Stable and Robust Fuzzy Control for Uncertain Nonlinear Systems

H. K. Lam, F. H. F. Leung, and P. K. S. Tam

Abstract—This paper presents the stability and robustness analysis for multivariable fuzzy control systems subject to parameter uncertainties based on a single-grid-point (SGP) approach. To perform the analysis, we represent a multivariable nonlinear system using a TS-fuzzy plant model. Three design approaches of fuzzy controllers are introduced to close the feedback loop. By estimating the matrix measures of the system parameters and parameter uncertainties, stability and robustness conditions for different cases are derived. Application examples will be given to show the design procedures and the merits of the proposed fuzzy controller.

Index Terms—fuzzy control, nonlinear system robustness, parameter uncertainty, stability.

I. INTRODUCTION

Control of nonlinear systems is a difficult problem because we do not have systematic ways to find a necessary and sufficient stability condition, and to guarantee good robustness and performance. The problem becomes more complex when some parameters of the plant are uncertain. Fuzzy control is one of the techniques to deal with this class of systems. Many successful applications of fuzzy control in industrial processes [1]–[3] and domestic products have been reported.

We can classify the studies on fuzzy control systems into three approaches, namely, model-free, mathematical model, and fuzzy model approaches. The model-free approach, as its name tells, does not require a model for the plant. A fuzzy controller for the complex plant is obtained by incorporating human experience or expert knowledge into a fuzzy controller through some linguistic rules [1]–[3]. This process makes the design simple and the linguistic rules make the control process to be understood easily. However, the heuristic design comes without considering the system stability, robustness and performance. To realize a systematic design process, neural and neural-fuzzy networks were employed to construct the fuzzy controllers. By using some training algorithms [16], [17], [20], [21] [56], parameters of the fuzzy controller can be obtained automatically. Although this design process is systematic, the global closed-loop system stability may not be guaranteed and the training process is quite time consuming. Stability conditions of the neural-network and neural-fuzzy control systems can be found in [29], [30], and [57]. A fuzzy PID controller was also proposed to control a plant based on the model-free approach. This fuzzy PID controller takes the output error, derivative of the output error and the integral of the output error as the inputs. Prior knowledge of the plant is not required. Under a particular design of the membership functions, a fuzzy PID controller is proved to be equivalent to a conventional PID controller [18], [19], [26], or a nonlinear PID controller [7]. Some methodologies on tuning adaptive fuzzy PI, PD, and PID controllers can be found in [9], [27], [31], [77], and [78]. Based on the model-free approach, most of the studies were

on obtaining the fuzzy controllers, while the stability and robustness of the closed-loop system are seldom considered.

For the mathematical model approach, the closed-loop system is analyzed based on a mathematical model of the plant. The closed-loop system stability of Fuzzy PI, PD, and PID control systems were analyzed by using the small gain theorem [11], [28], [59]. Adaptive techniques were applied in the design of fuzzy controllers [10], [12], [32], [40], [63], [79], [80], [83]. The shapes of the membership functions of the fuzzy controller were adjusted according to the adaptive rule derived based on the mathematical model of the plant. Model-reference adaptive [22], [60], [81] and self-organizing adaptive [33] fuzzy control approach were also reported. By applying adaptive fuzzy controllers, the stability of the closed-loop system is usually guaranteed, and the system is robust to the parameter uncertainties. However, the computational demand and the complexity of the adaptive fuzzy controller are always high. The robust sliding mode control technique was also applied in designing the fuzzy controllers. Based on sliding mode control theory [5], the stability can be guaranteed and the system is robust to parameter uncertainties within given bounds. The fuzzy sliding mode controller behaves like a conventional sliding mode controller with a boundary layer about the sliding plane [13], [23], [41], [61], when the discontinuous control signal is replaced by a fuzzy gain. The chattering problem of the control signal is reduced under this case. However, this fuzzy sliding mode controller suffers from a finite steady state error due to the boundary layer. To alleviate the chattering problem and eliminate the finite steady state error, some fuzzy tuning algorithms [35], [42], and adaptive techniques [14], [34] were introduced to the fuzzy sliding mode controller. Other stability analyses can be found in [43], [62], and [91]. It can be seen that the studies of the fuzzy control system based on the mathematical model approach were mainly on the system stability and other system characteristics through combining conventional control theories with fuzzy logic.

Under the fuzzy-model based approach, system analysis is carried out via a fuzzy plant model. One well-known fuzzy plant model is the TS-fuzzy plant model [4]. There are two ways to obtain the TS-fuzzy plant model of a nonlinear system. First, we can convert the mathematical model of the nonlinear system into a TS-fuzzy plant model directly using, for example, the method in [36]. Second, we can obtain TS-fuzzy plant model using some system identification or modeling techniques [4], [6], [24], [44], [45], [82]. The TS-fuzzy plant model expresses the nonlinear system as a weighted sum of some simple sub-systems. This special structure of the TS-fuzzy plant model facilitates the analysis of the systems. Because the TS-fuzzy plant model gives a standard form for general nonlinear systems, the analysis can be carried out systematically. Moreover, as the sub-systems of the TS-fuzzy plant model are usually linear systems, some linear design techniques can be applied for the fuzzy controller, which is a weighted sum of many sub-controllers. PID controller is used as the sub-controller in [58], [64]–[66]. It can be shown that this kind of fuzzy controller is a nonlinear PID controller. Sufficient stability conditions were derived by using the Lyapunov's method [58]. Fuzzy controllers using linear state feedback controllers as sub-controllers were also reported. A sufficient stability condition was derived based on the Lyapunov direct method [37], [67], [86]. Stability analysis of discrete-time fuzzy state feedback controller can be found in [8], [25], [38], [46]–[48], [72], [73], and [85]. Linear matrix inequality (LMI) techniques were also employed to analyze the system numerically [74]–[76], [87]. Adaptive [39], [53], [68], [84], [90] and sliding mode [69] techniques were applied to design this kind of fuzzy controller. As mentioned early, these two techniques will inevitably increase the computational demand and complexity of the controller. Robustness analysis results for fuzzy control systems based on the fuzzy model based

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approach were relatively hard to find in the literature. We derived the robustness conditions based on the Lyapunov stability theory reported in [49]–[52], and by estimating the matrix measures of the system parameters and the parameter uncertainties reported in [54], [55], [71], and [89]. control theory [36], and sliding mode control theory [70] were employed to analyze the robustness of the fuzzy control system.

In this paper, we concentrate on the fuzzy model based approach, and report a general analysis for fuzzy model based control systems subject to parameter uncertainties. A stable and robust single-grid-point (SGP) fuzzy controller is proposed. The idea of this SGP approach is shown in Fig. 1 which shows a system with two uncertain parameters a and b. The dot at the center of the circle denotes the nominal plant parameters (the SGP) of the fuzzy control system of which the plant is represented by a TS-fuzzy plant model. According to the nominal parameters, a stable and robust SGP fuzzy controller will be designed based on the TS-fuzzy plant model to close the feedback loop. The system stability is guaranteed by this SGP fuzzy controller if the uncertain parameters are inside a robust area denoted by the circle. We shall derive the stability conditions of the fuzzy control system, and the robustness conditions that define the robust area. Through a general and systematic stability and robustness analysis, a procedure for finding SGP fuzzy controllers can be obtained. Compare with other works on robust control such as the fuzzy sliding mode control technique [13], [32], adaptive H_∞, fuzzy control technique [32], self-tuning fuzzy control technique [77], our approach is simpler and easier to understand, and the structure of the fuzzy controller is not so complicated.

This paper is organized as follows. In Section II, the TS-fuzzy plant model and the fuzzy controller will be introduced. In Section III, three design approaches, namely, general design approach, parallel design approach and simplified design approach will be proposed to close the feedback loop. Stability and robustness analyses will then be carried out for the fuzzy control system, and the results for different approaches will be presented. The stability and robustness conditions will be derived by estimating the matrix measures of the system matrices of the linear sub-systems in the consequent parts of the fuzzy rules, and the norms of the parameter uncertainties. In Section IV, the finding of the stability and robustness conditions under different design approaches will be formulated into a nonlinear matrix inequality (NMI) (general case) and LMI (special case) [15] problems. Section V summarizes the procedures for fnding the SGP fuzzy controller. In Section VI, application examples will be given to illustrate the stabilizability and robustness property of the proposed SGP fuzzy controller. In Section VII, a conclusion will be drawn.

II. TS-FUZZY PLANT MODEL AND FUZZY CONTROLLER

Consider an uncertain multivariable nonlinear fuzzy control system comprising a TS-fuzzy plant model with parameter uncertainties, and a fuzzy controller closing the feedback loop.
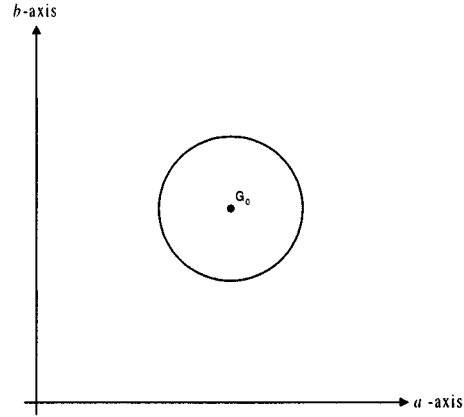

Fig. 1. Idea of the SGP approach.

A. TS-Fuzzy Plant Model with Parameter Uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The ith rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND $\cdots$ AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i
THEN $\dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{u}(t)$

$$(1)$$

where M_α^i is a fuzzy term of rule i corresponding to the function, $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \cdots, \Psi$, $i = 1, 2, \cdots, p$, Ψ is a positive integer; $\Delta\mathbf{A}_i \in \Re^{n \times n}$ and $\Delta\mathbf{B}_i \in \Re^{n \times m}$ are the uncertainties of the constant system matrices $\mathbf{A}_i \in \Re^{n \times n}$ and $\mathbf{B}_i \in \Re^{n \times m}$, respectively; $\mathbf{x}(t) \in \Re^{n \times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m \times 1}$ is the input vector. The plant dynamics is then described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \left[(\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{u}(t) \right] \quad (2)$$

where (see (3) and (4) at the bottom of the page) is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ is the membership function corresponding to M_α^i.

B. Fuzzy Controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The jth rule of the fuzzy controller is of the following format:

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j AND $\cdots$ AND $g_\Omega(\mathbf{x}(t))$ is N_Ω^j
THEN $\mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) + \mathbf{r}$ (5)

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \cdots, \Omega$, $j = 1, 2, \cdots, c$, Ω is a positive integer;

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0, 1] \quad \text{for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p} \left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t))) \right)} \tag{4}$$

$\mathbf{G}_j \in \Re^{m \times n}$ is the feedback gain of rule j, $\mathbf{r} \in \Re^{n \times 1}$ is the reference input vector (set-point). The inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t)) (\mathbf{G}_j \mathbf{x}(t) + \mathbf{r}) \qquad (6)$$

where (see (7) and (8) at the bottom of the page) is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{N_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the membership function corresponding to N_β^j.

Substituting (6) into (2), a closed-loop fuzzy control system with uncertain plant parameters can be formed. In this section, we shall analyze the stability and robustness of such an uncertain fuzzy control system. Three design approaches of the fuzzy controller, namely, general design approach [50], parallel design approach [37], and simplified design approach [50], can be used to close the feedback loop.

