Chaos

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In this paper, we study the controllability of networks with different numbers of communities and various strengths of community structure. By means of simulations, we show that the degree descending pinning scheme performs best among several considered pinning schemes under a small number of pinned nodes, while the degree ascending pinning scheme is becoming more powerful by increasing the number of pinned nodes. It is found that increasing the number of communities or reducing the strength of community structure is beneficial for the enhancement of the controllability. Moreover, it is revealed that the pinning scheme with evenly distributed pinned nodes among communities outperforms other kinds of considered pinning schemes. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4816009]

Synchronization of large ensembles of interacting units is a typical phenomenon in many biological and social networks. Meanwhile, the problem of synchronization is generally considered as a paradigmatic example of phase transitions that may occur when a large number of dynamical systems are coupled. Therefore, it is important to understand the related regulatory mechanisms in physics and applied science. The study of pinning control on networks will help to interpret the nature of the underlying mechanisms and give an insight into network dynamics. In this paper, we focus on an analysis of how to control networks with community structure efficiently. Considering random, degree descending and degree ascending pinning schemes, a comparison study of controllability is reported. Effects of the numbers of communities, strength of community structure, type of pinning schemes, and distributions of pinned nodes on controllability will be investigated thoroughly.

I. INTRODUCTION

Synchronization phenomena are widely existing in biological and social networks.[1–3] In physiology, the heart cells beat synchronously and the beating rhythm is generated by pacemaker cells situated at the sinoatrial node.[4] In social networks, key individuals termed as opinion leaders often drive the opinion dynamics.[5] It is therefore of great importance to understand the fundamental nature of regulatory mechanisms. Pinning control is an effective method to provide an insight into the regulation of networks of coupled dynamical systems. By inputting control signals to only a portion of the network nodes, the states of several specific complex networks can be controlled to a desired state.

In recent years, controllability of networks of coupled oscillators has been extensively studied.[6–10] Through a master stability function (MSF) approach, pinning controllability was defined and quantified to evaluate the controllability of complex networks.[7] The effects of network's structural properties on controllability were studied and it was found that high heterogeneity in degree distribution reduces controllability.[8] In Ref. 9, it was revealed that minimizing the distances between the driver nodes and other uncontrolled nodes can lead to a better control performance. From the viewpoint of control theory, several analytical tools were developed and applied to identify an optimum set of driver nodes. It was also unveiled that dense and homogeneous networks can be controlled only using a few driver nodes.[10] Further studies on pinning control of networks have been reported in Refs. 11–18.

Community structure in networks is of great importance for theoretical studies and various applications. Usually, the connections inside communities are dense, while the connections between communities are sparse. For instance, groups within the worldwide web might correspond to sets of web pages on related topics;[19] groups within social networks might correspond to social units or communities.[20] Several studies showed that a pronounced community structure influences the network dynamics such as packet delivery, local synchronization, and global synchronization.[21–25] However, until now, the effect of community structure on networks' controllability has been only partly investigated in the literature, despite its importance for theoretical and practical studies.

Motivated by the above discussion, by means of MSF, we will analyze effects of control schemes, control gains,

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number of pinned nodes, and distributions of driver nodes on controllability. The remainder of this paper is organized as follows. In Sec. II, a community network model is introduced. Then, based on the MSF approach, the network controllability is analyzed in Sec. III. In Sec. IV, our main results are given. The conclusions are drawn in Sec. V.

II. PRELIMINARIES

In order to obtain a specific complex network with community structure, we quote a growth model to create a network with a tunable parameter denoting the strength of community structure.[21] Starting with a given number of communities, we add new vertices to each community and adjust the fraction of within-community connections to create networks with various strengths of community structure. The main steps are listed as follows:

Step I: The network starts from C communities. Assume that each community has the same number of vertices m_0. The initial $m_0 \times C$ vertices link to each other and therefore the network is fully connected.

Step II: At each time step, a new vertex with $m\,(m < m_0)$ edges is added into each community. According to the preferential attachment mechanism, connections to $t\,(t < m)$ different vertices are created in the community $s\,(s \in \{1, 2, \ldots, C\})$.

