On the Theoretical Lower Bound of the Multiplicative Complexity for DCT

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Abstract: *In this paper, we present a discuccion on DCT algorithms based on convolutions. It is shown that these approaches can provide effective solutions to realize DCTs. Besides, they provide a theoretical lower bound on the number of multiplications required for the realization of DCTs. Some fast short length DCT algorithms have also been derived from these approaches.*

Introduction

It is well known that by making use of the Largrange's interpolation formula one can achieve the theoretical lower bound on the number of multiplications required for the realization of a convolution[1]. Based on this result, Duhamel shows that the theoretical minimum number of multiplications required to realize a 2^m-point DCT is $2^{m+1}-m-2$ by converting the DCT into a number of skew cyclic convolutions with different sizes[2]. In this paper, we will show that the same lower bound can be derived by two entirely different approaches. In particular, one of them is based on the algorithm proposed in [3] while another one is based on a modification of the results of the algorithm proposed in [4].

This paper is organised in the following way: First, a modification of the algorithm proposed in [4] is given such that no more multiplicative overhead is required to convert a p^m-point DCT into convolutions, where p is an odd prime. Note that convolutions differ from correlations only by virtue of a simple inversion of one of the input sequences. Hence, though some of the development in this paper refer to correlations, they apply equally well to convolutions. Second, theoretical minimum multiplicative requirements for the realization of the DCT are derived based on the modification of the results and the algorithm proposed in [3] respectively. Then we will provide some short length DCT algorithms based on the derived results and some other effective DCT algorithms[5], which support the PFM algorithm[6] with reduced arithmetic complexity.

Solution for p^m-point DCT

An interesting algorithm is proposed in [4] to realize a p^m-point DCT via convolutions. However, multiplicative overhead is still required during the conversion. In this and the following two sections, we shall show that the DCT can also be converted into cyclic convolutions by an entirely different method without any multiplicative overhead.

The DCT[7] of a sequence $\{y(n)\}$ is defined as

$$Y(k) = \sum_{i=0}^{N-1} y(i) \cos\left(\frac{\pi(2i+1)k}{2N}\right) \qquad \text{for } k = 0,1...N-1 \quad (1)$$

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Basically, if N is an odd number, there exists a bijective mapping on the set $\{i:i=0,1...N-1\}$[3]:

$$\xi(i) = \frac{\langle N-2i\rangle_{2N}-1}{2} \qquad \text{for } i=0,1...N-1 \quad (2)$$

By making use of this bijective mapping, we can split (1) and rewrite it as

$$\begin{cases} Y(0) = \sum_{i=0}^{N-1} f(i) \\ Y(2k) = (-1)^k \left\{ A(k) + f(0) \right\} \\ Y(N-2k) = (-1)^{k+1} \; B(k) \end{cases} \qquad \text{for } k=1,2..(N-1)/2 \quad (3)$$

$$\text{where } A(k) = \sum_{i=1}^{N-1} f(i) \cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k=1,2..(N-1)/2 \quad (4a)$$

$$B(k) = \sum_{i=1}^{N-1} h(i) \sin\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k=1,2..(N-1)/2 \quad (4b)$$

$$\text{and } \begin{cases} f(i) = y(\xi(i)) \\ h(i) = (-1)^{\xi(i)} y(\xi(i)) \end{cases} \qquad \text{for } i=0,1...N-1 \quad (5)$$

Note that both $\{A(k)\}$ and $\{B(k)\}$ are then in a standard form. This standard form can be easily converted into a convolution or convolutions if N is an appropriate value of p^m, where p is an odd prime. In the following sections, we will make a detailed discussion on cases with various values of N.

