Design and Stability Analysis of Fuzzy Model Based Nonlinear Controller for Nonlinear Systems Using Genetic Algorithm[1]

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Abstract – This paper presents the stability analysis of fuzzy model based nonlinear control systems, and the design of nonlinear gains and feedback gains of the nonlinear controller using genetic algorithm with arithmetic crossover and non-uniform mutation. A stability condition will be derived based on Lyapunov's stability theory with a smaller number of Lyapunov conditions. The solution of the stability conditions are also determined using GA. An application example of stabilizing a cart-pole type inverted pendulum system will be given to show the stabilizability of the nonlinear controller.

I. INTRODUCTION

Fuzzy control has been a hot research topic. Despite the lack of a concrete theoretical basis, many successful applications on fuzzy control were reported in various areas such as, sludge wastewater treatment [1], control of cement kiln [2], etc. However, without an in-depth analysis, the design may come with no guarantees of system stability and good system performance. Recently, stability analysis of fuzzy control systems based on a TSK (Takagi-Sugeno-Kang) fuzzy plant model [3, 7] was reported. The advantage of using the fuzzy model is that a nonlinear plant can be represented as a weighted sum of linear sub-systems, so that some linear or nonlinear control theories can possibly be applied to design the controller. Different stability conditions for this class of fuzzy control systems were derived. In [4, 5-7, 16, 19], the Lyapunov stability theory was employed to analyze the system stability. Sliding mode theory was employed in [8] to help the analysis. In [13-15], the stability conditions were derived in terms of some matrix measures of the system matrices. An LMI-based design of fuzzy controllers can be found in [10-12]. A switching controller [17] and other controller [16, 18] were also proposed to tackle nonlinear systems based on the TSK fuzzy plant model.

Genetic Algorithm (GA) is a powerful random search technique to handle optimization problems [1-6, 17]. This is especially useful for complex optimization problems with a large number of parameters that make global analytical solutions difficult to obtain. It has been widely applied in different areas such as fuzzy control [20], tuning of parameters of neural networks [21], eBook applications [22], load forecasting [23], etc.

In this paper, we focus on the system stability and present a stability analysis of fuzzy model based nonlinear control systems. A nonlinear controller is proposed to control a system represented by a TSK fuzzy plant model [3]. The proposed controller has a similar structure of the fuzzy controller reported in [6]. The main difference is that the weights in the nonlinear controller are signed but those in the fuzzy controller [6] must be positive (because they are the membership function values). Wang *et. al*. derived a stability condition for a TSK fuzzy model based systems using Lyapunov stability theory [6]. A sufficient condition for the system stability is obtained by finding a common Lyapunov function for all the fuzzy sub-control systems. For a TSK fuzzy plant model with p rules, a fuzzy controller with p rules (p sub-controllers) is used to close the feedback loop, and $p(p+1)/2$ Lyapunov conditions are required. In this paper, the numbers of sub-controllers of the nonlinear controller need not be the same as that of the TSK fuzzy plant model. By allowing both positive and negative weighting values in the proposed controller, the number of Lyapunov conditions can be reduced to p. We also provide a way of designing the nonlinear gains and the feedback gains of the nonlinear controller. The task of finding the common Lyapunov function can readily be formulated into a linear matrix inequality (LMI) problem [9]. The GA with arithmetic crossover and non-uniform mutation [27] will be used to help finding the solution of the derived stability conditions, and determine the feedback gains of the sub-controllers.

II. FUZZY PLANT MODEL AND NONLINEAR CONTROLLER

We consider a multivariable nonlinear control system comprising a TSK fuzzy plant model and a nonlinear controller connected in closed-loop.