Step III: For each one of the other $(m - t)$ between-community connections of the newly added vertex, we choose one community from the remainder $(C - 1)$ communities randomly. Then, using the preferential attachment mechanism, a new edge is created between this added vertex and one vertex in the chosen community.

It should be noted that, based on the above strategy, the degree distribution of the global network, as well as the degree distribution of each community, follows a power-law distribution.[21]

For a given partition of nodes of a network into communities, the strength of the community structure is quantified as follows:[26–29]

$$Q = \sum_{s=1}^{C} \left[\frac{l_s}{L} - \left(\frac{d_s}{2L} \right)^2 \right], \tag{1}$$

where C is the number of communities, L is the number of edges in the network, l_s denotes the number of edges between vertices in community s, and d_s stands for the sum of degrees of vertices in community s. A network with prominent communities would have a large value of Q and fewer between-community connections.

In addition to the parameter Q, the within-community connection strength r is used to represent the ratio of the within-community connections and the total connections of the newly added vertex. From step I to step III, r is formulated as follows:

$$r = \frac{t}{m}. \tag{2}$$

A small value of r implies a network with a weak strength of community structure. Here, a generated network is characterized by r by adjusting the parameter t when m and C are fixed.

III. OVERVIEW OF PINNING CONTROLLABILITY

To analyze the controllability of a specific network, the following model is considered:

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij}[H(x_j) - H(x_i)], \; i = 1, 2, \ldots, N, \tag{3}$$

where $f(x_i)$ is the dynamics at each node; σ is the overall coupling strength; $H(x)$ is the coupling function and gives the coupling term of two connected nodes. In this paper, the network is supposed to be undirected and unweighted. The coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ relates to the topology of the network. If there is a connection between nodes i and $j\,(j \neq i)$, then $a_{ij} = a_{ji} = 1$. Otherwise, $a_{ij} = a_{ji} = 0\,(j \neq i)$, and $a_{ii} = 0, i = 1, 2, \ldots, N$.

We consider an isolated node $s(t)$ given *a priori* and it satisfies $\dot{s} = f(s)$. In order to drive the states of Eq. (3) to the desired reference evolution $s(t)$, the following equation is obtained:

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij}[H(x_j) - H(x_i)] + \sigma \delta_i k_i (H(s) - H(x_i)), \; i = 1, 2, \ldots, N, \tag{4}$$

where k_i is the control gain. Suppose that all k_i are equal, e.g., $k_i = k$, $i = 1, 2, \ldots, N$. Let p denote the ratio between the pinned nodes and all nodes of a network $(0 \le p \le 1)$. Then the set of pinned nodes is $I = \{i_1, i_2, \ldots, i_n\}$, where $n = \lfloor p \times N \rfloor$.

Equation (4) can be rewritten as

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H(x_j) + \sigma \delta_i k_i (H(s) - H(x_i)), \; i = 1, 2, \ldots, N, \tag{5}$$

where the elements L_{ij} of the Laplacian matrix L are as follows: $L_{ij} = -a_{ij}$ if $j \neq i$ and $L_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}, i = 1, 2, \ldots, N$. Thus, the matrix has $\sum_{j=1}^{N} L_{ij} = 0, i = 1, 2, \ldots, N$. If $i \in I$, then $\delta_i = 1$, otherwise $\delta_i = 0$.

Following the method proposed in Ref. 7, we can convert Eq. (5) to an extended network composed of $(N + 1)$-dynamical systems y_i, where $y_i = x_i$ for $i = 1, 2, \ldots, N$ and $y_{N+1} = s$. Then, it can be written as

$$\dot{y}_i = f(y_i) - \sigma \sum_{j=1}^{N+1} M_{ij} H(y_j), \; i = 1, 2, \ldots, N + 1, \tag{6}$$