The case of m=2

If N = p^2 and p is an odd prime, (4a) and (4b) can be rewritten as

$$A(k) = \begin{cases} J_1(k) + J_2(k) & \text{for } k\in\Omega;\ k<N/2 \\ J_3(k) + J_4(k) & \text{for } k\in\Psi;\ k<N/2 \end{cases} \quad (6a)$$

$$B(k) = \begin{cases} J_5(k) + J_6(k) & \text{for } k\in\Omega;\ k<N/2 \\ J_7(k) + J_8(k) & \text{for } k\in\Psi;\ k<N/2 \end{cases} \quad (6b)$$

where $\Omega = \{pn \mid n=1,2..p-1\}$

$\Psi = \{ n \mid n=1,2...N-1 \text{ and } n\notin \Omega \}$

$$J_1(k) = \sum_{i\in\Psi} f(i) \cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Omega, k<N/2 \quad (7a)$$

$$J_2(k) = \sum_{i\in\Omega} f(i) \cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Omega, k<N/2 \quad (7b)$$

$$J_3(k) = \sum_{i\in\Psi} f(i) \cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Psi, k<N/2 \quad (7c)$$

$$J_4(k) = \sum_{i\in\Omega} f(i) \cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Psi, k<N/2 \quad (7d)$$

$$J_5(k) = \sum_{i\in\Psi} h(i)\sin\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Omega, k<N/2 \quad (7e)$$

$$J_6(k) = \sum_{i\in\Omega} h(i)\sin\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Omega, k<N/2 \quad (7f)$$

$$J_7(k) = \sum_{i\in\Psi} h(i)\sin\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Psi, k<N/2 \quad (7g)$$

$$J_8(k) = \sum_{i\in\Omega} h(i)\sin\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Psi, k<N/2 \quad (7h)$$

For reference propose, we also define $\Psi_n = \{np+i \mid i=1,2..p\text{-}1\}$, where $n=0,1,..p\text{-}1$. Then, it can be proved that $J_1(k)$, $J_2(k)$, $J_3(k)$ and $J_4(k)$ can all be realized through simple additions and cyclic correlations. In particular, we have the following solution:

For $J_2(k)$:

$$J_2(pk) = \sum_{i\in\Psi_o} f(pi) \qquad \text{for } k\in\Psi_o,\ k<p/2 \quad (8)$$

For $J_3(k)$:

By expanding the domain of $J_3(k)$ to Ψ, we have

$$J_3(k) = \sum_{i\in\Psi} f(i)\cos\left(\frac{2\pi ik}{N}\right) \qquad \text{for } k\in\Psi \quad (9)$$

If p is an odd prime, Ψ is cyclic under multiplication modulo N. Let g be a primitive element such that $\{\langle g^n\rangle_N \mid n=1,2...p(p\text{-}1)\} = \Psi$, then we have

$$J_3'(p(p-1)/2+k) = J_3'(k) = \sum_{i=1}^{p(p-1)/2} \{ f'(i)+f'(p(p-1)/2+i) \}\, C(i+k) \qquad \text{for } k=1,2...p(p-1)/2 \quad (10)$$

where $C(n) = \cos[(2\pi/N)\langle g^n\rangle_N]$ for n is an integer

$f'(i) = f(\langle g^i\rangle_N)$

$J_3'(k) = J_3(\langle g^k\rangle_N)$

For $J_1(k)$:

After expanding the domain of $J_1(k)$ to $k\in\Omega$, (7a) can be rewritten as

$$J_1(pk) = \sum_{i\in\Psi_o} \left\{ \sum_{n=0}^{p-1} f(np+i) \right\} \cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k\in\Psi_o \quad (11)$$

In other words, we have

$$J_1'((p-1)/2+k) = J_1'(k) = \sum_{i=1}^{(p-1)/2} \{\chi'(i)+\chi'((p-1)/2+i)\}\, C_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (12)$$

where $C_o(n) = \cos[(2\pi/p)\langle g_o^n\rangle_p]$ for n is an integer

$J_1'(k) = J_1(p\langle g_o^k\rangle_p)$

$\chi'(i) = \chi(\langle g_o^i\rangle_p) = \sum_{n=0}^{p-1} f(np+\langle g_o^i\rangle_p)$ and

$\{\langle g_o^i\rangle_p \mid i\in\Psi_o\} = \Psi_o$.