A. TSK Fuzzy Plant Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format,

Rule i : IF $f_1(\mathbf{x}(t))$ is M_1^i and … and $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \ldots, \Psi$, $i = 1, 2, \ldots, p$, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big), \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } i \tag{3}$$

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$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(x_1(t)) \times \mu_{\mathrm{M}_2^i}(x_2(t)) \times \cdots \times \mu_{\mathrm{M}_n^i}(x_n(t))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(x_1(t)) \times \mu_{\mathrm{M}_2^k}(x_2(t)) \times \cdots \times \mu_{\mathrm{M}_n^k}(x_n(t))\right)} \quad (4)$$

is a known nonlinear function and $\mu_{\mathrm{M}_\alpha^i}(x_\alpha(t))$, $\alpha = 1, 2, \ldots, n$, are known membership functions corresponding to the fuzzy terms M_α^i. (Thus, we assume that the TSK fuzzy plant model is known.)

B. Nonlinear Controller

A nonlinear controller consisting of c sub-controllers is proposed to close the feedback loop. The control output of the nonlinear controller is defined as,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \quad (5)$$

where $\mathbf{G}_j \in \Re^{m\times n}$, $j = 1, 2, \ldots, c$, are the feedback gain vectors that are to be designed, and

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1 \quad (6)$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}^j}(\mathbf{x}(t))}{\sum_{k=1}^{c}\mu_{\mathrm{N}^k}(\mathbf{x}(t))} \quad (7)$$

is a nonlinear function of $\mathbf{x}(t)$, and $\mu_{\mathrm{N}^j}(\mathbf{x}(t))$, $j = 1, 2, \ldots, c$, are nonlinear gains to be designed. It should be noted that the nonlinear controller does not require $m_j(\mathbf{x}(t)) \in [0 \;\; 1]$ for all j.

III. Stability Analysis

A closed-loop system can be obtained by combining (2) and (5). Writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, the fuzzy model based nonlinear control system then becomes,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t) \quad (8)$$

where

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j \quad (9)$$

To investigate the stability of the fuzzy model based nonlinear control system of (8), we consider the following Lyapunov function in quadratic form,

$$V(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \quad (10)$$

where $\mathbf{P} \in \Re^{n\times n}$ is a symmetric positive definite matrix. Then,

$$\dot{V}(\mathbf{x}(t)) = \frac{1}{2}\left(\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\right) \quad (11)$$

From (8), (11) and the property that $\sum_{i=1}^{p} w_i = \sum_{j=1}^{c} m_j = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j = 1$, we have,

$$\begin{aligned}
&\dot{V}(\mathbf{x}(t)) \\
&= \frac{1}{2}\left[\left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right] \\
&= \frac{1}{2}\left\{\begin{aligned}&\left[\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m\right)\mathbf{x}(t)\right]^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \\ &+ \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m\right)\mathbf{x}(t)\end{aligned}\right\} \\
&= \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{H}_m^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right)\mathbf{x}(t) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{H}_{ij} - \mathbf{H}_m\right)\right]\mathbf{x}(t) \\
&= -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)
\end{aligned} \quad (12)$$

where $\mathbf{H}_m \in \Re^{n\times n}$ is a stable symmetric matrix, which will be discussed later. $\mathbf{Q}_m \in \Re^{n\times n}$ is a symmetric positive definite matrix and $\mathbf{Q}_{ij} \in \Re^{n\times n}$ is a symmetric matrix. They are defined as,

$$\mathbf{Q}_m = -\left(\mathbf{H}_m^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right) \quad (13)$$

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right),\; i = 1, 2, \ldots, p;\; j = 1, 2, \ldots, c \quad (14)$$

From (12),

$$\dot{V}(\mathbf{x}(t)) = -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{j=1}^{c} m_j \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t) \quad (15)$$

and we set,

$$m_j = \frac{\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)}{\sum_{k=1}^{c}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]} \text{ for } j = 1, 2, \ldots, c \quad (16)$$