C. General Design Approach (GDA)

General design approach (GDA) allows differences in the number of rules and the rule antecedents between the TS-fuzzy plant model and the fuzzy controller. This approach gives the largest freedom on finding the fuzzy controller. More importantly, as the fuzzy controller (which depends on $m_j(\mathbf{x}(t))$) is not affected by the membership function values of the TS-fuzzy plant model ($w_i(\mathbf{x}(t))$), the TS-fuzzy plant model membership functions can be unknown or uncertain, as long as they satisfy the condition (3). This is an inherent robustness properties of GDA. In order to carry out the analysis, the closed-loop fuzzy system should be obtained first. From (2) and (6), the fuzzy control system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}(t)) m_j(\mathbf{x}(t))$$
$$\cdot \left[(\mathbf{H}_{ij} + \Delta \mathbf{H}_{ij}) \mathbf{x}(t) + (\mathbf{B}_i + \Delta \mathbf{B}_i) \mathbf{r} \right] \qquad (9)$$

where

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j \qquad (10)$$
$$\Delta \mathbf{H}_{ij} = \Delta \mathbf{A}_i + \Delta \mathbf{B}_i \mathbf{G}_j. \qquad (11)$$

D. Parallel Design Approach (PDA)

Parallel design approach (PDA) uses the same number of rules (p) and rule antecedents of the TS-fuzzy plant model to design the fuzzy controller. Hence, some of the terms in (9) can be grouped together. The number of terms of the closed-loop system is $p(p+1)/2$ instead of $p \times c$ for GDA. This makes the stability criterion to be satisfied more easily. The fuzzy control system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \left[w_i(\mathbf{x}(t)) (\mathbf{H}_{ii} + \Delta \mathbf{H}_{ii}) \mathbf{x}(t) \right.$$
$$\left. + (\mathbf{B}_i + \Delta \mathbf{B}_i) \mathbf{r} \right] + 2 \sum_{i<j}^{p} w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t))$$
$$\cdot (\mathbf{J}_{ij} + \Delta \mathbf{J}_{ij}) \mathbf{x}(t) \qquad (12)$$

where

$$\mathbf{J}_{ij} = \frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}, \quad \Delta \mathbf{J}_{ij} = \frac{\Delta \mathbf{H}_{ij} + \Delta \mathbf{H}_{ji}}{2} \qquad (13)$$
$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j \quad \Delta \mathbf{H}_{ij} = \Delta \mathbf{A}_i + \Delta \mathbf{B}_i \mathbf{G}_j. \qquad (14)$$

E. Simplified Design Approach (SDA)

Simplified design approach (SDA) requires that the sub-system inside the fuzzy control system has a common input matrix $\mathbf{B}$, and the fuzzy controller has the same number of rules and rule antecedents as those of the TS-fuzzy plant model. The number of summation terms of the fuzzy control system is further reduced to p. The fuzzy control system is given by

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t)) \left[(\mathbf{H}_j + \Delta \mathbf{H}_j) \mathbf{x}(t) + (\mathbf{B} + \Delta \mathbf{B}) \mathbf{r} \right] \qquad (15)$$

where

$$\mathbf{H}_j = \mathbf{A}_j + \mathbf{B} \mathbf{G}_j \qquad (16)$$
$$\Delta \mathbf{H}_j = \Delta \mathbf{A}_j + \Delta \mathbf{B} \mathbf{G}_j. \qquad (17)$$

It should be noted that in order to simplify (9) to (15), either one of the following conditions should hold:

$$\mathbf{B}_j = \mathbf{B}, \quad \Delta \mathbf{B}_j = \Delta \mathbf{B} \qquad (18)$$

or

$$\sum_{j=1}^{p} w_j(\mathbf{x}(t)) \mathbf{B}_j = \mathbf{B}, \quad \sum_{j=1}^{p} w_j(\mathbf{x}) \Delta \mathbf{B}_j = \Delta \mathbf{B}. \qquad (19)$$

In (18), $\mathbf{B}$ is a constant matrix. In (19), $\mathbf{B} = \mathbf{B}(\mathbf{x}(t))$ and $\Delta \mathbf{B} = \Delta \mathbf{B}(\mathbf{x}(t))$ vary during the operation as $w_j(\mathbf{x}(t))$ in each rule varies. Still, in both cases, when $\mathbf{G}_j$ is designed such that $\mathbf{H}_j = \mathbf{A}_j + \mathbf{B} \mathbf{G}_j = \mathbf{H}$ which is a constant matrix for all j, and the system has no parameter uncertainty (i.e., $\Delta \mathbf{A}_j = \Delta \mathbf{B}_j = \Delta \mathbf{H}_j = \mathbf{0}$ for all j), a linear closed-loop system can be obtained. If (18) holds, the linear system is obtained by feedback compensation (i.e., pole placement technique); otherwise, it is obtained by feedback linearization with respect to linear sub-systems satisfying (19). The structure of the fuzzy controller for the latter case is more complicated than that of the former case.

In summary, every design approach has its own advantages. GDA is applicable to those TS-fuzzy plant models with unknown membership functions and/or the number of rules of the fuzzy controller is different from that of the TS-fuzzy plant model. When the membership functions are known and the same rule antecedents of the TS-fuzzy plant model are used in the fuzzy controller, PDA allows the stability criterion to be satisfied more easily as compared with GDA. When each subsystem of the plant model can be compensated to become a common linear system using a suitable control law, SDA is recommended. The resulting closed-loop system will then become a linear system when the plant has no parameter uncertainty.

III. STABILITY AND ROBUSTNESS ANALYSES OF UNCERTAIN FUZZY CONTROL SYSTEMS

In the following paragraph, we proceed to the stability and robustness analyses with reference to an uncertain fuzzy control system

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1, \quad m_j(\mathbf{x}(t)) \in [0, 1] \quad \text{for all } j \qquad (7)$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{N_1^i}(g_1(\mathbf{x}(t))) \times \mu_{N_2^j}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{N_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{c} \left(\mu_{N_1^k}(g_1(\mathbf{x}(t))) \times \mu_{N_2^k}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{N_\Omega^k}(g_\Omega(\mathbf{x}(t))) \right)} \qquad (8)$$

under GDA. The analysis procedures for uncertain fuzzy control systems under PDA and SDA are similar to those under GDA, and the results will be given without proof. Consider the Taylor series

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \dot{\mathbf{x}}(t)\Delta t + \mathbf{o}(\Delta t) \qquad (20)$$

where $\mathbf{o}(\Delta t) = -\mathbf{x}(t) - \dot{\mathbf{x}}(t)\Delta t + \mathbf{x}(t + \Delta t)$ is the error term and $\Delta t > 0$

$$\lim_{\Delta t \to 0^+} \frac{\|\mathbf{o}(\Delta t)\|}{\Delta t} = 0. \qquad (21)$$

From (9) and (20), writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, and multiplying a transformation matrix $\mathbf{T} \in \Re^{n \times n}$ of rank n to both sides, we have

$$\mathbf{T}\mathbf{x}(t + \Delta t) = \mathbf{T}\mathbf{x}(t) + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}(\mathbf{H}_{ij} + \Delta\mathbf{H}_{ij})\mathbf{x}(t)$$
$$+ \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t + \mathbf{T}\mathbf{o}(\Delta t)$$
$$= \left(\mathbf{I} + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t \right) \mathbf{T}\mathbf{x}(t)$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t)$$
$$+ \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t + \mathbf{T}\mathbf{o}(\Delta t).$$

The reason for introducing $\mathbf{T}$ will be given at the end of this section. Taking norm on both sides of the above equation, we have

$$\|\mathbf{T}\mathbf{x}(t + \Delta t)\| \leq \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j (\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t) \right\|$$
$$\cdot \|\mathbf{T}\mathbf{x}(t)\| + \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \right.$$
$$\cdot [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t \left\| \right.$$
$$+ \|\mathbf{T}\mathbf{o}(\Delta t)\| \qquad (22)$$

where $\| \cdot \|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. From (22),

$$\|\mathbf{T}\mathbf{x}(t + \Delta t)\|$$
$$\leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \|\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\| \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t \right\|$$
$$+ \|\mathbf{T}\mathbf{o}(\Delta t)\|$$
$$\Rightarrow \|\mathbf{T}\mathbf{x}(t + \Delta t)\| - \|\mathbf{T}\mathbf{x}(t)\|$$
$$\leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j (\|\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\| - 1)$$
$$\cdot \|\mathbf{T}\mathbf{x}(t)\| + \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) \right.$$
$$+ \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t \left\| + \|\mathbf{T}\mathbf{o}(\Delta t)\| \right.$$
$$\Rightarrow \lim_{\Delta t \to 0^+} \frac{\|\mathbf{T}\mathbf{x}(t + \Delta t)\| - \|\mathbf{T}\mathbf{x}(t)\|}{\Delta t}$$
$$\leq \lim_{\Delta t \to 0^+} \left\{ \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \right.$$

$$\cdot (\|\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\| - 1) \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) \right.$$
$$+ \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}]\Delta t \left\| + \|\mathbf{T}\mathbf{o}(\Delta t)\| \right\} \Big/ \Delta t. \qquad (23)$$

From (21) and (23),

$$\frac{d\|\mathbf{T}\mathbf{x}(t)\|}{dt} \leq \lim_{\Delta t \to 0^+} \frac{\sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j (\|\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\| - 1)}{\Delta t}$$
$$\cdot \|\mathbf{T}\mathbf{x}(t)\| + \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \right.$$
$$+ [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}] \left\| \right.$$
$$\leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \left\| \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j [\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{x}(t) \right.$$
$$+ \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}] \left\| \right. \qquad (24)$$

where

$$\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] = \lim_{\Delta t \to 0^+} \frac{\|\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\| - 1}{\Delta t}$$
$$= \lambda_{\max} \left(\frac{\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1} + (\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1})^*}{2} \right) \qquad (25)$$

is the corresponding matrix measure [88] of the induced matrix norm of $\|\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\|$ (or the logarithmic derivative of $\|\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\|$); $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue, * denotes the conjugate transpose. From (24),

$$\frac{d\|\mathbf{T}\mathbf{x}(t)\|}{dt} \leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\| \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \left\| \sum_{i=1}^{p} w_i \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r} \right\|$$
$$\leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j (\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}]$$
$$+ \|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|) \|\mathbf{T}\mathbf{x}(t)\|$$
$$+ \left\| \sum_{i=1}^{p} w_i \mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}) \right\|. \qquad (26)$$

Let $\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}]$ be designed such that

$$\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \leq -\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max} - \varepsilon \text{ for all } i \text{ and } j \qquad (27)$$

where $\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max}$ is the maximum value of $\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|$, and ε is a designed nonzero positive constant. Increasing the value of ε will usually result in a system with improved performance but degraded robustness. In the next section, we shall show that the finding of $\mathbf{T}$ can be formulated into a NMI or a LMI problem. From (26) and (27),

$$
\begin{aligned}
\frac{d\|\mathbf{T}\mathbf{x}(t)\|}{dt} &\le -\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \varepsilon \|\mathbf{T}\mathbf{x}(t)\| \\
&\quad + \sum_{i=1}^{p} w_j \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\| \\
&= -\varepsilon\|\mathbf{T}\mathbf{x}(t)\| + \sum_{i=1}^{p} w_i \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\| \\
&\Rightarrow \left(\frac{d\|\mathbf{T}\mathbf{x}(t)\|}{dt} + \varepsilon\|\mathbf{T}\mathbf{x}(t)\|\right) e^{\varepsilon(t-t_o)} \\
&\le \sum_{i=1}^{p} w_i \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\| e^{\varepsilon(t-t_o)} \\
&\Rightarrow \frac{d}{dt}\left(\|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)}\right) \\
&\le \sum_{i=1}^{p} w_i |\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}| e^{\varepsilon(t-t_o)}
\end{aligned}
\tag{28}
$$

where $t_o < t$ is an arbitrary initial time. Based on (28), there are two cases to investigate the system behavior: 1) $\mathbf{r} = \mathbf{0}$ and 2) $\mathbf{r} \ne \mathbf{0}$. For the former case, it can be shown that if the condition (27) is satisfied, the closed-loop system (9) is stable, and $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