For $J_4(k)$:

After expanding the domain of $J_4(k)$ to $k\in\Psi$, (7d) can be rewritten as

$$J_4(np+k) = J_4(k) = \sum_{i\in\Psi_o} f(pi)\cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k\in\Psi_o \quad (13)$$

By permuting the input sequence, we have

$$J_4'((p-1)/2+k) = J_4'(k) = \sum_{i=1}^{(p-1)/2} \{\alpha'(i)+\alpha'((p-1)/2+i)\}\, C_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (14)$$

where $J_4'(k) = J_4(\langle g_o^k\rangle_p) = J_4(np+\langle g_o^k\rangle_p)$

$\alpha'(i) = \alpha(\langle g_o^i\rangle_p) = f(p\langle g_o^i\rangle_p)$ and

$\{\langle g_o^i\rangle_p \mid i\in\Psi_o\} = \Psi_o$.

In such case, one can realize $\{A(k)\}$ though two $(p-1)/2$-point and one $p(p-1)/2$-point cyclic correlations. Similarly, by applying the same technique to $J_5(k)$, $J_6(k)$, $J_7(k)$ and $J_8(k)$, we can realize $\{B(k)\}$ through two $(p-1)/2$-point and one $p(p-1)/2$-point skew cyclic correlations as follows:

For $J_6(k)$:

$$J_6(pk) = 0 \qquad \text{for } k\in\Psi_o,\ k<p/2 \quad (15)$$

For $J_7(k)$:

$$J_7'(k) = -J_7'(p(p-1)/2+k) = \sum_{i=1}^{p(p-1)/2} \{h'(i)-h'(p(p-1)/2+i)\}\, S(i+k) \qquad \text{for } k=1,2...p(p-1)/2 \quad (16)$$

where $S(n) = -S(p(p-1)/2+n) = \sin[(2\pi/N)\langle g^n\rangle_N]$ for n is integer

$h'(i) = h(\langle g^i\rangle_N)$

$J_7'(k) = J_7(\langle g^k\rangle_N)$

For $J_5(k)$:

$$J_5'(k) = -J_5'((p-1)/2+k) = \sum_{i=1}^{(p-1)/2} \{\eta'(i)-\eta'((p-1)/2+i)\}\, S_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (17)$$

where $S_o(n) = -S_o((p-1)/2+n) = \sin[(2\pi/p)\langle g_o^n\rangle_p]$ for n is an integer

$J_5'(k) = J_5(p\langle g_o^k\rangle_p)$

$\eta'(i) = \sum_{n=0}^{p-1} h(np+\langle g_o^i\rangle_p)$

For $J_8(k)$:

$$J_8'(k) = -J_8'((p-1)/2+k) = \sum_{i=1}^{(p-1)/2} \{\beta'(i)-\beta'((p-1)/2+i)\}\, S_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (18)$$

where $J_8'(k) = J_8(\langle g_o^k\rangle_p) = J_8(np+\langle g_o^k\rangle_p)$

$\beta'(i) = h(p\langle g_o^i\rangle_p)$

The case of m>2

In the previous section, we have shown how to realize a p^2-point DCT through correlations. Actually, this technique can be generalized and easily applied to realize a p^n-point DCT, where $n>2$.

We first extend the domain of (4a) and (4b) from $\{1,2..(N-1)/2\}$ to $\Theta^n=\{1,2..N\text{-}1\}$. Then, we select all integers that contain a factor p from the set Θ^n to form the set Ω^n. Those elements left form another set Ψ^n. In such case, (4a) can be realized through the realization of the following equations.