where $M = \{M_{ij}\}$ is an $(N + 1)$ dimensional square matrix defined by

$$M = \begin{pmatrix} L_{11}+\delta_1 k_1 & L_{12} & \dots & L_{1N} & -\delta_1 k_1 \\ L_{21} & L_{22}+\delta_2 k_2 & \dots & L_{2N} & -\delta_2 k_2 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ L_{N1} & L_{N2} & \dots & L_{NN}+\delta_N k_N & -\delta_N k_N \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $\{\mu_i^r\}$ be the eigenvalues of the matrix M and assume that they are sorted as $\mu_1^r \le \mu_2^r \le \dots \le \mu_{N+1}^r$. From graph theory, we get $\mu_i^r \ge 0$, and μ_1 is the only null eigenvalue of the matrix M.[30] Since the network is undirected, the symmetrical coupling matrix A ensures that the matrix M is diagonalizable. Therefore, the spectrum of M can be decomposed into the spectrum of a symmetric matrix containing its first N rows and first N columns, plus one zero eigenvalue.

Through the transformation from Eq. (5) to Eq. (6), the problem of controllability is converted into the analysis of synchronizability of the extended network M. Hence, the function f, the coupling function H, and the coupling matrix A influence the synchronizability of the network M. The range of stability of the synchronous state is assumed to be a bounded zone of the complex plane, and the method of eigenvalue ratio can be used to assess synchronizability of Eq. (6).[31] Here, the matrix M has a real spectrum and it is found to affect the stability of the synchronous manifold by applying the MSF approach to analyze the stability of Eq. (6). Then, the smaller the eigenvalue ratio $R = \frac{\mu_{N+1}^r}{\mu_2^r}$ is, the better the synchronizability is.[7] Hence, we need to suppress R as small as possible to enhance controllability.

IV. CONTROLLABILITY OF THE NETWORKS WITH COMMUNITY STRUCTURE

In this section, we will analyze the impact of community structure on controllability. The considered network with community structure consisting of $N = 200$ nodes is generated by the procedure described in Sec. II. The community structure varies from weak to strong by adjusting the within-community connection strength r in Eq. (2). For the results demonstrated in this section, we carry out 100 realizations. Three kinds of pinning schemes are compared here:

(i) Scheme 1: the nodes are randomly pinned, i.e., each node has a uniform probability to be chosen as a driver node over the network.