Proof: For $\mathbf{r} = \mathbf{0}$, from (28),

$$
\begin{aligned}
&\frac{d}{dt}\|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le 0 \\
&\Rightarrow \|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le \|\mathbf{T}\mathbf{x}(t_o)\| \\
&\quad\Rightarrow \|\mathbf{T}\mathbf{x}(t)\| \le \|\mathbf{T}\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)}.
\end{aligned}
\tag{29}
$$

Since ε is a positive value, $\|\mathbf{T}\mathbf{x}(t)\| \to 0$ as $t \to \infty$, and

$$
\begin{aligned}
\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\|\mathbf{x}(t)\|^2 &\le \|\mathbf{T}\mathbf{x}(t)\|^2 = \mathbf{x}(t)^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{T}\mathbf{x}(t) \\
&\le \sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\|\mathbf{x}(t)\|^2
\end{aligned}
\tag{30}
$$

where $\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ and $\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ denote the maximum and minimum singular values of $\mathbf{T}^{\mathrm{T}}\mathbf{T}$, respectively. As $\mathbf{T}^{\mathrm{T}}\mathbf{T}$ is symmetric positive definite ($\mathbf{T}$ has rank n), from (30), we have $\|\mathbf{T}\mathbf{x}(t)\| \to 0$ only when $\|\mathbf{x}(t)\| \to 0$. **QED**

For the latter case of $\mathbf{r} \ne \mathbf{0}$, the system states are bounded if the condition (27) is satisfied and $\mathbf{r}$ is bounded.

Proof: For $\mathbf{r} \ne \mathbf{0}$, from (28)

$$
\begin{aligned}
\|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} &\le \|\mathbf{T}\mathbf{x}(t_o)\| + \int_{t_o}^{t} \sum_{i=1}^{p} w_i \\
&\quad \cdot \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\| e^{\varepsilon(\tau-t_o)}\,d\tau \\
\Rightarrow \|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} &\le \|\mathbf{T}\mathbf{x}(t_o)\| + \|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\| \\
&\quad \cdot \int_{t_o}^{t} e^{\varepsilon(\tau-t_o)}\,d\tau
\end{aligned}
$$

where

$$
\begin{aligned}
\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\| &\le \max_i \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\|_{\max} \\
&\le \|\mathbf{T}(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{r}\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\mathbf{T}\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le \|\mathbf{T}\mathbf{x}(t_o)\| + \frac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}\left(e^{\varepsilon(t-t_o)} - 1\right) \\
&\Rightarrow \|\mathbf{T}\mathbf{x}(t)\| \le \|\mathbf{T}\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon} \\
&\qquad\qquad\qquad \cdot \left(1 - e^{-\varepsilon(t-t_o)}\right).
\end{aligned}
\tag{31}
$$

Since the right-hand side of (31) is finite if $\mathbf{r}$ is bounded, the system states are also bounded. **QED**.

The above analysis gives an upper bound of $\|\mathbf{T}\mathbf{x}(t)\|$ under different cases ((29) and (31)). Similarly, a lower bound of $\|\mathbf{x}(t)\|$ can be obtained by following the same analysis procedure with $\mathbf{x}(t - \Delta t) = \mathbf{x}(t) - \dot{\mathbf{x}}(t)\Delta t + \vartheta(\Delta t)$ and $\vartheta(\Delta t) = -\mathbf{x}(t) + \dot{\mathbf{x}}(t)\Delta t + \mathbf{x}(t - \Delta t)$. Hence, from (30), $\|\mathbf{x}(t)\|$ satisfies the following conditions:

$$
\frac{\|\mathbf{T}\mathbf{x}(t_o)\| e^{-\eta(t-t_o)}}{|\sqrt{\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}|} \le \|\mathbf{x}(t)\| \le \frac{\|\mathbf{T}\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)}}{|\sqrt{\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}|}
$$
$$
\text{for } \mathbf{r} = \mathbf{0}
\tag{32}
$$

or

$$
\begin{aligned}
&\max\left[\frac{\|\mathbf{T}\mathbf{x}(t_o)\| e^{-\eta(t-t_o)} - \dfrac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\eta}\left(1 - e^{-\eta(t-t_o)}\right)}{|\sqrt{\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}|}, 0\right] \\
&\le \|\mathbf{x}(t)\| \quad \text{for } \mathbf{r} \ne \mathbf{0} \\
&\le \frac{\|\mathbf{T}\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)} + \dfrac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}\left(1 - e^{-\varepsilon(t-t_o)}\right)}{|\sqrt{\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}|}
\end{aligned}
\tag{33}
$$

where

$$
\mu[-\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max} + \eta \quad \text{for all } i \text{ and } j
\tag{34}
$$

and η is a designed positive constant. Equation (34) is a condition governing η under GDA. For PDA, η is governed by

$$
\begin{cases}
\mu[-\mathbf{T}\mathbf{H}_{ii}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max} + \eta & \text{for all } i \\
\mu[-\mathbf{T}\mathbf{J}_{ij}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max} + \eta & \text{for all } i < j
\end{cases}.
\tag{35}
$$

For SDA, η is governed by

$$
\mu[-\mathbf{T}\mathbf{H}_j\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max} + \eta \quad \text{for all } j.
\tag{36}
$$

From (32) or (33),[1] the dynamic performance of the closed-loop system can be predicted. Also, the condition (27) is a sufficient criterion of stability for the system (9). In conclusion, the stability criterion and the robustness of the closed-loop fuzzy system under the three design approaches can be summarized by the following lemmas.

Lemma 1: Under GDA, the fuzzy control system as given by (9) without parameter uncertainty, i.e., $\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\| = 0$, is stable if $\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}$ is designed such that

$$
\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \le -\varepsilon \quad \text{for all } i \text{ and } j.
$$

---

[1]It should be noted that (32) or (33) is applicable to the three design approaches, but the conditions governing ε and η as given by (27) and (34), respectively, for GDA have to be modified for PDA and SDA accordingly.

Under PDA, the fuzzy control system as given by (12) without parameter uncertainty, i.e., $\|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\| = 0$ and $\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\| = 0$, is stable if $\mathbf{TH}_{ii}\mathbf{T}^{-1}$ and $\mathbf{TJ}_{ij}\mathbf{T}^{-1}$ are designed such that

$$\begin{cases} \mu[\mathbf{TH}_{ii}\mathbf{T}^{-1}] \le -\varepsilon & \text{for all } i \\ \mu[\mathbf{TJ}_{ij}\mathbf{T}^{-1}] \le -\varepsilon & \text{for all } i < j. \end{cases}$$

Under SDA, the fuzzy control system (15) without parameter uncertainty, i.e., $\|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\| = 0$, is stable if $\mathbf{TH}_j\mathbf{T}^{-1}$ is designed such that

$$\mu[\mathbf{TH}_j\mathbf{T}^{-1}] \le -\varepsilon \quad \text{for all } j.$$

Definition 1: The robust area of a fuzzy control system is defined as the area in the parameter space inside which uncertainties are allowed to exist without affecting the system stability.

Lemma 2: Under GDA, with the uncertain fuzzy control system given by (9), the robust area is governed by the following conditions:

$$\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max} \le -\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}] - \varepsilon \quad \text{for all } i \text{ and } j$$

where $\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max}$ is the maximum possible value of $\|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|$.

Under PDA, with the uncertain fuzzy control system given by (12), the robust area is governed by the following conditions:

$$\begin{cases} \|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max} \le -\mu[\mathbf{TH}_{ii}\mathbf{T}^{-1}] - \varepsilon & \text{for all } i \\ \|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max} \le -\mu[\mathbf{TJ}_{ij}\mathbf{T}^{-1}] - \varepsilon & \text{for all } i < j \end{cases}$$

where $\|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max}$ and $\|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max}$ are the maximum possible values of $\|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|$ and $\|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|$, respectively.

Under SDA, with the uncertain fuzzy control system given by (15), the robust area is governed by the following conditions:

$$\|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max} \le -\mu[\mathbf{TH}_j\mathbf{T}^{-1}] - \varepsilon \quad \text{for all } j$$

where $\|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max}$ is the maximum possible value of $\|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|$..

Both Lemmas 1 and 2 can be proved easily using condition (27) for GDA. Also, from (32) or (33), we can see that $\|\mathbf{x}(t)\|$ will go to its steady state faster if we have a larger ε. Hence, the system performance under a larger ε is expected to be better than that with a smaller ε. On the other hand, Lemma 2 implies that the robust area has to be smaller if we have a larger ε. Thus, we can conclude that a system with a larger ε is less robust than that with a smaller ε.

Finially, one should note that with the use of a suitable transformation matrix $\mathbf{T}$, we can transform any Hurwitz matrix having a positive or zero matrix measure into another matrix having a negative matrix measure. The stability and robustness conditions derived can then be applied. The problem left is how to find such a matrix $\mathbf{T}$ for a given system. This will be given in the next section.

IV. THE FINDING OF MATRIX $\mathbf{T}$

In this section, we formulate the task of finding $\mathbf{T}$ in Lemmas 1 and 2 into a NMI problem. The transformation matrix $\mathbf{T}$ should be found such that the uncertainty free system is stable (due to Lemma 1). Moreover, on minimizing the matrix measure of the system matrices, the transformation matrix should give us the maximum robust area. In view of these properties and Lemma 2, the NMI problem can be stated as follows.

For GDA, find $\mathbf{T}$ that minimizes $\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max}$ for all i and j subject to

$$\mathbf{TH}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} < \mathbf{0} \quad \text{for all } i \text{ and } j. \quad (37)$$

For PDA, find $\mathbf{T}$ that minimizes

$$\begin{cases} \mu[\mathbf{TH}_{ii}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max} & \text{for all } i \\ \mu[\mathbf{TJ}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max} & \text{for al } i < j \end{cases}$$

subject to

$$\begin{cases} \mathbf{TH}_{ii}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ii}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} < \mathbf{0} & \text{for all } i \\ \mathbf{TJ}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{J}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} < \mathbf{0} & \text{for all } i < j \end{cases}. \quad (38)$$

For SDA, find $\mathbf{T}$ that minimizes $\mu[\mathbf{TH}_j\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max}$ for all j subject to

$$\mathbf{TH}_j\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_j^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} < \mathbf{0} \quad \text{for all } j. \quad (39)$$

The proof of condition (37) under GDA can be obtained by considering that $\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}]$ is negative only when, from (25), the maximum eigenvalue of $(\mathbf{TH}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}})/2$ is negative, i.e., when $(\mathbf{TH}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}})/2$ is negative definite. This is because the $\mathbf{H}_{ij}$ of the control problem we consider are real, hence, $\mathbf{H}_{ij}^{\mathrm{T}} = \mathbf{H}_{ij}^*$. Similarly, (38) and (39) can be obtained for PDA and SDA, respectively. In particular, if $\mathbf{H}_{ij} + \mathbf{H}_{ij}^{\mathrm{T}}$ are negative definite for all i and j, the identity matrix $\mathbf{I} \in \Re^{n \times n}$ is the transformation matrix. For an uncertain fuzzy control system, the stability can be guaranteed if the conditions in Lemma 2 are satisfied. $\mathbf{T}$ is needed to determine the norm of the transformed parameter uncertainties.