$$J_{n,1}(k) = \sum_{i\in\Psi^n} f(i) \cos\left(\frac{2\pi ik}{N}\right) \quad \text{for } k\in\Omega^n \quad (19a)$$

$$J_{n,2}(k) = \sum_{i\in\Omega^n} f(i) \cos\left(\frac{2\pi ik}{N}\right) \quad \text{for } k\in\Omega^n \quad (19b)$$

$$J_{n,3}(k) = \sum_{i\in\Psi^n} f(i) \cos\left(\frac{2\pi ik}{N}\right) \quad \text{for } k\in\Psi^n \quad (19c)$$

$$J_{n,4}(k) = \sum_{i\in\Omega^n} f(i) \cos\left(\frac{2\pi ik}{N}\right) \quad \text{for } k\in\Psi^n \quad (19d)$$

where $\Theta^n = \left\{1,2...p^n-1\right\}$, $\Omega^n = \left\{ pm \mid m\in\Theta^{n-1}\right\}$ and

$\Psi^n = \left\{ n \mid n\in\Theta^n \text{ and } n\notin\Omega^n \right\}$

For the sake of simplicity, we call these formulations as $J_{n,1}(k)$, $J_{n,2}(k)$, $J_{n,3}(k)$ and $J_{n,4}(k)$ with order n respectively for future reference. Note that the input sequence is not necessary to be $\{f(i)\}$ when these structures are referred.

As Ψ^n forms a cyclic group with $p^{n-1}(p-1)$ elements under multiplication modulo p^n, we can realize $J_{n,3}(k)$ by correlations using exactly the same technique used in the p^2 case.

On the other hand, note that $J_{n,1}(k)$ and $J_{n,4}(k)$ can be expressed in the form of $J_{n-1,1}(k)$, $J_{n-1,3}(k)$ and $J_{n-1,4}(k)$ accordingly. This fact can be easily observed by the following deduction:

For $J_{n,1}(k)$,

$$J_{n,1}(pk) = \sum_{i\in\Psi^{n-1}} \left\{ \sum_{d=0}^{p-1} f(dp^{n-1}+i) \right\} \cos\left(\frac{2\pi ik}{p^{n-1}}\right) \quad \text{for } k\in\Theta^{n-1} \quad (20)$$

For $J_{n,4}(k)$,

$$J_{n,4}(k+dp^{n-1}) = J_{n,4}(k) = \sum_{i\in\Theta^{n-1}} f(pi) \cos\left(\frac{2\pi ik}{p^{n-1}}\right) \quad \text{for } k\in\Psi^{n-1} \quad (21)$$

These equations involve two $J_{n-1,3}(k)$, a $J_{n-1,1}(k)$ and a $J_{n-1,4}(k)$ structures. As $J_{n-1,3}(k)$ can be realized through a correlation, this decomposition technique suggests an approach for us to realize $J_{n,1}(k)$ and $J_{n,4}(k)$. One can decompose $J_{n,1}(k)$ and $J_{n,4}(k)$ to $J_{2,1}(k)$ and $J_{2,4}(k)$ by using this technique recursively and then realize them with the same technique mentioned in p^2 case.

For $J_{n,2}(k)$,

$$J_{n,2}(kp) = \sum_{i\in\Theta^{n-2}} \left\{ \sum_{d=0}^{p-1} f(dp^{n-1} + pi) \right\} \cos\left(\frac{2\pi ik}{p^{n-2}}\right) + \sum_{d=1}^{p-1} f(dp^{n-1}) \quad \text{for } k\in\Theta^{n-1} \quad (22)$$

However, as we have

$$\begin{cases} J_{n,2}(dp^{n-1} + kp) = J_{n,2}(kp) & \text{for } k\in\Theta^{n-2};\ d=0,1..p-1 \\ J_{n,2}(dp^{n-1}) = \sum_{i\in\Theta^{n-1}} f(pi) & \text{for } d=1,2..p-1 \end{cases} \quad (23)$$

it is only necessary for us to compute

$$J_{n,2}(kp) = \sum_{i\in\Theta^{n-2}} \left\{ \sum_{d=0}^{p-1} f(dp^{n-1} + pi) \right\} \cos\left(\frac{2\pi ik}{p^{n-2}}\right) + \sum_{d=1}^{p-1} f(dp^{n-1}) \quad \text{for } k\in\Theta^{n-2} \quad (24)$$

and $$J_{n,2}(p^{n-1}) = \sum_{i\in\Theta^{n-1}} f(pi) \quad (25)$$

Note that (24) can be decomposed into $J_{n-2,1}(k)$, $J_{n-2,2}(k)$, $J_{n-2,3}(k)$ and $J_{n-2,4}(k)$. Therefore, by making use of the above techniques recursively, the problem can be finally resolved.