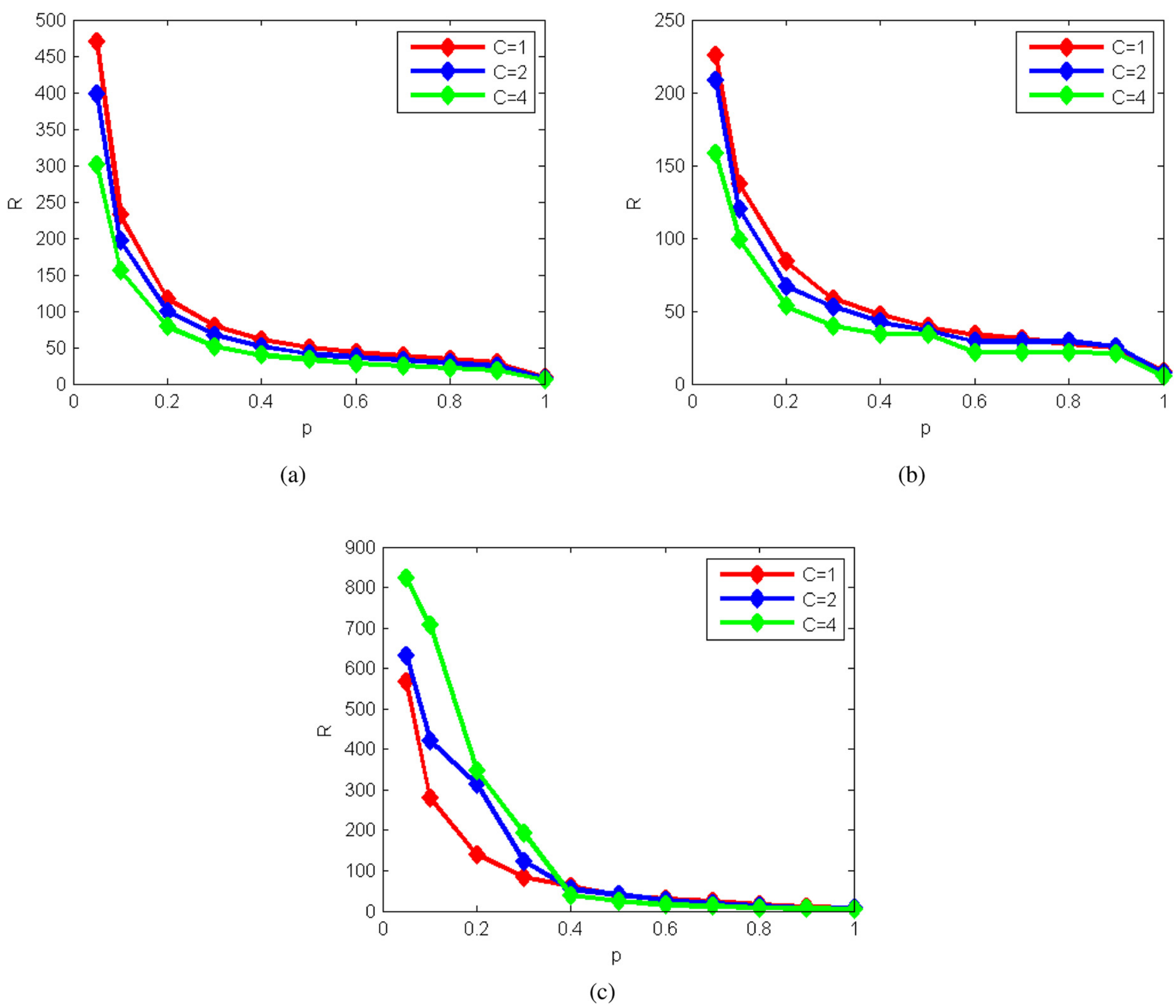

FIG. 1. Eigen value ratio R as a function of the pinning ratio p for networks with different numbers of communities $C = 1$, 2, and 4. The within-community connection strength is $r = 0.8$ and the control gain is $k = 5$. (a) Controllability analysis with scheme 1, (b) controllability analysis with scheme 2 and (c) controllability analysis with scheme 3.

(ii) Scheme 2: the nodes are pinned according to their descending degrees, starting by the node with the highest degree.

(iii) Scheme 3: the pinned nodes are selected according to their ascending degrees, starting by the node with the smallest degree.

A. Analysis of controllability of networks with different numbers of communities

A comparison study between different numbers of communities is shown in Figs. 1(a)–1(c). By increasing the pinning ratio p, one can always observe a better controllability,

