B_2^*(\xi^{21}) |0^{\otimes 4}\rangle \\ &+ \frac{\eta\sqrt{\eta(1-\eta)}}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11}) B_1^*(\xi^{12}) B_1^*(\xi^{21}) B_2^*(\xi^{22}) |0^{\otimes 4}\rangle \\ &- \frac{\sqrt{\eta(1-\eta)}(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{21}) B_1^*(\xi^{22}) B_1^*(\xi^{12}) B_2^*(\xi^{11}) |0^{\otimes 4}\rangle \\ &- \frac{\sqrt{\eta(1-\eta)}(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{21}) B_1^*(\xi^{22}) B_1^*(\xi^{11}) B_2^*(\xi^{12}) |0^{\otimes 4}\rangle \\ &+ \frac{\eta^2}{\sqrt{N^{21}N_{2_2}}} B_1^*(\xi^{11}) B_1^*(\xi^{12}) B_2^*(\xi^{21}) B_2^*(\xi^{22}) |0^{\otimes 4}\rangle \\ &+ \frac{(1-\eta)^2}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{21}) B_1^*(\xi^{22}) B_2^*(\xi^{11}) B_2^*(\xi^{12}) |0^{\otimes 4}\rangle \\ &- \frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} \left(B_1^*(\xi^{11}) B_2^*(\xi^{12}) + B_1^*(\xi^{12}) B_2^*(\xi^{11})\right) \\ &\quad \times \left(B_1^*(\xi^{21}) B_2^*(\xi^{22}) + B_1^*(\xi^{22}) B_2^*(\xi^{21}\right) |0^{\otimes 4}\rangle \\ &- \frac{\eta\sqrt{\eta(1-\eta)}}{\sqrt{N_{2_1}N_{2_2}}} \left(B_1^*(\xi^{11}) B_2^*(\xi^{12}) + B_1^*(\xi^{12}) B_2^*(\xi^{11})\right) \\ &\quad \times B_2^*(\xi^{21}) B_2^*(\xi^{22}) |0^{\otimes 4}\rangle \\ &+ \frac{\sqrt{\eta(1-\eta)}(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} \left(B_1^*(\xi^{21}) B_2^*(\xi^{22}) + B_1^*(\xi^{22}) B_2^*(\xi^{21})\right) \\ &\quad \times B_2^*(\xi^{11}) B_2^*(\xi^{12}) |0^{\otimes 4}\rangle \\ &+ \frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_2^*(\xi^{11}) B_2^*(\xi^{12}) B_2^*(\xi^{21}) B_2^*(\xi^{22}) |0^{\otimes 4}\rangle. \end{aligned} \tag{38}$$

Assume $\eta = \frac{1}{2}$, $\xi^{11}(t) \equiv \xi^{12}(t) \equiv \xi^{21}(t) \equiv \xi^{22}(t)$ and $\int_{-\infty}^{\infty} |\xi^{11}(t)|^2 dt = 1$. Let $\frac{1}{\sqrt{i!}\sqrt{k!}}|i,k\rangle$ be the state with i photons in the first channel and k photons in the second channel respectively, $(i = 0, \dots, 4)$. (38) reduces to

$$|\Psi_{\rm out}\rangle = \sqrt{\frac{3}{8}}|4,0\rangle - \frac{1}{2}|2,0\rangle|0,2\rangle + \sqrt{\frac{3}{8}}|0,4\rangle. \tag{39}$$

(39) is the same as (15) in (Ou, 2007). In a similar way, (17) in (Ou, 2007) can also be re-produced.

4 The unfactorizable case

The factorizable multi-photon states studied in Section 3 form a subclass of more general multi-photon states, e.g., Gheri, Ellinger, Pellizzari & Zoller, 1998, (58) and Baragiola, Cook, Brańczyk & Combes, 2012, Section 2. In this section, we study the response of quantum linear systems to general multi-channel multi-photon states where there may exist correlation among photons in channels.