Similarly, (4b) can be decomposed into formulations with lower order and then realized through cyclic correlations by applying the same technique used in realizing (4a). In such case, a p^m-point DCT can be realized through correlations.

Short length DCT algorithm

Recall that the minimal number of real multiplications required to compute a polynomial product with real coefficients modulo another polynomial Q(z) is 2N-d, where d is the number of factorial polynomials of Q(z) and N is the order of the resultant polynomial[8]. In particular, the cyclic convolution of two sequences $\{g(i):i=0,1..N-1\}$ and $\{c(i):i=0,1..N-1\}$ can be determined from the coefficients of the polynomial $F(z) = C(z)G(z)\bmod(z^N-1)$, where $G(z) = g(N-1)z^{N-1} + ... + g(1)z + g(0)$ and $C(z) = c(N-1)z^{N-1} + ... + c(1)z + c(0)$, while the skew-cyclic convolution of these two sequences can be determined from the coefficients of the polynomial $F'(z) = C(z)G(z)\bmod(z^N+1)$. Hence, the cyclic convolution and the skew-cyclic convolution results can be theoretically determined at a cost of 2N-2 and 2N-1 multiplications respectively.

In [3], we proved that any DCT of odd prime size P can be decomposed into one $(P-1)/2$-point cyclic convolution and one $(P-1)/2$-point skew-cyclic convolution at no cost of multiplication. As the minimum number of factorial polynomials of the polynomial $z^{(P-1)/2}-1$ is 2, that implies 2P-5 real multiplications are required for the realization of the P-point DCT. For some values of P, this figure can be further reduced. For instance, when P=13, there are three factorial polynomials of the polynomial z^6-1 (z^3+1, z^2+z+1 and $z-1$) and therefore only $2\times13-6=20$ multiplications are necessary. Similarly, by using the derivation results obtained in earlier sections, one can realize a p^2-point DCT at a cost of $2p^2+2p-13$ multiplications. Fig. 1 shows the derivated results of the theoretical minimum number of the multiplications necessary for the DCT realization. Note that the results based on the two new algorithms are exactly the same as that derived by Duhamel[2].

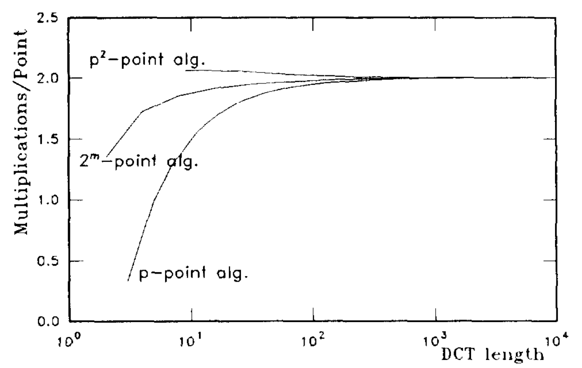

Figure 1. Theoretical lower bound of the number of multiplications required for the realization of the DCT by using different algorithms.

Table 1. Arithmetic complexity of short length DCT algorithms.

	New Algorithms		[10]		[11]	
N	M	A	M	A	M	A
2	1	2	-	-	-	-
3	1	4	3	4	7	24
4	4	9	5	9	6	8
5	4	14	6	15	15	54
6	4	16	-	-	-	-
7	9	29	9	34	23	100
8	12	29	14	32	16	26
9	10	34	13	46	29	126
10	13	42	-	-	-	-

Obviously, these figures are theoretical derivation results only. When the size of the DCT gets larger, it is hardly possible to apply the Lagrange's interpolation formula to compute the convolution result effectively. However, when the value of N is small, this approach is practically possible and therefore these theoretical limits become real figures. On the other hand, as a number of short convolution algorithms have been developed[8,9], one can make use of them to implement short length DCT algorithms. Some examples are presented in the appendices.