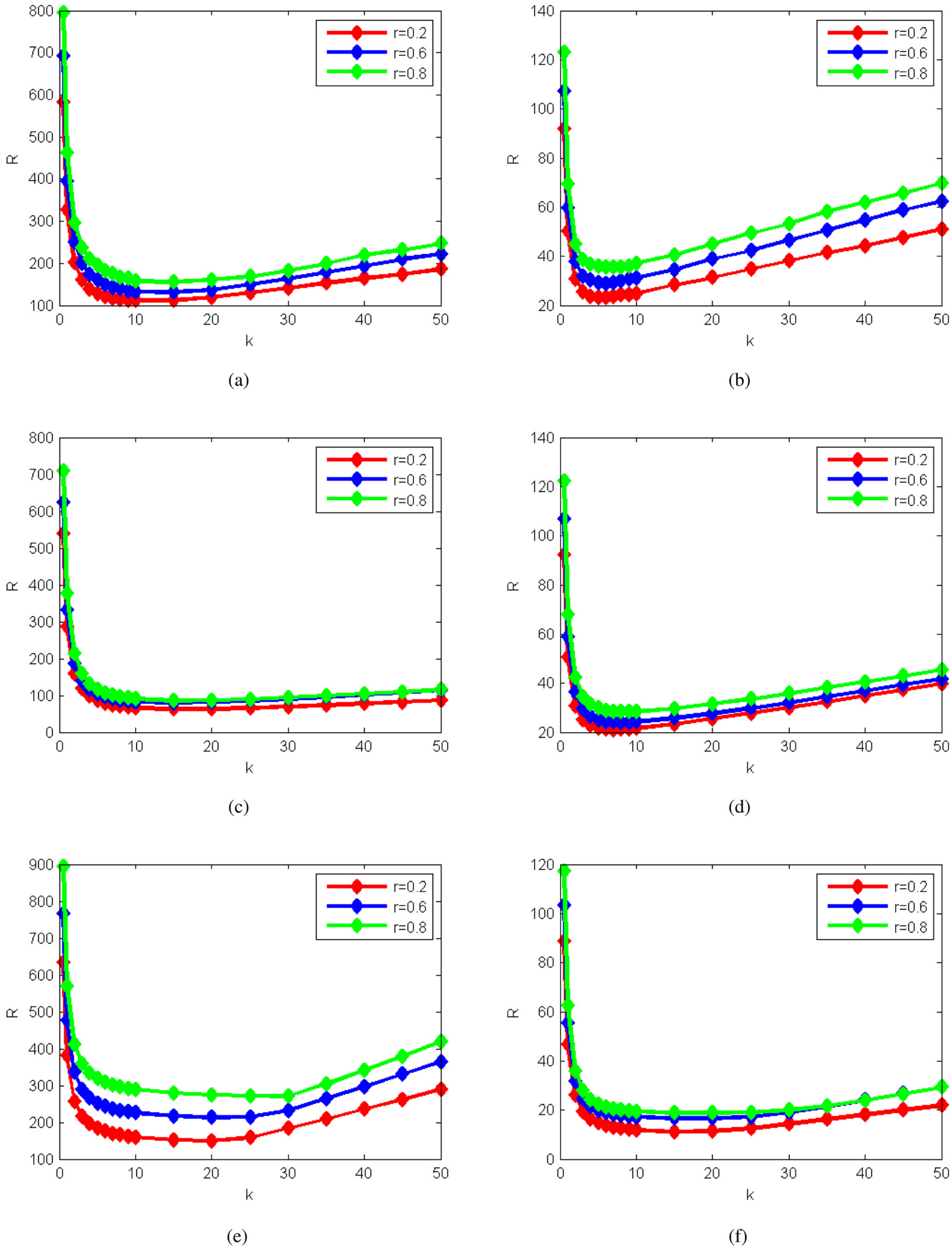

FIG. 2. R as a function of the control gain k for networks with different within-community connection strengths $r = 0.2$, 0.6, and 0.8. The pinning ratio is chosen as follows: $p = 0.1$ in (a), (c), (e) and $p = 0.6$ in (b), (d), (f), respectively. (a)–(b) Controllability analysis of networks with scheme 1, (c)–(d) with scheme 2, and (e)–(f) with scheme 3.

especially when the number of pinning nodes is small. From Figs. 1(a) and 1(b), when random pinning and degree descending pinning schemes are used, it can be seen that networks with a larger number of communities will produce a better control performance. From Fig. 1(c), it can be observed that R with $C=1$ is smaller than R with $C=2$ and 4 when the pinning ratio p less than or equal to 0.4. For the degree ascending pinning scheme, the pinned nodes with a small degree are distributed evenly in the network with $C=1$, but in networks with $C=2$ and $C=4$, the small degree nodes are pinned in ascending order and they are from nearly the same communities instead of being distributed evenly in the network. In addition, from Figs. 1(a)–1(c), with a growth of p, particularly when $p \geq 0.5$, the values of R achieved by $C=1$ and $C=2$ gradually move closely to those obtained on networks with $C=4$.

B. Analysis of controllability of networks with different within-community connection strengths

A comparison study of varying within-community connection strengths is shown in Figs. 2(a)–2(f). Here, we consider networks with the community number $C=2$. It is worth mentioning that the results are similar to networks with $C=4$. We find that R first decreases for a very small value of the control gain k and then increases as k increases. The value of R achieves a minimum around a specific value of the control gain k. This phenomenon indicates that an appropriate selection of the control gain k can achieve the best controllability. Consequently, one should avoid a too large or a too small value of k, which may reduce the controllability of a network. Note that for the pinning schemes considered (random pinning scheme, degree descending pinning scheme, and degree ascending pinning scheme), a network with a large within-community connection strength r has been found to weaken controllability. This can be explained as follows. As r increases, the network demonstrates a heterogeneous feature, i.e., the network has more newly added within-community connections. Thus, the pinned nodes can influence more easily the neighbors in the same community than those in different communities. The propagation of control information is hindered by the fewer between-community connections. This finding is consistent with the result that a homogeneous feature of networks benefits the controllability of networks.

