Stability Analysis of Fuzzy Control Systems Subject to Uncertain Grades of Membership

H. K. Lam, *Member, IEEE* and F. H. F. Leung, *Senior Member, IEEE*

***Abstract*—This paper presents relaxed stability conditions for fuzzy control systems subject to parameter uncertainties. As the parameter uncertainties introduce uncertain grades of membership to the fuzzy control systems, the favorable property offered by sharing the same premises in the fuzzy plant models and fuzzy controllers cannot be employed to enhance the stabilization ability of the fuzzy control systems. To widen the applicability of the fuzzy control approach, fuzzy control systems subject to uncertain grades of membership will be investigated. New relaxed stability conditions will be derived to guarantee the stability of this class of fuzzy control systems. A numerical example will be given to show the effectiveness of the proposed approach.**

***Index Terms*—Fuzzy control, parameter uncertainty, stability.**

I. Introduction

The fuzzy-model-based control approach offers a systematic way to analyze nonlinear systems. In general, this approach first represents the nonlinear plant with a TSK-fuzzy plant model [1], [2] in the form of weighted sums of some linear subsystems. The TSK-fuzzy plant model extracts the linear and nonlinear characteristics of the nonlinear plant. It provides a general framework to represent the nonlinear plant and exhibits semi-linear characteristics which facilitates the system analysis and controller synthesis. Similarly, a fuzzy controller in the form of weighted sums of some linear subcontrollers [3], [4] was proposed to close the feedback loop. As a result, the closed-loop fuzzy control system is also represented as weighted sums of some linear subcontrol systems. The weights are regarded as the grades of membership, which are used to measure the contributions of the linear subsystems/subcontrollers to the modeling/controlling task. The grades of membership are characterized by membership functions, which play an important role in the system stability and performance.

The Lyapunov-based approach is the most common approach to investigate the stability of the fuzzy control systems. The fuzzy control systems nonlinearly combine some linear subcontrol systems. By investigating the linear part of the fuzzy control system, it was shown that the fuzzy control system is stable if all linear subcontrol systems are stable subject to a common Lyapunov function [3], [4]. The advantage of this stability analysis result is that the system stability is governed by the linear subcontrol systems only and independent of the membership functions of the fuzzy controllers. Hence, the membership functions of the fuzzy controllers can be freely designed. Furthermore, as the membership functions of both the fuzzy plant model and the fuzzy controller contribute nothing to the stability conditions, the fuzzy controller can handle a nonlinear system subject to parameter uncertainties that appear in the membership functions of the fuzzy plant model. However, these design flexibility and the robustness property may lead to conservative stability conditions as the effects of the membership functions are not taken into the stability consideration, which can in fact play an important role in the stability of the system. The stability conditions were relaxed in [4] when the fuzzy controller shares the same premises as those of the fuzzy plant model. The relaxed stability conditions are achieved as the contribution of some linear subcontrol systems compensate with others. Different relaxed stability conditions were given in [6]–[10]. Relaxed stability conditions making using this property and based on other stability analysis approach were reported in [11], [12]. One more advantage of sharing the same premises in the fuzzy plant models and fuzzy controllers is that the stabilization ability of the fuzzy controller is enhanced. However, this design criterion implies that the fuzzy plant model must be known as its membership functions must be known. Hence, the fuzzy controller under such design criterion is not suitable for handling nonlinear system subject to parameter uncertainties. In other words, the stabilization ability of the fuzzy controllers is enhanced by scarifying its robustness property. It is important to balance the stabilization ability and robustness property of the fuzzy controller in order to widen its applicability to nonlinear systems subject to parameter uncertainties.

Most of the practical applications are subject to parameters uncertainties. On tackling the stability and robustness issues, the highly nonlinear nature of the plant makes the analysis difficult. The parameter uncertainties make the problem more complex. In this paper, a fuzzy controller is proposed to handle this class of fuzzy control system. Lyapunov approach will be employed to investigate the system stability. Limited knowledge of the nonlinear plant subject to parameter uncertainties will be employed to investigate the system stability. Relaxed stability conditions for this class of fuzzy control systems will be derived to guarantee the system stability. The relaxed stability conditions offer a larger stability region to fuzzy-model-based control systems when compared with the stability analysis results in [3], [4]. Consequently, the fuzzy controller can be applied to handle nonlinear systems subject to relatively larger parameter uncertainties.