4.1 More general multi-photon states

Assume the j-th channel has ℓ_j photons, and the state for this channel is

$$|\Psi_j\rangle := \frac{1}{\sqrt{N_{\ell_j}}} \int_{\ell_j} \Psi_j(t_1, \dots, t_{\ell_j}) b_j^*(t_1) \cdots b_j^*(t_{\ell_j}) dt_{1\to\ell_j} |0\rangle. \tag{40}$$

Then the state for the m-channel input field can be defined as

$$|\Psi\rangle = \prod_{j=1}^{m} |\Psi_j\rangle. \tag{41}$$

Remark 3. When $\Psi_j(t_1, \dots, t_{\ell_j}) = \prod_{k=1}^{\ell_j} \Psi^{jk}(t_k)$, $(j = 1, \dots, m)$, (40) reduces to (22), and correspondingly (41) reduces to (23), the factorizable case.

4.2 The passive case

In this subsection we study the response of the quantum linear passive system G to an m-channel input field in the state $|\Psi\rangle$ defined in (41).

We first rewrite the m-channel multi-photon state $|\Psi\rangle$ into an alternative form; this will enable us to present the input and output states in a unified form. For $j = 1, \dots, m$, $i = 1, \dots, \ell_j$, and $k_i = 1, \dots, m$, define

$$\Psi_{j,k_1,\dots,k_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) := \begin{cases} \Psi_j(\tau_1, \dots, \tau_{\ell_j}), & k_1 = \cdots = k_{\ell_j} = j, \\ 0, & \text{otherwise}. \end{cases} \tag{42}$$

We define a class of pure states

$$\mathcal{F}_1 = \left\{ |\Psi\rangle = \prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \sum_{k_1,\dots,k_{\ell_j}=1}^{m} \int_{\ell_j} \Psi_{j,k_1,\dots,k_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) \times \prod_{i=1}^{\ell_j} b_{k_i}^*(\tau_i) d\tau_{1\to\ell_j} |0^{\otimes m}\rangle : \langle\Psi|\Psi\rangle = 1 \right\}. \tag{43}$$

Clearly, $|\Psi\rangle$ in (41) belongs to $\mathcal{F}_1$.

Theorem 12 *Suppose that the quantum linear system G is asymptotically stable and passive. The steady-state output state of G driven by a state $|\Psi_{\rm in}\rangle \in \mathcal{F}_1$ is another state $|\Psi_{\rm out}\rangle \in \mathcal{F}_1$ with wave packet transfer*

$$\Psi_{{\rm out},j} = \Psi_{{\rm in},j} \times_1 g_{G^-} \times_2 \cdots \times_{\ell_j} g_{G^-}, \quad \forall j = 1, \dots, m,$$

where the operation between the matrix and tensor is defined in (16).

Due to page limitation, the proof of Theorem 12 is omitted.

In particular, for the single-channel case, we have

Corollary 13 *The steady-state output state of a quantum linear passive system G driven by the ℓ-photon state $|\psi_\ell\rangle$ is an ℓ-photon state*

$$|\psi_{\rm out}\rangle = \frac{1}{\sqrt{N_\ell}} \int_\ell \psi_{\rm out}^-(\iota_1, \dots, \iota_\ell) b^*(t_1) b^*(t_2) \cdots b^*(t_\ell) dt_{1\to\ell} |0\rangle,$$

where the multi-variable function $\psi_{\rm out}^-$ is

$$\psi_{\rm out}^-(t_1, \dots, t_\ell) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_{G^-}(t_1 - \tau_1) \cdots g_{G^-}(t_\ell - \tau_\ell) \times \psi_\ell(\tau_1, \dots, \tau_\ell) d\tau_1 \cdots d\tau_\ell.$$

4.3 The active case

In this subsection we study the response of the quantum linear system G to the m-channel input field in the state $|\Psi\rangle$ defined in (41). Here G is not necessarily passive, i.e., $g_{G^+} \neq 0$.