C. Analysis of controllability with different pinning schemes

In this subsection, we further compare different pinning schemes on networks with $r=0.2$ and $r=0.8$, respectively. From Figs. 3(a) and 3(b), it can be seen that the degree descending pinning scheme performs best when the number of the pinned nodes is small. As p increases, R of the degree ascending pinning scheme is smaller than those of the random pinning and the degree descending pinning schemes. This finding indicates that the degree descending pinning scheme is most powerful in enhancing the controllability when the pinning proportion p is small. With increasing of p, it is better to convert to pin the nodes with a small degree. Hence, the types of pinning schemes, the pinning proportion, and the strength of community structure should be fully considered in studying controllability of networks with a certain community structure.

D. Analysis of controllability with different distributions of pinned nodes

For schemes 1 to 3, the pinned nodes are chosen according to the degree information. From Figs. 4(a)–4(c), we find that the schemes where the pinned nodes are evenly distributed in the communities $(p_1 = 0.5, p_2 = 0.5)$ are found to perform best, no matter what kinds of pinned schemes are considered. With an even distribution of pinned nodes, the networks become more homogeneous which thus leads to a better controllability. From Figs. 4(a) and 4(b), it can also be seen that the value of R obtained by scheme 1 is close to that by scheme 1 with $(p_1 = 0.5, p_2 = 0.5)$ with increasing pinning ratio p. The same is true with scheme 2 and scheme 2 with $(p_1 = 0.5, p_2 = 0.5)$. It is worth pointing out that scheme 3 combined with $(p_1 = 0.5, p_2 = 0.5)$ performs much better than other kinds of schemes, especially when the number of pinned nodes is small.

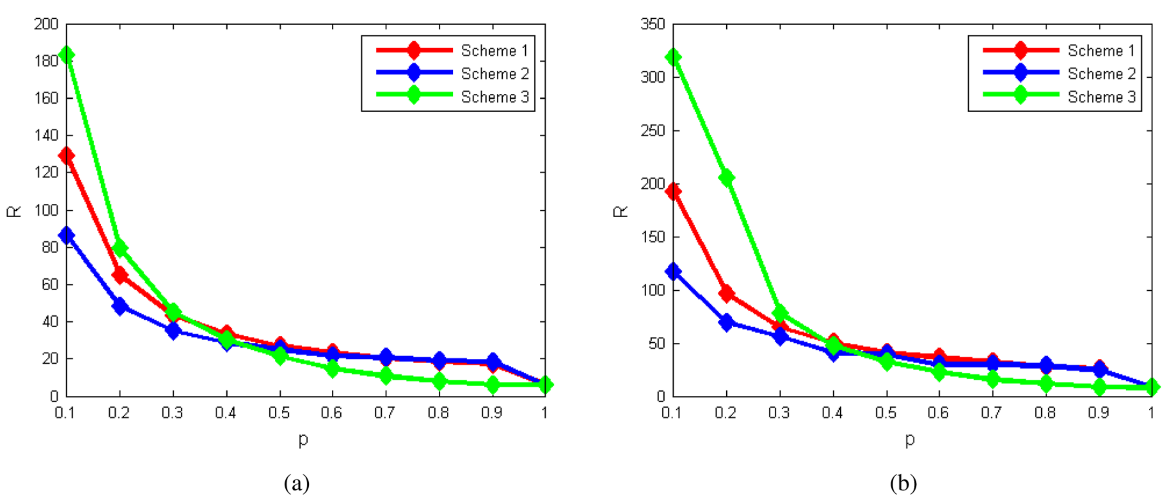

FIG. 3. R as a function of the pinning ratio p with the community number $C=2$ and the control gain $k=5$. (a) $r=0.2$ and (b) $r=0.8$.