The above NMI problem can be reduced to a LMI problem if $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. In this case, Lemma 1 will be reduced to the stability condition in [37] when an uncertainty free fuzzy control system is considered.

Proof: Under GDA,

$$\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}] < 0 \Rightarrow (\mathbf{TH}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}})/2 \text{ is negative definite}.$$

Then, if $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$

$$\mathbf{T} \times (\mathbf{TH}_{ij}\mathbf{T}^{-1} + \mathbf{T}^{-1^{\mathrm{T}}}\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) \times \mathbf{T} \text{ is negative definite}.$$
$$\Rightarrow \mathbf{TTH}_{ij} + \mathbf{H}_{ij}^{\mathrm{T}}\mathbf{TT} \text{ is negative definite for all } i \text{ and } j. \quad (40)$$

Let $\mathbf{TT} = \mathbf{P}$ ($\mathbf{P}$ is symmetric positive definite), the problem of finding the stability condition for GDA can be formulated as follows:

Find $\mathbf{T}$ that minimizes $\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max}$

for all i and j subject to $\mathbf{PH}_{ij} + \mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} < \mathbf{0}$

for all i and j. \quad (41)

Similarly, the problem of finding the stability conditions for PDA and SDA can be formulated as follows:

For PDA, find $\mathbf{T}$ that minimizes

$$\begin{cases} \mu[\mathbf{TH}_{ii}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max} & \text{for all } i \\ \mu[\mathbf{TJ}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max} & \text{for all } i < j \end{cases}$$

subject to

$$\begin{cases} \mathbf{PH}_{ii} + \mathbf{H}_{ii}^{\mathrm{T}}\mathbf{P} < 0 & \text{for all } i \\ \mathbf{PJ}_{ij} + \mathbf{J}_{ij}^{\mathrm{T}}\mathbf{P} < 0 & \text{for all } i < j \end{cases}. \tag{42}$$

For SDA, find $\mathbf{T}$ that minimizes $\mu[\mathbf{TH}_j\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max}$ for all j subject to

$$\mathbf{PH}_j + \mathbf{H}_j^{\mathrm{T}}\mathbf{P} < 0 \quad \text{for all } j \tag{43}$$

QED

In particular, we can also formulate the finding of $\mathbf{G}_i$'s of the fuzzy controller that satisfy the stability and robustness conditions as an LMI problem. This is done by letting the feedback gains of the fuzzy controller be $\mathbf{G}_i = -\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}$, where $\mathbf{R} \in \Re^{m \times m}$ is a symmetric positive definite matrix. Condition (41) can then be restated as follows:

$$\mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}) + (\mathbf{A}_i - \mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P})^{\mathrm{T}}\mathbf{P}$$
$$\text{is negative definite for all } i \text{ and } j$$
$$\Rightarrow \mathbf{PA}_i + \mathbf{A}_i^{\mathrm{T}}\mathbf{P} - 2\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P} \text{ is negative}$$
$$\text{definite for all } i \text{ and } j. \tag{44}$$

Multiplying $\mathbf{P}^{-1}$ to the left and the right sides, condition (44) becomes

$$\mathbf{P}^{-1}(\mathbf{PA}_i + \mathbf{A}_i^{\mathrm{T}}\mathbf{P} - 2\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P})\mathbf{P}^{-1} \text{ is negative}$$
$$\text{definite for all } i \text{ and } j$$
$$\Rightarrow \mathbf{A}_i\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_i^{\mathrm{T}} - 2\mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}} \text{ is negative definite}$$
$$\text{for all } i \text{ and } j. \tag{45}$$

Hence, the problem of finding $\mathbf{T}$ under $\mathbf{G}_i = -\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}$ for the three design approaches can be summarized as follows.

For GDA and $\mathbf{G}_i = -\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}$.

Find $\mathbf{T}$ and $\mathbf{R}$ that minimizes $\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ij}\mathbf{T}^{-1}\|_{\max}$ for all i and j such that

$$\mathbf{A}_i\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_i^{\mathrm{T}} - 2\mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}} < 0 \quad \text{for all } i \text{ and } j. \tag{46}$$

For PDA and $\mathbf{G}_i = -\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}$.

Find $\mathbf{T}$ and $\mathbf{R}$ that minimizes

$$\begin{cases} \mu[\mathbf{TH}_{ii}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_{ii}\mathbf{T}^{-1}\|_{\max} & \text{for all } i \\ \mu[\mathbf{TJ}_{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{J}_{ij}\mathbf{T}^{-1}\|_{\max} & \text{for all } i < j \end{cases}$$

such that

$$\begin{cases} \mathbf{A}_i\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_i^{\mathrm{T}} - 2\mathbf{B}_i\mathbf{RB}_i^{\mathrm{T}} < 0, \\ \quad\quad\quad\quad\quad \text{for all } i \\ (\mathbf{A}_i + \mathbf{A}_j)\mathbf{P}^{-1} + \mathbf{P}^{-1}(\mathbf{A}_i + \mathbf{A}_j)^{\mathrm{T}} - 2\mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}} \\ \quad - 2\mathbf{B}_j\mathbf{RB}_j^{\mathrm{T}} < 0, \quad \text{for all } i < j \end{cases}. \tag{47}$$

For SDA and $\mathbf{G}_i = -\mathbf{RB}_i^{\mathrm{T}}\mathbf{P}$

Find $\mathbf{T}$ and $\mathbf{R}$ that minimizes $\mu[\mathbf{TH}_j\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}_j\mathbf{T}^{-1}\|_{\max}$ for all j such that

$$\mathbf{A}_j\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_j^{\mathrm{T}} - 2\mathbf{B}_j\mathbf{RB}_j^{\mathrm{T}} < 0 \quad \text{for all } j. \tag{48}$$

V. PROCEDURE FOR FINDING SINGLE-GRID-POINT FUZZY CONTROLLERS

The procedure for developing SGP fuzzy controllers is summarized as follows.

1) Obtain the mathematical model for the uncertain multivariable nonlinear system. (Skip this step if a TS-fuzzy plant model is already at hand or obtainable by other ways.)
2) Obtain the TS-fuzzy plant model.
3) Determine the ranges of the parameter uncertainties, $\Delta\mathbf{A}_i$ and $\Delta\mathbf{B}_i$.
4) Decide the number of rules and membership functions of the SGP fuzzy controller. Then, choose one of the three design approaches to design the state feedback control law in each rule of the SGP fuzzy controller.
5) Apply Lemma 1 to check the stability of the uncertainty free fuzzy control system. If the fuzzy control system is stable, apply Lemma 2 to check the stability and robustness of the fuzzy control system after the defined ranges of parameter uncertainties have been introduced. The finding of the transformation matrix $\mathbf{T}$ in Lemmas 1 and 2 can be formulated as an NMI or an LMI problem.
6) If the stability and robustness tests are failed, redesign the fuzzy controller by going back to Step 4).

VI. APPLICATION EXAMPLES

Application examples on stabilizing a ball-and-beam system [61], an uncertain nonlinear mass-spring-damper system [37] and a two-inverted pendulum system [40] are given in this section. We shall find SGP fuzzy controllers under different design approaches by following the design procedures given in the previous section. Simulation results will be given.

A. Ball-and-Beam System

A ball-and-beam system with uncertain system parameters will be considered in this example by following the procedure in the previous section. GDA will be employed to design the SGP fuzzy controller.

1) A ball-and-beam system is shown in Fig. 2 [61]. Its dynamics equations are described by the following:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= B(x_1(t)x_4(t)^2 + g\sin(x_3(t))) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= u(t) \end{aligned} \tag{49}$$

where $x_1(t)$ is the position of the ball measured from the centre of the beam, $x_2(t)$ is the velocity of the ball, $x_3(t)$ is the angle of the beam with respect to the horizontal axis, $x_4(t)$ is the angular velocity of the beam, $u(t)$ is the input torque; $M \in [0.05\ 0.5]$ kg is the mass of the ball, $R = 0.01$ m is the radius of the ball, $g = 9.8$ ms^{-2} is the acceleration due to gravity, $B - (MR^2/J_b + MR^2) \leq 1$, J_b (in kgm^2) is the moment of inertia of the ball about the centre of the ball and is not necessary to be known in this example. The objective of this application example is to drive the ball to the centre of the beam such that $x_1(t) = 0$.
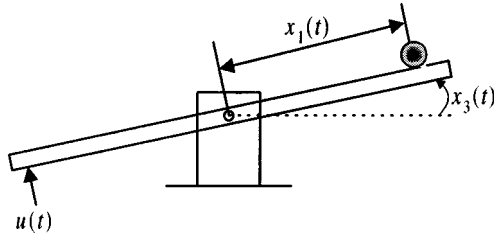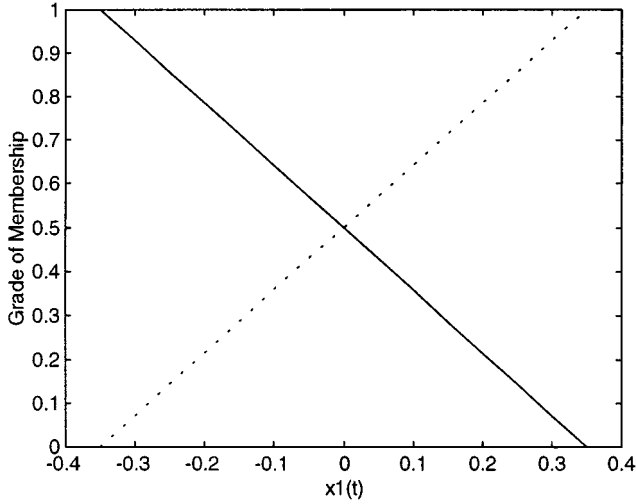
Fig. 2.   Ball-and-beam system.

Fig. 3.   Membership functions of the fuzzy controller for the ball-and-beam system: Solid line: $\mu_{N_1^1}(x_1(t)) = (-x_1(t) + x_{1\max})/(x_{1\max} - x_{1\min})$, Dotted line: $\mu_{N_1^2}(x_1(t)) = 1 - \mu_{N_1^1}(x_1(t))$.

2) The ball-and-beam system can be represented by a TS-fuzzy plant model having four rules. The ith rule can be written as follow:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND $f_2(\mathbf{x}(t))$ is M_2^i
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t),$
$$i = 1, 2, 3, 4 \qquad (50)$$

so that the system dynamics is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)) \qquad (51)$$

where

$$\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^{\mathrm{T}},$$
$$x_1(t) \in [x_{1\min} \quad x_{1\max}] = [-0.35 \quad 0.35],$$
$$x_2(t) \in [x_{2\min} \quad x_{2\max}] = [-1 \quad 1],$$
$$f_1(\mathbf{x}(t)) = B x_4(t)^2 \quad \text{and} \quad f_2(\mathbf{x}(t)) = -B \frac{\sin(x_3(t))}{x_3(t)};$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\min} & 0 & -g f_{2\min} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\min} & 0 & -g f_{2\max} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\max} & 0 & -g f_{2\min} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\max} & 0 & -g f_{2\max} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

we choose $f_{1\min} = -1$ and $f_{1\max} = -f_{1\min}$, $f_{2\min} = 0.6$ and $f_{2\max} = 1$;

$$w_i = \frac{\mu_{M_1^i}(f_1(\mathbf{x}(t))) \times \mu_{M_2^i}(f_2(\mathbf{x}(t)))}{\sum_{l=1}^{4}(\mu_{M_1^l}(f_1(\mathbf{x}(t))) \times \mu_{M_2^l}(f_2(\mathbf{x}(t))))};$$

$$\mu_{M_1^\beta}(f_1(\mathbf{x}(t))) = \frac{-f_1(\mathbf{x}(t)) + f_{1\max}}{f_{1\max} - f_{1\min}} \quad \text{for } \beta = 1, 2 \quad \text{and}$$

$$\mu_{M_1^\delta}(f_1(\mathbf{x}(t))) = 1 - \mu_{M_1^1}(f_1(\mathbf{x}(t))) \quad \text{for } \delta = 3, 4;$$

$$\mu_{M_2^\varepsilon}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t)) + f_{2\max}}{f_{2\max} - f_{2\min}} \quad \text{for } \varepsilon = 1, 3 \quad \text{and}$$

$$\mu_{M_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{M_2^1}(f_2(\mathbf{x}(t))) \quad \text{for } \phi = 2, 4$$

are the membership functions related to the uncertain system parameters.