We first introduce some more notations. Define

$$\operatorname{sgn}(d_i) := \begin{cases} 1, & d_i = 1, \\ 0, & d_i = -1, \end{cases} \quad \forall i = 1, \dots, \max\{\ell_1, \dots, \ell_m\}.$$

For each $j=1,\dots,m$ and $k_i=1,\dots,m$, define

$$\Psi^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\tau_1,\dots,\tau_{\ell_j}) \tag{44}$$
$$:=\begin{cases}\Psi_j(\tau_1,\dots,\tau_{\ell_j}), & k_1=\cdots=k_{\ell_j}=j,\ d_1=\cdots=d_{\ell_j}=-1,\\ 0, & \text{otherwise.}\end{cases}$$

Accordingly, for each $j=1,\dots,m$, and $i=1,\dots,\ell_j$ define operators

$$b_j^{d_i}(t):=\begin{cases}b_j^*(t), & d_i=-1,\\ b_j(t), & d_i=1.\end{cases}$$

For each $j,k=1,\dots,m$, define

$$g^{kj}_{G^d}(t):=\begin{cases}g^{kj}_{G^-}(t), & d=-1,\\ g^{kj}_{G^+}(t)^*, & d=1.\end{cases}$$

Moreover, for each $j=1,\dots,m$, $i=1,\dots,\ell_j$ and $k_i=1,\dots,m$, define operators

$$b^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\Psi_j):=\Psi^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(t_1,\dots,t_{\ell_j})b_j^{d_1}(t_1)\cdots b_j^{d_{\ell_j}}(t_{\ell_j}), \tag{45}$$

where the $2\ell_j$-way tensor $\Psi^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\tau_1,\dots,\tau_{\ell_j})$ is that defined in (44). Then the multi-channel state $|\Psi\rangle$ in (41) can be re-written as

$$|\Psi\rangle=\prod_{j=1}^m\frac{1}{\sqrt{N_{\ell_j}}}\sum_{k_1,\dots,k_{\ell_j}=1}^m\sum_{d_1,\dots,d_{\ell_j}=\pm1}(-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(d_i)}\times\int_{\ell_j}b^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\Psi_j)dt_1\dots dt_{\ell_j}|0^{\otimes m}\rangle. \tag{46}$$

The above motivates us to define a class of states.

Definition 14 *A state $\rho_{\Psi,R}$ is said to be a* photon-Gaussian *state if it belongs to the set*

$$\begin{aligned}\mathcal{F}_2&\\ :=\Bigg\{&\rho_{\Psi,R}=\prod_{j=1}^m\frac{1}{\sqrt{N_{\ell_j}}}\sum_{k_1,\dots,k_{\ell_j}=1}^m\sum_{d_1,\dots,d_{\ell_j}=\pm1}\\ &\times(-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(d_i)}\int_{\ell_j}b^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\Psi_j)dt_1\dots dt_{\ell_j}|0^{\otimes m}\rangle\rho_R\\ &\times\Bigg(\prod_{j=1}^m\frac{1}{\sqrt{N_{\ell_j}}}\sum_{k_1,\dots,k_{\ell_j}=1}^m\sum_{d_1,\dots,d_{\ell_j}=\pm1}(-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(d_i)}\\ &\times\int_{\ell_j}b^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\Psi_j)dt_1\dots dt_{\ell_j}|0^{\otimes m}\rangle\Bigg)^*\Bigg\},\end{aligned} \tag{47}$$

where the operator $b^{d_1,\dots,d_{\ell_j}}_{k_1,\dots,k_{\ell_j}}(\Psi_j)$ is defined in (45), and ρ_R is a zero-mean Gaussian field state with covariance function R. It is assumed that $\mathrm{Tr}[\rho_{\Psi,R}]=1$.

Remark 4. Clearly, the m-channel multi-photon state $|\Psi\rangle$ defined in (41) belongs to $\mathcal{F}_2$. Moreover, when G is passive, $\mathcal{F}_1=\mathcal{F}_2$.

Next we study how the input state in $\mathcal{F}_2$ is transformed by the quantum linear system G.