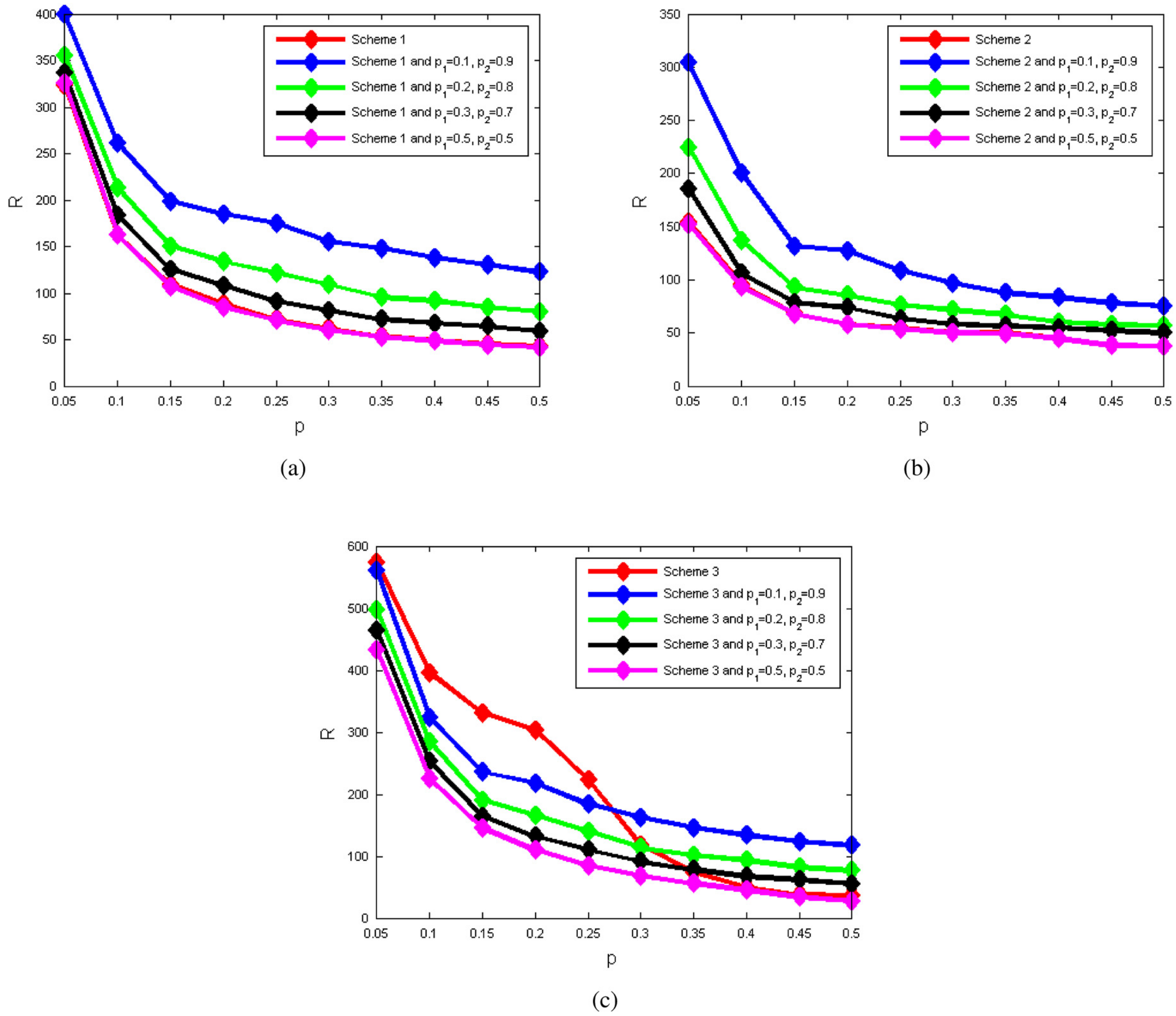

FIG. 4. R as a function of the pinning ratio p with the community number $C = 2$. The within-community connection strength is $r = 0.9$ and the control gain is $k = 5$. The parameters p_1 and p_2 are the proportions of pinned nodes located in the two communities, respectively. The following relationship $p_1 * p + p_2 * p = p$ holds. (a) Scheme 1. (b) Scheme 2. (c) Scheme 3.