3) We can see that the uncertain parameters are inside the membership functions. Hence, the parameter uncertainties of $\Delta\mathbf{A}_i$ and $\Delta\mathbf{B}_i$, $i = 1, 2, 3, 4$, are all zero in this example.

4) A two-rule fuzzy controller is designed for the nonlinear plant (49) under GDA. The rules are listed as follows:

Rule j: IF $x_1(t)$ is N_1^j THEN $u(t) = \mathbf{G}_j\mathbf{x}(t)$
$$= -\mathbf{R}\mathbf{B}_j^{\mathrm{T}}\mathbf{P}\mathbf{x}(t), \qquad j = 1, 2. \qquad (52)$$

We choose the membership functions of N_1^j, $j = 1, 2$, as follows:

$$\mu_{N_1^1}(x_1(t)) = \frac{-x_1(t) + x_{1\max}}{x_{1\max} - x_{1\min}}$$
$$\mu_{N_1^2}(x_1(t)) = 1 - \mu_{N_1^1}(x_1(t)). \qquad (53)$$

The membership functions are shown in Fig. 3.

5) By solving the LMI problem as described in (46), we have

$$\mathbf{P} = \begin{bmatrix} 29.5656 & -21.3312 & 3.4445 & -11.0125 \\ -21.3312 & 55.1292 & 7.4880 & 11.1344 \\ 3.4445 & 7.4880 & 4.4989 & -7.3562 \\ -11.0125 & 11.1344 & -7.3562 & 36.2820 \end{bmatrix} \quad \text{and}$$

$$\mathbf{R} = 87.4582.$$

As $\mathbf{P} = \mathbf{T}\mathbf{T}$, we have

$$\mathbf{T} = \begin{bmatrix} 0.3461 & 0.2003 & -0.5841 & -0.0967 \\ 0.2003 & 0.3311 & -0.6914 & -0.1572 \\ -0.5841 & -0.6914 & 2.7532 & 0.5618 \\ -0.0967 & -0.1572 & 0.5618 & 0.2926 \end{bmatrix}.$$

Table I summarizes the stability and robustness analysis results. By Lemma 1, we can conclude that the closed-loop system is asymptotically stable.

Figs. 4–7 show the responses of the system states under the initial condition of $\mathbf{x}(0) = [0.35 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. From (32), the system performance should lie inside the range

$$0.35e^{-\eta t} \leq \|\mathbf{x}(t)\| \leq 0.35e^{-\varepsilon t} \qquad (54)$$

where ε and η are chosen as 0.1272 (the smallest absolute value among $\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}]$ in Table I) and 34.8506 (the largest value among $\mu[-\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}]$ in Table I), respectively.
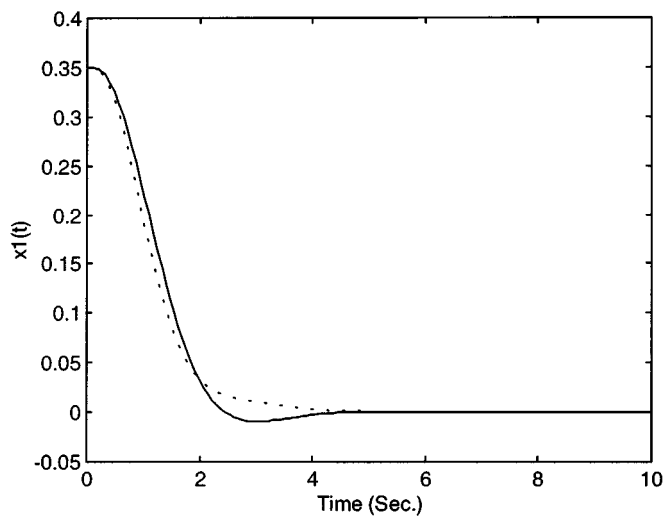
Fig. 4.   Responses of $x_1(t)$ of the ball-and-beam system under $M = 0.05$ kg (solid line) and $M = 0.5$ kg (dotted line).
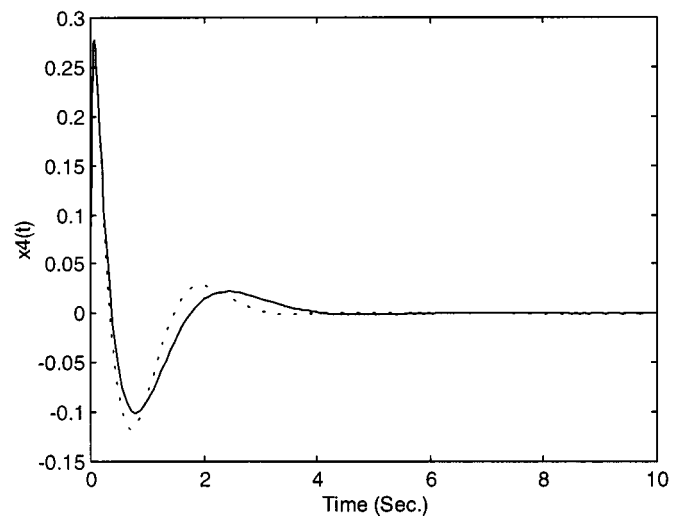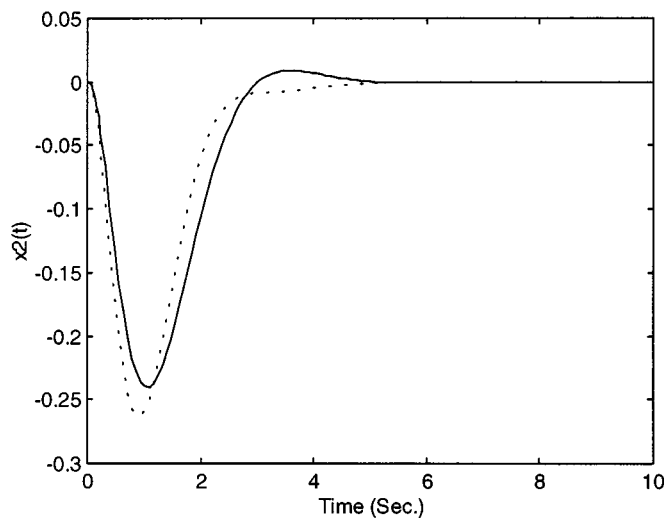

Fig. 5.   Responses of $x_2(t)$ of the ball-and-beam system under $M = 0.05$ kg (solid line) and $M = 0.5$ kg (dotted line).
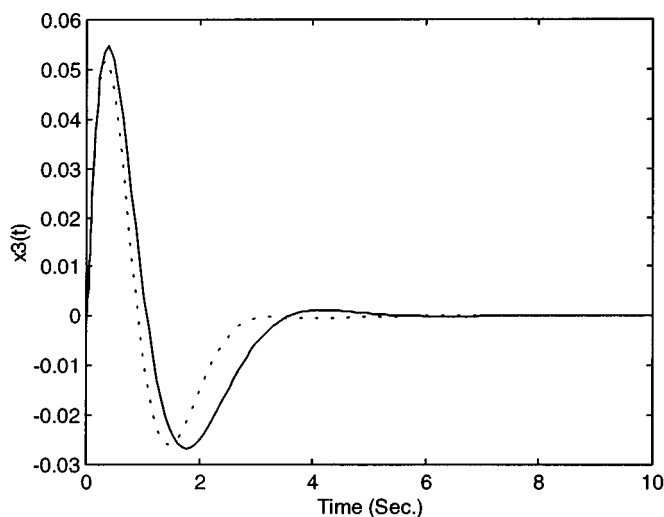

Fig. 6.   Responses of $x_3(t)$ of the ball-and-beam system under $M = 0.05$ kg (solid line) and $M = 0.5$ kg (dotted line).

Fig. 7.   Responses of $x_4(t)$ of the ball-and-beam system under $M = 0.05$ kg (solid line) and $M = 0.5$ kg (dotted line).

TABLE I
STABILITY AND ROBUSTNESS RESULTS OF
THE BALL-AND-BEAM SYSTEM

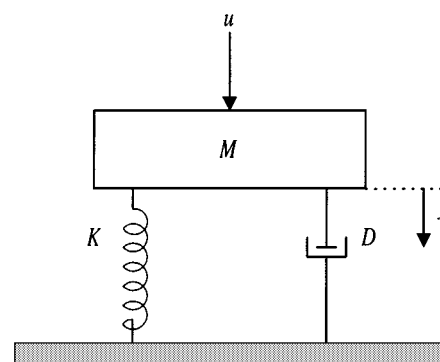| i, j	$\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}]$	$\mu[-\mathbf{TH}_{ij}\mathbf{T}^{-1}]$
1, 1	−0.1976	34.6090
1, 2	−0.1976	34.6090
2, 1	−0.1354	34.8506
2, 2	−0.1354	34.8506
3, 1	−0.1272	34.6590
3, 2	−0.1272	34.6590
4, 1	−0.5832	34.6529
4, 2	−0.5832	34.6529

Fig. 8.   Mass-spring-damper system.

B. Mass-Spring-Damper System

We shall follow the deign procedure in the previous section to obtain a fuzzy controller under PDA for a mass-spring-damper system subject to parameter uncertainties.