This paper is organized as follows. In Section II, the fuzzy plant model with uncertain grades of membership and the fuzzy controller are presented. In Section III, the stability of the fuzzy control systems subject to uncertain grades of membership will be analyzed. Relaxed stability conditions will be derived based on the Lyapunov stability theory. In Section IV, a numerical example will be presented. A conclusion will be drawn in Section V.

II. TSK Fuzzy Plant Model and Fuzzy Controller

A multivariable nonlinear control system comprising a TSK fuzzy plant model subject to uncertain grades of membership and a fuzzy controller connected in closed-loop will be considered.

A. TSK Fuzzy Plant Model

Let p be the number of fuzzy rules describing the nonlinear plant. The ith rule is of the following format:

$$\begin{aligned}\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } \mathrm{M}_\Psi^i \\ \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \quad (1)\end{aligned}$$

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \ldots, \Psi$; $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n \times n}$ and $\mathbf{B}_i \in \Re^{n \times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n \times 1}$ is the system state vector; and $\mathbf{u}(t) \in \Re^{m \times 1}$ is the input vector. The system dynamics are described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \quad (2)$$

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The authors are with the Centre for Multimedia Signal Processing, Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

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where

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(x_\alpha(t))$ is the grade of membership corresponding to the fuzzy term M_α^i. The nonlinear plant is subject to parameter uncertainties such that the grades of membership $\mu_{\mathrm{M}_\alpha^i}(x_\alpha(t))$ are uncertain.

B. Fuzzy Controller

A fuzzy controller with p fuzzy rules is to be designed for the nonlinear plant. As the nonlinear plant is subject to parameter uncertainties which lead to uncertain grades of membership, the premises of the fuzzy plant model cannot be employed by the fuzzy controller. The jth rule of the fuzzy controller is of the following format:

$$\text{Rule } j: \text{IF } g_1(\mathbf{x}(t)) \text{ is } \mathrm{N}_1^j \text{ AND} \cdots \text{AND } g_\Omega(\mathbf{x}(t)) \text{ is } \mathrm{N}_\Omega^j \quad \text{THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) \tag{5}$$

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \ldots, \Omega$; $j = 1, 2, \ldots, p$; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m \times n}$ is the feedback gain of rule j to be designed. The inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{6}$$

where

$$\sum_{j=1}^{p} m_j(\mathbf{x}(t)) = 1,\ m_j(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } j \tag{7}$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\right)} \tag{8}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_β^j.

C. Fuzzy Control System

The fuzzy control system is formed by the nonlinear plant in the form of (2) and the fuzzy controller of (6) connected in closed loop. The property that

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{j=1}^{p} m_j(\mathbf{x}(t)) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) = 1$$

will be employed in the following section for doing the stability analysis. From (2) and (6)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(\sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)\right)\right)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\mathbf{H}_{ij}\mathbf{x}(t)\end{aligned} \tag{9}$$

where

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j. \tag{10}$$

Remark 1: Stability conditions in the form of linear matrix inequalities (LMIs) shown in the following theorem have been given in [3], [4] to guarantee the system stability of (9).

Theorem 1 [3], [4]: The fuzzy control system formed by the nonlinear plant in the form of (2) and the fuzzy controller of (6)is guaranteed to be asymptotically stable if there exists a symmetric matrix $\mathbf{P}$ such that the following LMIs hold:

III. Stability Analysis

The system stability of the fuzzy control system of (9) will be investigated. In the following analysis, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are written as w_i and m_j, respectively, for simplicity. Consider the following Lyapunov function candidate:

$$V = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{11}$$

where $\mathbf{P} \in \Re^{n \times n}$ is a symmetric positive definite matrix. From (9) and (11)

$$\begin{aligned}\dot{V} &= \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\\ &= \left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)\\ &\quad + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right)\mathbf{x}(t)\\ &= -\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\end{aligned} \tag{12}$$

where

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right). \tag{13}$$

It can be seen from (12) that if the stability conditions in Theorem 1 are satisfied, the fuzzy control system of (9) is guaranteed to be asymptotically stable. To proceed further, let $m_l(\mathbf{x}(t)) = \rho_l(\mathbf{x}(t))w_l(\mathbf{x}(t))$ and