Table 1 shows the computational effort required for the realization of short length DCTs by using different algorithms. The figures provided in column 2 and column 3 are the optimal figures derived from the mixed-radix algorithm[5] and the algorithms presented in this paper. Columns 4 to 7 are copied from [10] for reference purpose. These figures illustrate the superiority of the proposed short length algorithms.

Conclusions

In this paper, we improve the algorithm proposed in [4] such that no multiplicative overhead is required to convert a p^m-point DCT into convolutions. Based on this modified algorithm, we derive the theoretical lower bound of the number of multiplications necessary for the realization of the DCT, which matchs the result derived by Duhamel[2]. In particular, there would be about two multiplications per point whatever the DCT size is. Though this theoretical derivation result can hardly be achieved practically when the DCT size is large, this figure is true for short length DCT. Consequently, these algorithms can be directly used as short length DCT algorithms and to support the PFM algorithm[6] to save arithmetic complexity.

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Appendix

A collection of short length DCT algorithms:

A1. 5-point discrete cosine trar.sform; 4 multiplications, 14 additions

$g_0 = x_0+x_4$ $\qquad h_0 = x_0-x_4$

$g_1 = x_1+x_3$ $\qquad h_1 = x_1-x_3$

$X_0 = g_0+g_1+x_2$

$m_0 = (g_0+g_1)/4$ $\qquad n_0 = (h_0+h_1)\times c_1$

$m_1 = (g_0-g_1)\times(c_2+c_4)/2$ $\qquad n_1 = h_1\times(c_1-c_3)$

$m_2 = m_0-x_2$ $\qquad n_2 = h_0\times(c_3+c_1)$

$X_2 = m_1+m_2$ $\qquad X_1 = n_0-n_1$

$X_4 = m_1-m_2$ $\qquad X_3 = n_2-n_0$

where $c_n = \cos\left(n\pi/10\right)$

A2. 7-point discrete cosine transform; 9 multiplications, 29 additions

$g_0 = x_0+x_6$ $\qquad h_0 = x_0-x_6$

$g_1 = x_5+x_1$ $\qquad h_1 = x_5-x_1$

$g_2 = x_4+x_2$ $\qquad h_2 = x_4-x_2$

$a_0 = g_0+g_1+g_2$

$a_1 = g_0-g_2$

$a_2 = g_1-g_2$

$a_3 = a_1+a_2$

$X_0 = a_0+x_3$

$m_0 = -a_0/6$ $\qquad n_0 = (h_0+h_2)\times(s_2+s_1)$

$m_1 = a_1\times(c_2-c_1)$ $\qquad n_1 = (h_0+h_1)\times(s_2+s_3)$

$m_2 = a_2\times(c_3-c_1)$ $\qquad n_2 = (h_1+h_2)\times(s_3+s_1)$

$m_3 = a_3\times(c_2-2c_1+c_3)/3$ $\qquad n_3 = (h_0-h_2)\times(s_2-s_1)$

$n_4 = h_1\times s_3$

$u_0 = m_1-m_3$ $\qquad n_5 = (n_0+n_3)/2$

$u_1 = m_2-m_3$ $\qquad n_6 = n_4+n_5$

$u_2 = m_0+x_3$ $\qquad n_7 = n_5-n_4$

$X_2 = u_0+u_1-u_2$ $\qquad X_1 = n_6-n_2$

$X_4 = u_1+u_2$ $\qquad X_3 = n_0-n_7$

$X_6 = -u_0-u_2$ $\qquad X_5 = n_1-n_6$

where $c_n = \cos\left(2n\pi/7\right)$ and $s_n = \sin\left(2n\pi/7\right)$