1) Fig. 8 shows a diagram of the mass-spring-damper system. Its dynamic equation is given by

$$M\ddot{x}(t) + g(x(t), \dot{x}(t)) + f(x(t)) = \phi(\dot{x}(t))u(t) \qquad (55)$$

where M is the mass and u is the force, $f(x(t))$ depicts the spring nonlinearity and uncertainty, $g(x(t), \dot{x}(t))$ depicts the damper nonlinearity, and $\phi(\dot{x}(t))$ uncertainty, and depicts the input nonlinearity and uncertainty

$$g(x(t), \dot{x}(t)) = D(c_1 x(t) + c_2 \dot{x}(t)^3 + c_3(t)\dot{x}(t))$$
$$f(x(t)) = K(c_4 x(t) + c_5 x(t)^3 + c_6(t)x(t))$$
$$\phi(\dot{x}(t)) = 1.4387 + c_7 \dot{x}(t)^2 + c_8(t)\cos(5\dot{x}(t)). \quad (56)$$

The operating range of the states is assumed to be within -1.5 and 1.5. The parameters are chosen as follows:

$$M = D = K = 1.0, \quad c_1 = 0, \quad c_2 = 1,$$
$$c_3(t) = \frac{c_3^U + c_3^L}{2} + \left(c_3^L - \frac{c_3^U + c_3^L}{2}\right)\sin(10t)$$

so that

$$c_3(t) \in [c_3^L, c_3^U], \quad c_4 = 0.01, \quad c_5 = 0.1,$$
$$c_6(t) = \frac{c_6^U + c_6^L}{2} + \left(c_6^L - \frac{c_6^U + c_6^L}{2}\right)\cos(5t)$$

so that

$$c_6(t) \in [c_6^L, c_6^U], \quad c_7 = -0.03,$$
$$c_8(t) = \frac{c_8^U + c_8^L}{2} + \left(c_8^L - \frac{c_8^U + c_8^L}{2}\right)\cos(5t)$$

so that

$$c_8(t) \in [c_8^L, c_8^U], \quad c_3^L = c_6^L = -0.3,$$
$$c_3^U = c_6^U = 0.3, \quad c_8^L = -0.2, \quad c_8^U = 0.2.$$

It can be seen that the uncertain parameters c_3, c_6 and c_8 are modeled as known functions of time t. Practically, they can be some uncertain parameters within the specified ranges. The nonlinear system then becomes

$$\ddot{x}(t) = -\dot{x}(t)^3 - 0.01 x(t) - 0.1 x(t)^3 - c_3(t)\dot{x}(t) - c_6(t)x(t)$$
$$+ (1.4387 - 0.13\dot{x}(t)^2 + c_8(t)\cos(5\dot{x}(t)))u(t). \quad (57)$$

2) The mass-spring-damper system (57) can exactly be represented by a TS-fuzzy plant model with the following fuzzy rules:

Rule i: IF $x(t)$ is M_1^i AND $\dot{x}(t)$ is M_2^i

THEN $\dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)u(t)$

$i = 1, 2, 3, 4 \quad (58)$

where the membership functions of M_α^i, $i = 1, 2$, $\alpha = 1, 2, 3, 4$, are given by

$$\mu_{M_1^1}(x(t)) = \mu_{M_1^2}(x(t)) = 1 - \frac{x(t)^2}{2.25}$$
$$\mu_{M_1^3}(x(t)) = \mu_{M_1^4}(x(t)) = \frac{x(t)^2}{2.25}$$
$$\mu_{M_2^1}(\dot{x}(t)) = \mu_{M_2^3}(\dot{x}(t)) = 1 - \frac{\dot{x}(t)^2}{6.75}$$
$$\mu_{M_2^2}(\dot{x}(t)) = \mu_{M_2^4}(\dot{x}(t)) = \frac{\dot{x}(t)^2}{6.75} \quad (59)$$

which are shown in Fig. 9. Also,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) = x(t)$$
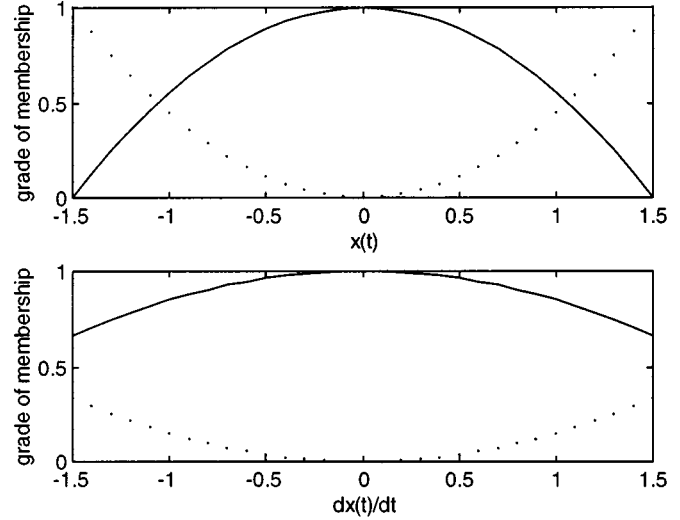

Fig. 9. Membership functions of the TS-fuzzy plant model of the uncertain nonlinear mass-spring-damper system: Upper: $\mu_{M_1^1}(x(t)) = \mu_{M_1^2}(x(t)) = 1 - (x(t)^2/2.25)$ (solid line), $\mu_{M_1^3}(x(t)) = \mu_{M_1^4}(x(t)) = (x(t)^2/2.25)$ (dotted line). Lower: $\mu_{M_2^1}(\dot{x}(t)) = \mu_{M_2^3}(\dot{x}(t)) = 1 - (\dot{x}(t)^2/6.75)$ (solid line), $\mu_{M_2^2}(\dot{x}(t)) = \mu_{M_2^4}(\dot{x}(t)) = (\dot{x}(t)^2/6.75)$ (dotted line).

TABLE II
STABILITY ANALYSIS RESULT OF THE MASS-SPRING-DAMPER SYSTEM FOR $i = j$

i	$\mu[\mathbf{H}_{ii}]$	$\mu[-\mathbf{H}_{ii}]$	$\|\Delta\mathbf{H}_{ii}\|_{max}$	$\mu[\mathbf{H}_{ii}] + \|\Delta\mathbf{H}_{ii}\|_{max}$	$\mu[-\mathbf{H}_{ii}] + \|\Delta\mathbf{H}_{ii}\|_{max}$
1	−2	2	1.2917	−0.7083	3.9217
2	−2	2	1.8561	−0.1439	3.8561
3	−2	2	1.2850	−0.7150	3.2850
4	−2	2	1.9206	−0.0794	3.9206

TABLE III
STABILITY ANALYSIS RESULT OF THE MASS-SPRING-DAMPER SYSTEM FOR $i \neq j$

i, j	$\mu[\mathbf{J}_{ij}]$	$\mu[-\mathbf{J}_{ij}]$	$\|\Delta\mathbf{J}_{ij}\|_{max}$	$\mu[\mathbf{J}_{ij}] + \|\Delta\mathbf{J}_{ij}\|_{max}$	$\mu[-\mathbf{J}_{ij}] + \|\Delta\mathbf{J}_{ij}\|_{max}$
1, 2	−1.8544	3.5756	1.5169	−0.3375	5.0925
1, 3	−2	2	1.2883	−0.7117	3.2883
1, 4	−1.8025	3.6275	1.5504	−0.2521	5.1179
2, 3	−1.8730	3.5570	1.5299	−0.3431	5.0869
2, 4	−2	2	1.8881	−0.1119	3.8881
3, 4	−1.8234	3.6066	1.5636	−0.2598	5.1702

$$x_2(t) = \dot{x}(t) + 2x_1(t); \quad \mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} -2 & 0 \\ -2.01 & 1 \end{bmatrix}$$
$$\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} -2 & 0 \\ -2.235 & 1 \end{bmatrix}; \quad \mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ 1.4387 \end{bmatrix}$$
$$\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ 0.5613 \end{bmatrix}$$
$$\Delta\mathbf{A}_1 = \Delta\mathbf{A}_2 = \Delta\mathbf{A}_3 = \Delta\mathbf{A}_4 = \begin{bmatrix} 0 & 0 \\ 2c_3(t) - c_6(t) & -c_3(t) \end{bmatrix}$$
$$\Delta\mathbf{B}_1 = \Delta\mathbf{B}_2 = \Delta\mathbf{B}_3 = \Delta\mathbf{B}_4 = \begin{bmatrix} 0 \\ c_8(t)\cos(5\dot{x}(t)) \end{bmatrix}.$$

3) The ranges of the parameter uncertainties are

$$\Delta \mathbf{A}_1 = \Delta \mathbf{A}_2 = \Delta \mathbf{A}_3 = \Delta \mathbf{A}_4 \in \left(\begin{bmatrix} 0 & 0 \\ -0.9 & -0.3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0.9 & 0.3 \end{bmatrix} \right)$$

$$\Delta \mathbf{B}_1 = \Delta \mathbf{B}_2 = \Delta \mathbf{B}_3 = \Delta \mathbf{B}_4 \subset \left(\begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \right).$$

4) PDA is employed to design the SGP fuzzy controller. We use four fuzzy rules to implement the SGP fuzzy controller and the membership functions of the fuzzy controller are chosen to be the same as those of the TS-fuzzy plant model (58), i.e., the membership functions (59). The rules of the SGP fuzzy controller are as follows:

Rule j: IF $x(t)$ is M_1^j AND $\dot{x}(t)$ is M_2^j THEN
$$u(t) = \mathbf{G}_j \mathbf{x}(t), \qquad j = 1, 2, 3, 4. \tag{60}$$

The feedback gains are then designed as

$$\mathbf{G}_1 = \begin{bmatrix} 1.3971 & -2.0852 \end{bmatrix}, \qquad \mathbf{G}_2 = \begin{bmatrix} 3.5810 & -5.3447 \end{bmatrix}$$
$$\mathbf{G}_3 = \begin{bmatrix} 1.5535 & -2.0852 \end{bmatrix}, \qquad \mathbf{G}_4 = \begin{bmatrix} 3.9818 & -5.3447 \end{bmatrix}$$

so that

$$\mathbf{H}_{11} = \mathbf{H}_{22} = \mathbf{H}_{33} = \mathbf{H}_{44} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

5) Tables II and III summarize the stability and robustness analysis results. From the numerical values listed in the tables, and by Lemmas 1 and 2 with $\mathbf{T} = \mathbf{I} \in \Re^{2 \times 2}$, the stability of the uncertain fuzzy control system is guaranteed. This is because both the second and the fifth columns of these two tables all contain only negative values.

The zero-input responses of the system under the initial conditions $\mathbf{x}(0) = \begin{bmatrix} -1 & -1 \end{bmatrix}^{\mathrm{T}}$ with parameter uncertainties (solid lines) and without parameter uncertainties (dotted lines) are shown in Fig. 10. From (32), the system performance should lie inside the range

$$\sqrt{2} e^{-\eta t} \le \|\mathbf{x}(t)\| \le \sqrt{2} e^{-\varepsilon t} \tag{61}$$

where ε and η are chosen as 0.0794 (the smallest absolute value among $\mu[\mathbf{H}_{ii}] + \|\Delta \mathbf{H}_{ii}\|_{\max}$ and $\mu[\mathbf{J}_{ij}] + \|\Delta \mathbf{J}_{ij}\|_{\max}$ in Tables II and III) and 5.1702 (the largest value among $\mu[\mathbf{H}_{ii}] + \|\Delta \mathbf{H}_{ii}\|_{\max}$ and $\mu[-\mathbf{J}_{ij}] + \|\Delta \mathbf{J}_{ij}\|_{\max}$ in Tables II and III), respectively. The simulation results verify the condition (61). It should be noted that the stability and robust conditions shown in Lemmas 1 and 2 are sufficient conditions. A system not satisfying these conditions may still be a stable system. This can be shown by increasing the ranges of the parameter uncertainties to

$$\Delta \mathbf{A}_1 = \Delta \mathbf{A}_2 = \Delta \mathbf{A}_3 = \Delta \mathbf{A}_4 \in \left(\begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \right)$$

and

$$\Delta \mathbf{B}_1 = \Delta \mathbf{B}_2 = \Delta \mathbf{B}_3 = \Delta \mathbf{B}_4 \in \left(\begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right).$$

In this case, the stability and robustness conditions in Lemmas 1 and 2 cannot be satisfied. However, the simulation results as shown in Fig. 11 display stable responses.

C. Two-Inverted Pendulum System

An SGP fuzzy controller will be designed to stabilize a two-inverted pendulum system under SDA by following the procedure in the previous section.