V. CONCLUSIONS

Community structure is one of the most important topological features of complex networks. In this paper, by taking into account the community number and the strength of community structure, we investigate effects of community structure on the controllability of complex networks. The pinned nodes are chosen according to both their degree information and the distributions over the network.

We find that a network with a large number of communities has better controllability when applying random pinning and degree descending pinning schemes. By considering a network with a fixed number of communities and different strengths of community structure, we find that these networks with a weak community structure have a better controllability than the ones with a strong community structure. The degree descending pinning scheme performs best when the number of pinned nodes is small. However, the degree ascending pinning scheme outperforms the random pinning and the degree descending pinning scheme when increasing the pinning ratio. Moreover, the pinning schemes whose pinned nodes are evenly distributed among the communities outperform other kinds of distributions of pinned nodes.

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[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[2] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[3] A. Arenas, A. Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
[4] M. Guevara and T. Lewis, Chaos **5**, 174 (1995).
[5] G. Weisbuch, G. Deffuant, and F. Amblard, Physica A **353**, 555 (2005).
[6] X. Li, X. F. Wang, and G. Chen, IEEE Trans. Circuits Syst. **51**, 2074 (2004).

[7]F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen, Phys. Rev. E **75**, 046103 (2007).
[8]F. Sorrentino, Chaos **17**, 033101 (2007).
[9]C. Wu, Chaos **18**, 037103 (2008).
[10]Y. Y. Liu, J. J. Slotine, and A. L. Barabasi, Nature **473**, 167 (2011).
[11]L. Y. Xiang, Z. X. Liu, Z. Q. Chen, F. Chen, and Z. Z. Yuan, Physica A **379**, 298 (2007).
[12]Y. Tang, H. Gao, W. Zou, and J. Kurths, PLoS ONE **7**, e41375 (2012).
[13]Q. Miao, Z. Rong, Y. Tang, and J. Fang, Physica A **387**, 6225 (2008).
[14]Y. Tang, Z. Wang, W. K. Wong, J. Kurths, and J. Fang, Chaos **21**, 025114 (2011).
[15]W. Yu, G. Chen, and J. Lv, Automatica **45**, 429 (2009).
[16]Y. Tang and W. K. Wong, IEEE Trans. Neural Networks Learn. Syst. **24**, 435 (2013).
[17]Y. Tang, Z. Wang, H. Gao, S. Swift, and J. Kurths, IEEE/ACM Trans. Computat. Biol. Bioinf. **9**, 1569 (2012).
[18]Y. Tang, H. Gao, J. Kurths, and J. Fang, IEEE Trans. Ind. Inform. **8**, 828 (2012).
[19]G. W. Flake, S. R. Lawrence, C. L. Giles, and F. M. Coetzee, IEEE Comput. **35**, 66 (2002).
[20]M. Girvan and M. E. J. Newman, Proc. Natl. Acad. Sci. U.S.A. **99**, 7821 (2002).
[21]G. Yan, Z. Q. Fu, J. Ren, and W. X. Wang, Phys. Rev. E **75**, 016108 (2007).
[22]S. Fortunato, Phys. Rep. **486**, 75 (2010).
[23]L. Danon, A. Arenas, and A. D. Guilera, Phys. Rev. E **77**, 036103 (2008).
[24]T. Zhou, M. Zhao, G. Chen, G. Yan, and B. H. Wang, Phys. Lett. A **368**, 431 (2007).
[25]Y. Wu, P. Li, M. Chen, J. Xiao, and J. Kurths, Physica A **388**, 2987 (2009).
[26]M. E. J. Newman and M. Girvan, Phys. Rev. E **69**, 026113 (2004).
[27]M. E. J. Newman, Phys. Rev. E **69**, 066133 (2004).
[28]R. Guimer, M. Sales-Pardo, and L. A. Nunes Amaral, Phys. Rev. E **70**, 025101(R) (2004).
[29]N. Kashtan and U. Alon, Proc. Natl. Acad. Sci. U.S.A. **102**, 13773 (2005).
[30]F. Chung, *Spectral Graph Theory* (American Mathematical Society, 1997).
[31]G. Hu, J. Yang, and W. Liu, Phys. Rev. E **58**, 4440 (1998).