$$\begin{aligned}0 < \frac{m_{l_{\min}}}{w_{l_{\max}}} = \rho_{l_{\min}} = \min_{\mathbf{x}(t)}\left(\frac{m_l(\mathbf{x}(t))}{w_l(\mathbf{x}(t))}\right) \le \rho_l(\mathbf{x}(t))\\ \le \max_{\mathbf{x}(t)}\left(\frac{m_l(\mathbf{x}(t))}{w_l(\mathbf{x}(t))}\right) = \rho_{l_{\max}} = \frac{m_{l_{\max}}}{w_{l_{\min}}} < \infty\end{aligned}$$

where $l = 1, 2, \ldots, p$; $w_{i_{\min}}$; and $w_{i_{\max}}$ denote the known minimum and maximum values of $w_i(\mathbf{x}(t))$ respectively during the operation, which exhibit the property that $0 < w_{l_{\min}} \le w_{l_{\max}} \le 1$. Similarly, $m_{l_{\min}}$ and $m_{l_{\max}}$ denote the known minimum and maximum values of $m_j(\mathbf{x}(t))$, respectively, which exhibit the property that $0 < m_{l_{\min}} \le m_{l_{\max}} \le 1$. $\min_{\mathbf{x}(t)}(\cdot)$ and $\max_{\mathbf{x}(t)}(\cdot)$ denotes the minimum and maximum values of the arguments for all $\mathbf{x}(t)$. In most of the practical applications, the boundary values of the parameters are known. Based on the known form of the membership functions and the boundary values of the parameters, the minimum and maximum grades of memberships can be obtained analytically or estimated by some searching algorithms, e.g., genetic algorithm. To ensure $0 < m_l(\mathbf{x}(t))/w_l(\mathbf{x}(t)) < \infty$, it is required that the minimum values of $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ (i.e., $w_{l_{\min}}$ and $m_{l_{\min}}$) must be greater than zero. $w_{l_{\min}} > 0$ can be achieved by properly building the fuzzy plant model for the nonlinear plant. (The procedure of obtaining the fuzzy plant model of the nonlinear plant is detailed in [4] and [13].) For instance, the operating domain is considered a bit larger than the actual one during the modeling process. $w_l(\mathbf{x}(t))$ will lie in the range of 0 and 1 if the nonlinear plant operates in this larger operating domain considered in the modeling process. When the nonlinear plant operates in the actual operating domain which is slightly smaller than that considered in the modeling process, $w_l(\mathbf{x}(t))$ will not go into the boundary values. The same idea can be applied to determine $m_l(\mathbf{x}(t))$, $m_{l_{\min}}$ and $m_{l_{\max}}$. From (12) and writing $\rho_j(\mathbf{x}(t))$ as ρ_j for $j = 1, 2, \ldots p$,

$$\dot{V}(t) = -\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_j w_i w_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t). \tag{14}$$

Let $\rho_j = \eta_{j1}\rho_{j1} + \eta_{j2}\rho_{j2}$ where $\rho_{j1} = \rho_{j_{\min}}$ and $\rho_{j2} = \rho_{j_{\max}}$ denote the minimum and maximum values of ρ_j, $\eta_{j2} = 1 - \eta_{j1}$ and $0 \le \eta_{j1}, \eta_{j2} \le 1$. Hence, we have

$$\rho_j = \eta_{j1}\rho_{j1} + (1-\eta_{j1})\rho_{j2} \Rightarrow \eta_{j1} = \frac{\rho_j - \rho_{j2}}{\rho_{j1} - \rho_{j2}} \ge 0$$
$$\text{and}$$
$$\eta_{j2} = 1 - \eta_{j1} = 1 - \frac{\rho_j - \rho_{j2}}{\rho_{j1} - \rho_{j2}} = \frac{\rho_{j1} - \rho_j}{\rho_{j1} - \rho_{j2}} \ge 0.$$