(a) A two-inverted pendulum system is shown in Fig. 12. It consists of two cart-pole inverted pendulums. The inverted pendulums are linked by a spring in the middle. The carts will move to and fro during the operation. The control objective is to balance the inverted pendulums vertically despite the movings of the spring and carts by applying forces to the tips of the pendulums. Referring to Fig. 12, M and m are the masses of the carts and the pendulums, respectively, $m = 10$ kg and $M = 100$ kg. $L = 1$ m is the length of the pendulums. The spring has a stiffness constant $k = 1$ N/m. $y_1(t) = \sin(2t)$ and $y_2(t) = L + \sin(3t)$ are the trajectories of the moving carts. $u_1(t)$ and $u_2(t)$ are the forces applied to the pendulums. $\theta_1(t)$ and $\theta_2(t)$ are the angular displacements of the pendulums measured from the vertical. The dynamic equation of the two-inverted pendulum system can be written as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{62}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \theta_1(t) \\ \dot{\theta}_1(t) \\ \theta_2(t) \\ \dot{\theta}_2(t) \end{bmatrix}$$

$$x_1 \in \begin{bmatrix} x_{1_{\min}} & x_{1_{\max}} \end{bmatrix} = \begin{bmatrix} -\dfrac{\pi}{2} & \dfrac{\pi}{2} \end{bmatrix}$$

$$x_3 \in \begin{bmatrix} x_{3_{\min}} & x_{3_{\max}} \end{bmatrix} = \begin{bmatrix} -\dfrac{\pi}{2} & \dfrac{\pi}{2} \end{bmatrix}$$

$$\mathbf{A}(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_1(x_1(t)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_2(x_3(t)) & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \\ 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \text{and} \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$f_1(t) = \frac{2}{L} - \frac{m}{M} \sin(x_1(t)) x_1(t)$$

$$f_2(t) = \frac{2}{L} - \frac{m}{M} \sin(x_3(t)) x_3(t); \quad \lambda = \frac{2}{mL^2}.$$

(b) A four-rule TS-fuzzy plant model is used to represent the two-inverted pendulum system. The ith rule of the TS-fuzzy plant model is given by

Rule i: IF $f_1(x_1(t))$ is M_1^i AND $f_2(x_3(t))$ is M_2^i
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$,
$$i = 1, 2, 3, 4 \tag{63}$$

where M_α^i is a fuzzy term of rule i, $i = 1, 2, 3, 4$, $\alpha = 1, 2$. Then, the system dynamics is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i [\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)] \tag{64}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1_{\min}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2_{\min}} & 0 \end{bmatrix}$$
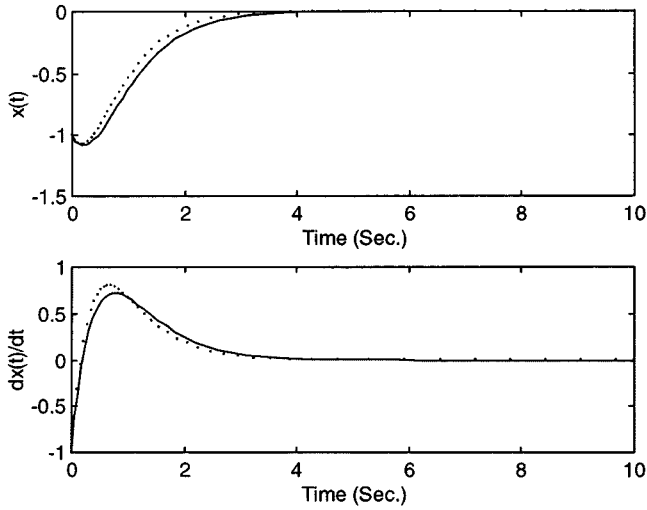
Fig. 10. System responses of the nonlinear mass-spring-damper system with parameter uncertainties (solid lines) and without parameter uncertainties (dotted lines).
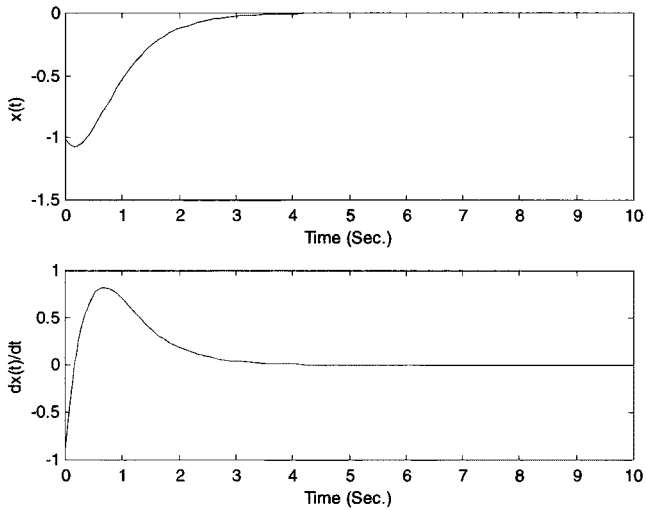

Fig. 11. System responses of the nonlinear mass-spring-damper system under larger ranges of parameter uncertainties.
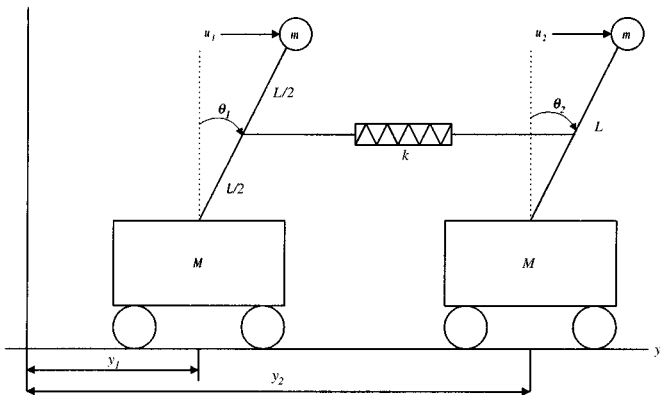

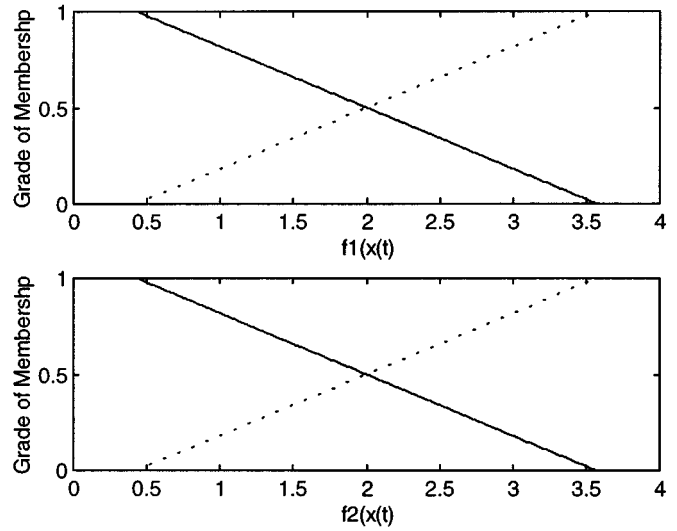Fig. 12. Two-inverted pendulum system.

Fig. 13. Membership functions of the TS-fuzzy plant model of the two-inverted pendulum system: Upper: $\mu_{\mathrm{M}_1^1}(f_1(x(t))) = \mu_{\mathrm{M}_1^2}(f_1(x(t))) = (-f_1(x(t)) + f_{1\max})/(f_{1\max} - f_{1\min})$ (solid line), $\mu_{\mathrm{M}_1^3}(f_1(x(t))) = \mu_{\mathrm{M}_1^4}(f_1(x(t))) = 1 - \mu_{\mathrm{M}_1^1}(f_1(x(t)))$ (dotted line); Lower: $\mu_{\mathrm{M}_2^1}(f_2(x_3(t))) = \mu_{\mathrm{M}_2^3}(f_2(x_3(t))) = (-f_2(x_3(t)) + f_{2\max})/(f_{2\max} - f_{2\min})$ (solid line), $\mu_{\mathrm{M}_2^2}(f_2(x_3(t))) = \mu_{\mathrm{M}_2^4}(f_2(x_3(t))) = 1 - \mu_{\mathrm{M}_2^1}(f_2(x_3(t)))$ (dotted line).

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\min} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2\max} & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\max} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2\min} & 0 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1\max} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2\max} & 0 \end{bmatrix}$$

$$w_i = \frac{\mu_{\mathrm{M}_1^i}(f_1(x_1(t))) \times \mu_{\mathrm{M}_2^i}(f_2(x_3(t)))}{\sum_{i=1}^{4} (\mu_{\mathrm{M}_1^i}(f_1(x_1(t))) \times \mu_{\mathrm{M}_2^i}(f_2(x_3(t))))}$$

$$\mu_{\mathrm{M}_1^\beta}(f_1(x_1(t))) = \frac{-f_1(x_1(t)) + f_{1\max}}{f_{1\max} - f_{1\min}} \quad \text{for } \beta = 1, 2 \quad \text{and}$$

$$\mu_{\mathrm{M}_1^\delta}(f_1(x_1(t))) = 1 - \mu_{\mathrm{M}_1^1}(f_1(x_1(t))) \quad \text{for } \delta = 3, 4$$

$$\mu_{\mathrm{M}_2^\varepsilon}(f_2(x_3(t))) = \frac{-f_2(x_3(t)) + f_{2\max}}{f_{2\max} - f_{2\min}} \quad \text{for } \varepsilon = 1, 3, \quad \text{and}$$

$$\mu_{\mathrm{M}_2^\phi}(f_2(x_3(t))) = 1 - \mu_{\mathrm{M}_2^1}(f_2(x_3(t))) \quad \text{for } \phi = 2, 4$$

$$f_{1\max} = \frac{2}{L} + x_{1\max}, \quad f_{1\min} = \frac{2}{L} - x_{1\max}$$

$$f_{2\max} = f_{1\max} \quad \text{and} \quad f_{2\min} = f_{1\min}.$$

The membership functions are shown in Fig. 13.

(c) $\Delta \mathbf{A}_i = \mathbf{0}$ and $\Delta \mathbf{B}_i = \mathbf{0}$, $i = 1, 2, 3, 4$.

(d) A four-rule SGP fuzzy controller is designed under SDA.

Rule j: IF $f_1(x_1(t))$ is $\mathbf{M}_1^i$ and $f_2(x_3(t))$ is $\mathbf{M}_2^i$

THEN $u(t) = \mathbf{G}_j \mathbf{x}(t), \qquad j = 1, 2, 3, 4.$ (65)

The feedback gains are then chosen as

$$\mathbf{G}_1 = \begin{bmatrix} -116.6410 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.0240 & -260.5216 & -180.2173 \end{bmatrix}$$

$$\mathbf{G}_2 = \begin{bmatrix} -116.6410 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.0240 & -323.3534 & -180.2173 \end{bmatrix}$$

$$\mathbf{G}_3 = \begin{bmatrix} -179.4729 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.0240 & -260.5216 & -180.2173 \end{bmatrix}$$

and

$$\mathbf{G}_4 = \begin{bmatrix} -179.4729 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.0240 & -323.3534 & -180.2173 \end{bmatrix}$$

so that

$$\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}_3 = \mathbf{H}_4$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6.4028 & -5.9891 & -4.7529 & -1.9823 \\ 0 & 0 & 0 & 1.0000 \\ -4.8515 & -2.0012 & -13.5969 & -9.0109 \end{bmatrix}.$$

(e) It can be seen that the closed-loop system is a linear system. The closed-loop system matrix is stable and has the eigenvalues of -1, -2, -3, and -4. Thus, we can conclude the closed-loop system is asymptotically stable.