Theorem 15 *Suppose that the quantum linear system G is asymptotically stable. The density function $\rho_{\Psi_{\rm out},R_{\rm out}}$ of the steady-state output field of G driven by the density operator $\rho_{\Psi,R}\in\mathcal{F}_2$ is*

$$\begin{aligned}&\rho_{\Psi_{\rm out},R_{\rm out}}\\ &=\Bigg(\prod_{j=1}^m\frac{1}{\sqrt{N_{\ell_j}}}\sum_{r_1,\dots,r_{\ell_j}=1}^m\sum_{f_1,\dots,f_{\ell_j}=\pm1}(-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(f_i)}\\ &\int_{\ell_j}b^{f_1,\dots,f_{\ell_j}}_{r_1,\dots,r_{\ell_j}}(\Psi_{out,j})d\tau_{1\to\ell_j}|0^{\otimes m}\rangle\Bigg)\rho_{R_{\rm out}}\\ &\times\Bigg(\prod_{j=1}^m\frac{1}{\sqrt{N_{\ell_j}}}\sum_{r_1,\dots,r_{\ell_j}=1}^m\sum_{f_1,\dots,f_{\ell_j}=\pm1}(-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(f_i)}\\ &\int_{\ell_j}b^{f_1,\dots,f_{\ell_j}}_{r_1,\dots,r_{\ell_j}}(\Psi_{out,j})d\tau_{1\to\ell_j}|0^{\otimes m}\rangle\Bigg),\end{aligned} \tag{48}$$

where

$$g^{kj}_{G^{d_i},f_i}(t):=\begin{cases}g^{kj}_{G^{-d_i}}(t), & f_i=-1,\\ g^{kj}_{G^{d_i}}(t), & f_i=1,\end{cases},$$

$$b_i^{d_k,f_k}(t) := \begin{cases} b_i^{-d_k}(t), & f_i = -1, \\ b_i^{d_k}(t), & f_i = 1, \end{cases},$$

$$\Psi_{k_{1\to\ell_j},r_{1\to\ell_j}}^{d_{1\to\ell_j},f_{1\to\ell_j}}(t_1,\ldots,t_{\ell_j}) \tag{49}$$
$$:= \int_{\ell_j} \prod_{i=1}^{\ell_j} g_{G^{d_i,f_i}}^{r_i k_i}(t_i-\tau_i)\Psi_{k_1,\ldots,k_{\ell_j}}^{d_1,\ldots,d_{\ell_j}}(\tau_1,\ldots,\tau_{\ell_j})d\tau_{1\to\ell_j},$$

$$b_{r_1,\ldots,r_{\ell_j}}^{f_1,\ldots,f_{\ell_j}}(\Psi_{out,j}) \tag{50}$$
$$:= \sum_{k_1,\ldots,k_{\ell_j}=1}^{m} \sum_{d_1,\ldots,d_{\ell_j}=\pm1} (-1)^{\sum_{i=1}^{\ell_j}\mathrm{sgn}(d_i)}$$
$$\times \Psi_{k_{1\to\ell_j},r_{1\to\ell_j}}^{d_{1\to\ell_j},f_{1\to\ell_j}}(t_1,\ldots,t_{\ell_j})\prod_{i=1}^{\ell_j} b_{r_i}^{d_i,f_i}(t_i),$$

and $\rho_{R_{\rm out}}$ is a Gaussian state whose covariance function is given by the Gaussian transfer $R_{\rm out}[i\omega] = G[i\omega]R[i\omega]G[i\omega]^\dagger$.

Due to page limitation, the proof of Theorem 15 is omitted.

Remark 5. It can be verified that the *factorizable* m-channel multi-photon state $|\Psi\rangle$ defined in (23) (equivalently (30)) can be re-written as

$$|\Psi\rangle\langle\Psi| \tag{51}$$
$$= \prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \sum_{i=1}^{m}\prod_{k=1}^{\ell_j}(B_j^*(\xi_{ijk}^-) - B_j(\xi_{ijk}^+))|0^{\otimes m}\rangle\langle 0^{\otimes m}|$$
$$\times \left(\prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_j}}} \sum_{i=1}^{m}\prod_{k=1}^{\ell_j}(B_j^*(\xi_{ijk}^-) - B_j(\xi_{ijk}^+)) \right)^*.$$

There is clear similarity between (23) and (46), or equivalently, between (51) and (47). The implication of such similarity is that all the results for the unfactorizable case can be reduced to those for the factorizable case.