It should be noted that $\sum_{k=1}^{2}\eta_{jk} = 1$ and $0 \le \eta_{jk} \le 1$, $k = 1, 2$ as $\rho_{j1} \le \rho_j \le \rho_{j2}$. From (14),

$$\begin{aligned}\dot{V}(t) &= -\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{2}\eta_{jk}\rho_{jk}w_i w_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\\ &= -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{2}\eta_{jk}w_i w_j\mathbf{x}(t)^{\mathrm{T}}\rho_{jk}\mathbf{Q}_{ij}\mathbf{x}(t)\\ &\quad -\frac{1}{2}\sum_{j=1}^{p}\sum_{i=1}^{p}\sum_{k=1}^{2}\eta_{ik}w_j w_i\mathbf{x}(t)^{\mathrm{T}}\rho_{ik}\mathbf{Q}_{ji}\mathbf{x}(t)\\ &= -\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{2}w_i w_j\mathbf{x}(t)^{\mathrm{T}}\left(\frac{\eta_{jk}\rho_{jk}\mathbf{Q}_{ij}+\eta_{ik}\rho_{ik}\mathbf{Q}_{ji}}{2}\right)\mathbf{x}(t)\\ &= -\sum_{i=1}^{p}w_i w_i\mathbf{x}(t)^{\mathrm{T}}\sum_{k=1}^{2}\eta_{ik}\rho_{ik}\mathbf{Q}_{ii}\mathbf{x}(t)\\ &\quad -2\sum_{i<j}^{p}w_i w_j\mathbf{x}(t)^{\mathrm{T}}\sum_{k=1}^{2}\left(\frac{\eta_{jk}\rho_{jk}\mathbf{Q}_{ij}+\eta_{ik}\rho_{ik}\mathbf{Q}_{ji}}{2}\right)\mathbf{x}(t).\end{aligned} \tag{15}$$

Let

$$\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii} > 0, \quad i = 1, 2, \ldots, p;\ k = 1, 2 \tag{16}$$

$$\frac{\rho_{jk}\mathbf{Q}_{ij} + \rho_{i\gamma}\mathbf{Q}_{ji}}{2} - \mathbf{P}_{ij} \ge 0, \quad i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, p;\ i < j;\ k = 1, 2;\ \gamma = 1, 2 \tag{17}$$

where $\mathbf{P}_{ij} \in \Re^{n\times n}$ is a symmetric matrix for all i and j. In the following, it will be shown that (16) and (17) are the stability conditions governing the system stability of (9). From (16) and using the property that $\sum_{k=1}^{2}\eta_{ik} = 1$, we have

$$\begin{aligned}&\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii} > 0\\ &\Rightarrow \sum_{k=1}^{2}\eta_{ik}(\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii}) > 0\\ &\Rightarrow \sum_{k=1}^{2}\eta_{ik}\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii} > 0,\ i = 1, 2, \ldots, p; k = 1, 2.\end{aligned} \tag{18}$$

From (17) and using the property that $\sum_{k=1}^{2}\eta_{jk} = \sum_{\gamma=1}^{2}\eta_{i\gamma} = \sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{jk}\eta_{i\gamma} = 1$

$$\begin{aligned}&\sum_{k=1}^{2}\eta_{jk}\left(\frac{\rho_{jk}\mathbf{Q}_{ij}+\rho_{i\gamma}\mathbf{Q}_{ji}}{2} - \mathbf{P}_{ij}\right) \ge 0\\ &\Rightarrow \sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}\left(\frac{\rho_{jk}\mathbf{Q}_{ij}+\rho_{i\gamma}\mathbf{Q}_{ji}}{2} - \mathbf{P}_{ij}\right) \ge 0\\ &\Rightarrow \left(\frac{\sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}(\rho_{jk}\mathbf{Q}_{ij}+\rho_{i\gamma}\mathbf{Q}_{ji})}{2}\right)\\ &\quad - \sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}\mathbf{P}_{ij} \ge 0\\ &\Rightarrow \left(\frac{\sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}(\rho_{jk}\mathbf{Q}_{ij}+\rho_{i\gamma}\mathbf{Q}_{ji})}{2}\right) - \mathbf{P}_{ij} \ge 0\\ &\Rightarrow \left(\frac{\sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}\rho_{jk}\mathbf{Q}_{ij} + \sum_{\gamma=1}^{2}\sum_{k=1}^{2}\eta_{jk}\eta_{i\gamma}\rho_{i\gamma}\mathbf{Q}_{ji}}{2}\right)\\ &\quad - \mathbf{P}_{ij} \ge 0\\ &\Rightarrow \left(\frac{\sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}\rho_{jk}\mathbf{Q}_{ij} + \sum_{k=1}^{2}\sum_{\gamma=1}^{2}\eta_{j\gamma}\eta_{ik}\rho_{ik}\mathbf{Q}_{ji}}{2}\right)\\ &\quad - \mathbf{P}_{ij} \ge 0\\ &\Rightarrow \sum_{k=1}^{2}\left(\frac{\sum_{\gamma=1}^{2}\eta_{i\gamma}\eta_{jk}\rho_{jk}\mathbf{Q}_{ij} + \sum_{\gamma=1}^{2}\eta_{j\gamma}\eta_{ik}\rho_{ik}\mathbf{Q}_{ji}}{2}\right) - \mathbf{P}_{ij} \ge 0\\ &\Rightarrow \sum_{k=1}^{2}\left(\frac{\eta_{jk}\rho_{jk}\mathbf{Q}_{ij}+\eta_{ik}\rho_{ik}\mathbf{Q}_{ji}}{2}\right) - \mathbf{P}_{ij} \ge 0.\end{aligned} \tag{19}$$