The zero-input responses of the system under the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} \dfrac{22\pi}{45} & 0 & -\dfrac{22\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$$

are shown in Figs. 14 and 15, respectively. In this example, it can be seen that an uncertainty free nonlinear system will become a linear system under SDA. We now consider that $M \geq m$ is an unknown value. GDA is employed to design a fuzzy controller with four rules described as follows:

> Rule j: IF $x_1(t)$ is N_1^j AND $x_3(t)$ is N_2^j
> THEN $\mathbf{u}(t) = \mathbf{G}_j \mathbf{x}(t)$, $i = 1, 2, 3, 4.$ (66)

The membership functions are designed as

$$\mu_{\mathrm{N}_1^\beta}(x_1(t)) = \frac{-x_1(t) + x_{1\max}}{x_{1\max} - x_{1\min}} \quad \text{for } \beta = 1, 2 \quad \text{and}$$

$$\mu_{\mathrm{N}_1^\delta}(x_1(t)) = 1 - \mu_{\mathrm{N}_1^1}(x_1(t)) \quad \text{for } \delta = 3, 4;$$

$$\mu_{\mathrm{N}_2^\varepsilon}(x_3(t)) = \frac{-x_3(t) + x_{3\max}}{x_{3\max} - x_{3\min}} \quad \text{for } \varepsilon = 1, 3, \quad \text{and}$$

$$\mu_{\mathrm{N}_2^\phi}(x_3(t)) = 1 - \mu_{\mathrm{N}_2^1}(x_3(t)) \quad \text{for } \phi = 2, 4.$$

Table IV lists the stability analysis results with

$$\mathbf{T} = \begin{bmatrix} 1.0159 & 0.0669 & 0.0431 & -0.0210 \\ 0.0669 & 0.3157 & -0.0226 & -0.0395 \\ 0.0431 & -0.0226 & 1.0833 & 0.0358 \\ -0.0210 & -0.0395 & 0.0358 & 0.2546 \end{bmatrix}.$$

By Lemma 1, as the values in the second column are all negative, it can be concluded that the closed-loop system is stable. Under the GDA, the zero-input responses of the system with $M = 10$ kg under the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} \dfrac{2\pi}{45} & 0 & -\dfrac{22\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$$

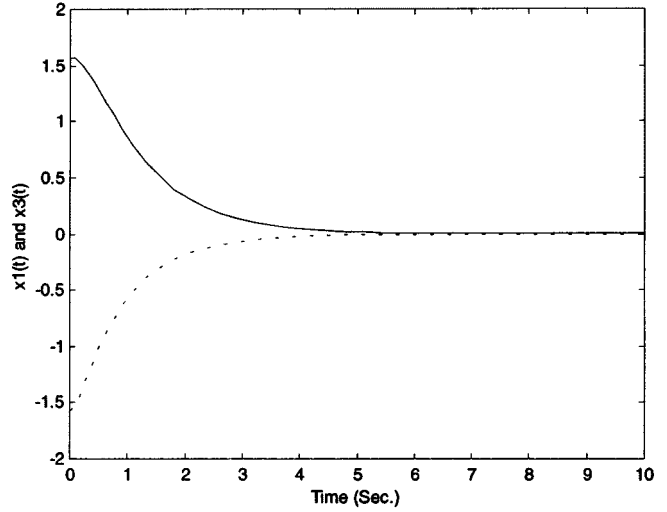are shown in Figs. 16 and 17, respectively.

Fig. 14.   Responses of $x_1(t)$ (solid line) and $x_3(t)$ (dotted line) of the two-inverted pendulum system under SDA with $M = 100$ kg.
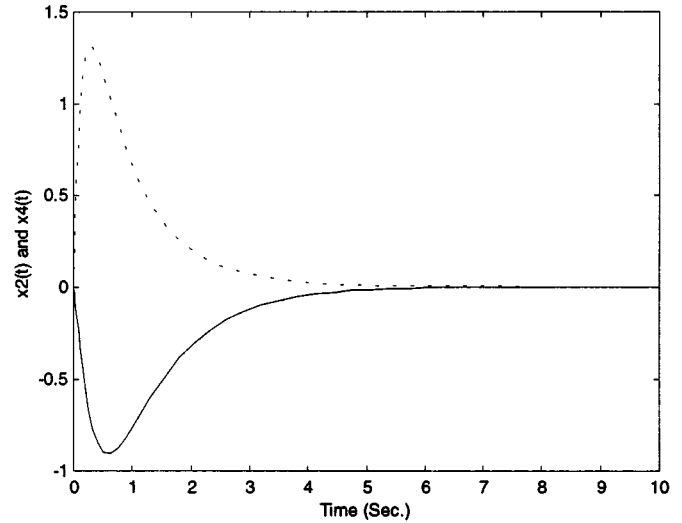

Fig. 15.   Responses of $x_2(t)$ (solid line) and $x_4(t)$ (dotted line) of the two-inverted pendulum system under SDA with $M = 100$ kg.
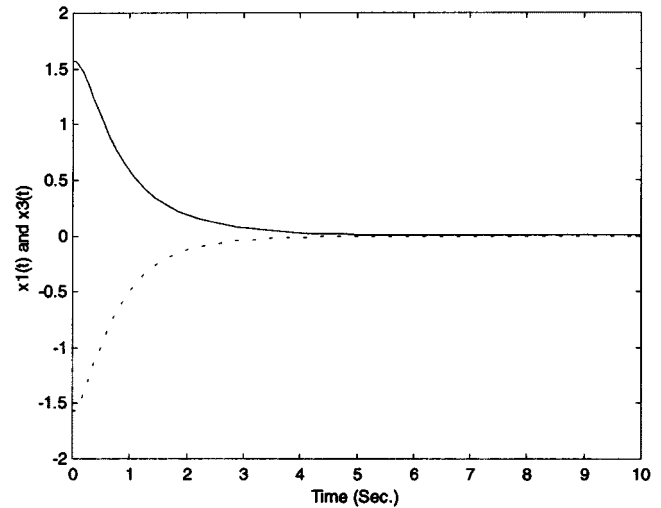

Fig. 16.   Responses of $x_1(t)$ (solid line) and $x_3(t)$ (dotted line) of the two-inverted pendulum system under GDA with $M = 10$ kg.
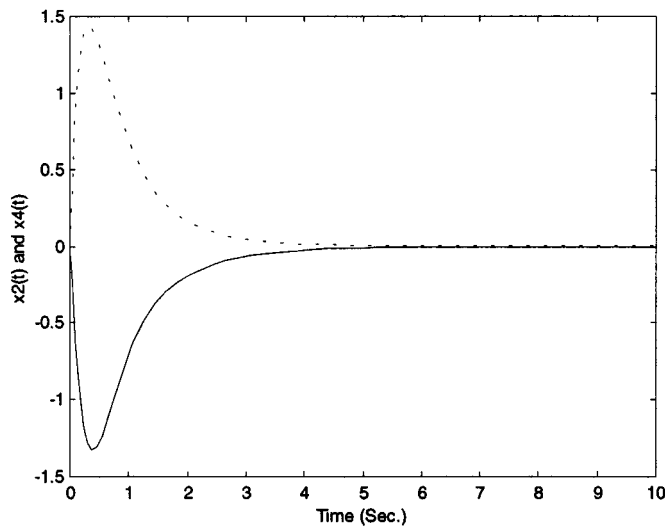
Fig. 17.   Responses of $x_2(t)$ (solid line) and $x_4(t)$ (dotted line) of the two-inverted pendulum system under GDA with $M = 10$ kg.

TABLE IV
STABILITY ANALYSIS RESULT OF THE TWO-INVERTED PENDULUM SYSTEM
UNDER GDA

i,j	$\mu[\mathbf{TH}_{ij}\mathbf{T}^{-1}]$	$\mu[-\mathbf{TH}_{ij}\mathbf{T}^{-1}]$
1, 1	−0.4125	9.5679
1, 2	−0.4045	9.5075
1, 3	−0.4231	9.5544
1, 4	−0.5081	9.4917
2, 1	−0.2522	9.6496
2, 2	−0.4125	9.5679
2, 3	−0.2751	9.6386
2, 4	−0.4231	9.5544
3, 1	−0.0693	9.5888
3, 2	−0.0836	9.5314
3, 3	−0.4125	9.5679
3, 4	−0.4045	9.5075
4, 1	−0.0157	9.6675
4, 2	−0.0693	9.5888
4, 3	−0.2522	9.6496
4, 4	−0.4125	9.5679

VII. CONCLUSION

The stability and robustness of an uncertain multivariable fuzzy control system that is designed based on a SGP approach have been investigated. The analyses are general and systematic. Stability and robustness conditions under three design approaches of the fuzzy controller have been derived. The resulting fuzzy controller is capable of tackling

multivariable nonlinear systems subject to small parameter uncertainties. Application examples on stabilizing various uncertain nonlinear systems have been given to illustrate the stabilizability and robustness property of the proposed SGP fuzzy controllers. As a note here, the results of the SGP approach is the basis for the development of a more general multiple-grid-point (MGP) fuzzy controller [50], [52], that can tackle nonlinear plants subject to large parameter uncertainties.

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Neighborhood Detection and Rule Selection from Cellular Automata Patterns

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Abstract—Using Genetic Algorithms (GAs) to search for cellular automation (CA) rules from spatio-temporal patterns produced in CA evolution is usually complicated and time-consuming when both the neighborhood structure and the local rule are searched simultaneously. The complexity of this problem motivates the development of a new search which separates the neighborhood detection from the GA search. In this paper, the neighborhood is determined by independently selecting terms from a large term set on the basis of the contribution each term makes to the next state of the cell to be updated. The GA search is then started with a considerably smaller set of candidate rules pre-defined by the detected neighborhood. This approach is tested over a large set of one-dimensional (1-D) and two-dimensional (2-D) CA rules. Simulation results illustrate the efficiency of the new algorithm.

Index Terms—Cellular automata, genetic algorithms, identification, spatio-temporal systems.

I. INTRODUCTION

A cellular automaton (CA) is a discrete system which evolves in discrete time over a lattice structure composed of a large quantity of cells. The states of the cells are discrete and are updated synchronously according to a local rule operating on a given neighborhood. The study of low-dimensional CAs has been the focus of attention from a wide range of researchers [1]–[6]. One of the most important topics in CA studies is the identification of the CA, that is, to extract the neighborhood and the governing local rule from a given set of spatio-temporal patterns produced by the CA evolution.

Ideally, the identification technique should be designed to produce an optimal CA expression which consists of a clear, minimal neighborhood structure and a correct local rule. In [7], although correct rules were generated by applying a set of sequential and parallel algorithms, the associated neighborhood was not clearly presented and the identification process was complicated and time-consuming. Genetic Algorithms (GAs) were employed in [8] in search for a matching rule from a large rule set of all possible rules. Again, no satisfactory neighborhood structure was obtained. In [9], the minimal neighborhood problem was addressed by introducing a second search objective to the GAs on the basis of reformulating the CA rules into a uniform Boolean expression. Because the assumed neighborhood which determines the run time is usually much larger than the actual neighborhood, the search process can be very long, sometimes taking several hours for a single run, even for a very simple one-dimensional (1-D) CA rule (see [9, Table VI]). However it might be possible to substantially reduce the run time if the assumed neighborhood for the GA search was correct and minimal. One way to achieve this would be to determine the neighborhood before starting the rule search and this is the main objective of the present study.

In this paper, a new neighborhood detection technique is introduced which is capable of extracting the correct and minimal neighborhood from a large set of candidate neighborhoods without having to define