Remark 6. When the quantum linear system G is passive and $\rho_R = |\phi\rangle\langle\phi|$, $\rho_{\Psi,R}$ in (47) becomes a pure state. Moreover, for the case case, $\mathrm{sgn}(d_i) = 0$ for all i. Therefore, in the passive case $\rho_{\Psi,R}$ is a pure state in the class $\mathcal{F}_1$ defined in (43). As a result, in the passive case Theorem 15 reduces to Theorem 12.

Example 2: The $(1+\ell)$-photon case. Consider a beamsplitter with parameter

$$S = \begin{bmatrix} \sqrt{1-R} & \sqrt{R} \\ \sqrt{R} & -\sqrt{1-R} \end{bmatrix}, \quad (0 < R < 1).$$

Let the input state be $|\Psi_{in}\rangle = B_1^*(\xi)\otimes\frac{1}{\sqrt{N_\ell}}\prod_{k=1}^{\ell} B_2^*(\xi_k)|00\rangle$. As with Example 1, the output state can be derived by means of Corollary 11. Alternatively, it can be derived via Theorem 12. By Theorem 12, the output state is

$$|\Psi_{\rm out}\rangle = (\sqrt{1-R}B_1^*(\xi) + \sqrt{R}B_2^*(\xi)) \tag{52}$$
$$\times \frac{1}{\sqrt{N_\ell}} \sum_{k_1,\ldots,k_\ell=1}^{2} B_{k_1}^*(S^{k_1 2}\xi_{k_1})\cdots B_{k_\ell}^*(S^{k_\ell 2}\xi_{k_\ell})|00\rangle.$$

In particular, assume $\xi_1(t) \equiv \cdots \equiv \xi_\ell(t) \equiv \xi(t)$ and $\int_{-\infty}^{\infty}|\xi(t)|^2 dt = 1$. Then (52) becomes

$$|\Psi_{\rm out}\rangle = \frac{1}{\sqrt{\ell!}}(\sqrt{1-R}B_1^*(\xi) + \sqrt{R}B_2^*(\xi))$$
$$\times (\sqrt{R}B_1^*(\xi) - \sqrt{1-R}B_2^*(\xi))^\ell|00\rangle.$$

The coefficient for the component $\frac{1}{\sqrt{\ell!}}B_1^*(\xi)^\ell B_2^*(\xi)|00\rangle = \frac{1}{\sqrt{\ell!}}|\ell,1\rangle$ is $\sqrt{R^{\ell-1}}(R-\ell(1-R))$, whose squared value is exactly (in) in Sanaka, Resch & Zeilinger, 2006.

Example 3: The photon-catalyzed optical coherent (PCOC) case. Consider a beamsplitter with parameter

$$S = \begin{bmatrix} T & -R \\ R & T \end{bmatrix}, \quad (R,T > 0, \quad R^2 + T^2 = 1).$$

Let the input be $|\psi_\ell\rangle\otimes|\alpha\rangle$, where $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ is a coherent state. The input state can be re-written as

$$|\Psi_{in}\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\prod_{j=1}^{2}\prod_{k=1}^{\ell_j} B_j^*(\xi)|0^{\otimes 2}\rangle,$$

where $\ell_1 = \ell$ and $\ell_2 = n$. When the first output channel is measured by means of the state $|\ell\rangle$, the state at the second output channel becomes

$$|\Psi_{\rm out,conditioned}\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}$$
$$\times \sum_{j=0}^{\min\{\ell,n\}} \binom{n}{n-j}\binom{\ell}{j}(-1)^j T^{n+\ell-2j}R^{2j}|n\rangle,$$

which is the key formula (1) in Bartley, et al., 2012.

5 Conclusion

In this paper we have studied the response of quantum linear systems to multi-channel multi-photon states. New types of tensors are defined to encode pulse information of multi-photon states, for both the factorizable case and the unfactorizable case. The steady-state action of quantum linear systems on multi-photon states are characterized in terms of tensor processing by transfer functions. Explicit forms of output states, output covariance functions and output intensities have been derived. In contrast to the discrete-variable (single-mode) treatments in most discussions on quantum information, we have presented a continuous-variable (multi-mode) treatment of multi-photon processing. As can be seen from Examples 1-3, the continuous-variable treatment is also applicable to many discrete-variable treatments. (Due to page limitation, some details have to be omitted. An extended version is arxiv.org/abs/1311.0357.)

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