From (15), (18) and (19), we have

$$\begin{aligned}\dot{V} &\le -\sum_{i=1}^{p}w_i w_i\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}_{ii}\mathbf{x}(t) - 2\sum_{i<j}^{p}w_i w_j\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}_{ij}\mathbf{x}(t)\\ &= -\begin{bmatrix} w_1\mathbf{x}(t)\\ w_2\mathbf{x}(t)\\ \vdots\\ w_p\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1p}\\ \mathbf{P}_{12} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{P}_{1p} & \mathbf{P}_{2p} & \cdots & \mathbf{P}_{pp}\end{bmatrix}\begin{bmatrix} w_1\mathbf{x}(t)\\ w_2\mathbf{x}(t)\\ \vdots\\ w_p\mathbf{x}(t)\end{bmatrix}\\ &= -\mathbf{z}(t)^{\mathrm{T}}\overline{\mathbf{P}}\mathbf{z}(t)\end{aligned} \tag{20}$$

where $\mathbf{z}(t) = \begin{bmatrix} w_1\mathbf{x}(t)\\ w_2\mathbf{x}(t)\\ \vdots\\ w_p\mathbf{x}(t)\end{bmatrix}$ and $\overline{\mathbf{P}} = \begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1p}\\ \mathbf{P}_{12} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{P}_{1p} & \mathbf{P}_{2p} & \cdots & \mathbf{P}_{pp}\end{bmatrix}$. It can be seen that $\dot{V} \le 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$) if $\overline{\mathbf{P}} > 0$. In conclusion, the fuzzy control system of (9) is asymptotically stable. The analysis results are summarized in the following Lemma.

Lemma 1: The fuzzy control system formed by the nonlinear plant in the form of (2) and the fuzzy controller of (6) is guaranteed to be asymptotically stable if there exist symmetric matrices $\mathbf{P}$ and $\overline{\mathbf{P}}$ such that the following LMIs hold:

$$\begin{aligned} &\mathbf{P} > 0,\ \ \overline{\mathbf{P}} > 0, \\ &\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii} > 0,\ \ i = 1,2,\ldots,p;\ k = 1,2 \\ \frac{\rho_{jk}\mathbf{Q}_{ij} + \rho_{i\gamma}\mathbf{Q}_{ji}}{2} &- \mathbf{P}_{ij} \geq 0,\ \ i = 1,2,\ldots,p;\ j = 1,2,\ldots,p; \\ &i < j;\ k = 1,2;\ \gamma = 1,2 \end{aligned}$$

where $\rho_{j1} = \rho_{j_{\min}}$ and $\rho_{j2} = \rho_{j_{\max}}$.

It can be shown that the stability conditions of Lemma 1 are less conservative than those of Theorem 1. Referring to Theorem 1 and considering that there exists $\mathbf{P} > 0$ such that $-\mathbf{Q}_{ij} = \mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij} < 0$, $i = 1,2,\ldots,p$; $j = 1,2,\ldots,p$. With these $\mathbf{P}$ and $\mathbf{Q}_{ij}$, and referring to the stability conditions of Lemma 1, it can be shown that there must exist $\mathbf{P}_{ii}$ and $\mathbf{P}_{ij}$ such that $\rho_{ik}\mathbf{Q}_{ii} - \mathbf{P}_{ii} > 0$, $((\rho_{jk}\mathbf{Q}_{ij} + \rho_{i\gamma}\mathbf{Q}_{ji})/2) - \mathbf{P}_{ij} \geq 0$ and $\overline{\mathbf{P}} > 0$. As $\mathbf{Q}_{ii} > 0$ and let $\mathbf{P}_{ii} = \delta\mathbf{P}$, there must exist a sufficiently small δ such that $\rho_{ik}\mathbf{Q}_{ii} - \delta\mathbf{P} > 0$. Similarly, as $\mathbf{Q}_{ij} > 0$ and let $\mathbf{P}_{ij} = \mathbf{0}$, it can be seen that $((\rho_{jk}\mathbf{Q}_{ij} + \rho_{i\gamma}\mathbf{Q}_{ji})/2) > 0$. Hence, we have $\overline{\mathbf{P}} = \begin{bmatrix} \delta\mathbf{P} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{P} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \delta\mathbf{P} \end{bmatrix} > 0$. However, the solution of Lemma 1 may not be the solution of Theorem 1 as there may exist a solution of Lemma 1 that some $\mathbf{Q}_{ij} < 0$. Hence, it can be concluded that the stability conditions of Lemma 1 are less conservative than those of Theorem 1.

IV. Numerical Example

A numerical example [8] will be given to show the merits of the stability conditions of Lemma 1. A fuzzy plant model with the following two fuzzy rules is considered.

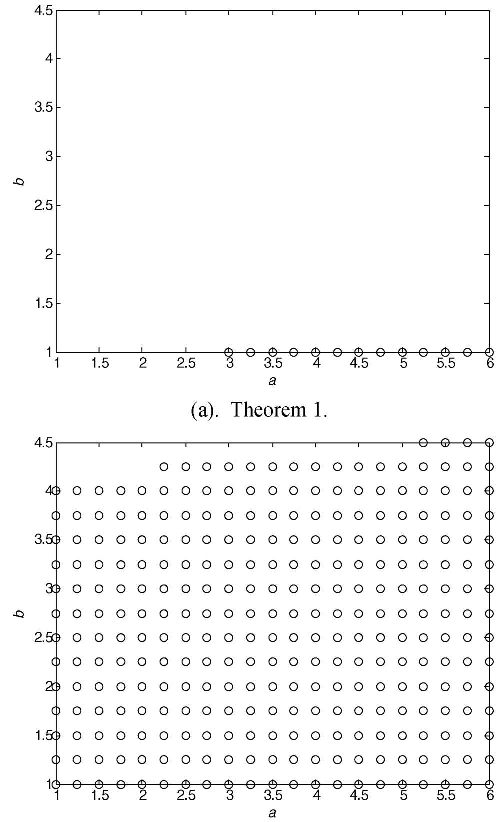
Rule 1: IF $x_1(t)$ is M_1^1 THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{B}_1 u$

Rule 2: IF $x_1(t)$ is M_1^2 THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t) + \mathbf{B}_2 u$

where $\mathbf{A}_1 = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{A}_2 = \begin{bmatrix} a & -10 \\ 1 & 1 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{B}_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}$. It is assumed that the membership functions of fuzzy controller are different from those of the fuzzy plant model. The feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ are designed under the same criteria as given in [8] such that the eigenvalues of $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ are located at -1 and -15. The stability conditions of Theorem 1 and Lemma 1 will be employed to testify the stability of the closed-loop system. For the proposed approach, it is assumed that $\rho_{j1} = \rho_{j_{\min}} = 0.6667$ and $\rho_{j2} = \rho_{j_{\max}} = 1.5$. Fig. 1 shows the feasible areas for the stability conditions of Theorem 1 and Lemma 1 respectively for $1 \leq a \leq 6$ and $1 \leq b \leq 4.5$. It can be seen from Fig. 1 that Lemma 1 provides more relaxed results than those of Theorem 1. Furthermore, the stability conditions given in [6]–[10] cannot be applied to testify the system stability as the premises of the fuzzy plant model and the fuzzy controller are different.

V. Conclusion

Stability of fuzzy control systems subject to parameter uncertainties has been analyzed. The fuzzy control system is formed by a nonlinear plant subject to parameter uncertainties and a fuzzy controller. The nonlinear plant subject to parameter uncertainties is represented by a fuzzy plant model with uncertain grades of memberships. Relaxed stability conditions have been derived to guarantee the stability of this class of fuzzy control system. A numerical has been presented to show the effectiveness of the proposed approach.



(a). Theorem 1.

(b). Lemma 1.

Fig. 1. Feasible area of the stability conditions for example 1.

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