By comparing (16) to (7), (16) gives the design of m_j, $j = 1, 2, \ldots, c$, such that $\mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)$ and satisfies the condition of (6). Considering the denominator at the right hand side of (16), we have,

$$\begin{aligned}
&\sum_{k=1}^{c}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right] \\
&= \sum_{i=1}^{p} w_i\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m\right)\mathbf{x}(t)\right]
\end{aligned} \quad (17)$$

We choose $\mathbf{Q}_{ik}$ and $\mathbf{Q}_m$ such that,

$$\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m > \mathbf{0} \text{ for } i = 1, 2, \ldots, p \quad (18)$$

As $w_i(\mathbf{x}(t)) \in [0 \;\; 1]$ for all i, and at least one of the $w_i \neq 0$ (a property of the TSK fuzzy plant model), (18) implies that (17) will be always greater than or equal to zero. It is equal to zero only when $\mathbf{x}(t)=\mathbf{0}$. Under this condition, the output of the nonlinear controller of (5) should be zero and we choose $m_j=\frac{1}{c}$ for satisfying the condition of (6). From (15) and (16),

$$\dot{V}(\mathbf{x}(t))=-\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t)-\frac{\sum_{j=1}^{c}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p}w_i\mathbf{Q}_{ij}-\mathbf{Q}_m\right)\mathbf{x}(t)\right]^2}{2\sum_{k=1}^{c}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}-\mathbf{Q}_m\right)\mathbf{x}(t)\right]} \tag{19}$$

As the second term at the right side of (19) is semi-positive definite, we have,

$$\dot{V}(\mathbf{x}(t))\le-\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t)\le 0 \tag{20}$$

Hence, we can conclude that the fuzzy model based nonlinear control system is asymptotically stable. The problem left is how to determine $\mathbf{Q}_m$. Considering (18), if $\sum_{k=1}^{c}\mathbf{Q}_{ik}>\mathbf{0}$ for i = 1 , 2, .,. p, it can be shown that there exists a $\mathbf{Q}_m$ such that $\sum_{k=1}^{c}\mathbf{Q}_{ik}-c\mathbf{Q}_m>\mathbf{0}$ for i = 1 , 2, .,. p using the following theorem.

Theorem 1 (Spectral Shift) [24]: Let λ_1, λ_2, $\cdots$, λ_n, be the eigenvalues of a matrix $\mathbf{A}\in\Re^{n\times n}$. The eigenvalues of $\mathbf{A}-\varepsilon\mathbf{I}$ are $\lambda_1-\varepsilon$, $\lambda_2-\varepsilon$, $\cdots$ $\lambda_n-\varepsilon$ where ε is a scalar.

Proof:

Let $\mathbf{Q}_i=\sum_{k=1}^{c}\mathbf{Q}_{ik}>\mathbf{0}$, i = 1, 2, .,. p. By using the spectral shift property of Theorem 1, it can be seen that $\sum_{k=1}^{c}\mathbf{Q}_{ik}-\varepsilon\mathbf{I}=\sum_{k=1}^{c}\left(\mathbf{Q}_{ik}-\frac{\varepsilon}{c}\mathbf{I}\right)>\mathbf{0}$ if $\min_i\lambda_{\min}(\mathbf{Q}_i)>\varepsilon>0$, where $\min_i\lambda_{\min}(\mathbf{Q}_i)$ denotes the smallest eigenvalues among $\mathbf{Q}_i$, $\mathbf{I}$ is the identity matrix. By comparing $\sum_{k=1}^{c}\mathbf{Q}_{ik}-c\mathbf{Q}_m>0$ of (18) with $\sum_{k=1}^{c}\mathbf{Q}_{ik}-\varepsilon\mathbf{I}>0$ term by term, we have $c\mathbf{Q}_m=\varepsilon\mathbf{I}\Rightarrow\mathbf{Q}_m=\frac{\varepsilon}{c}\mathbf{I}>\mathbf{0}$. Consequently, we can conclude that if $\sum_{k=1}^{c}\mathbf{Q}_{ik}>\mathbf{0}$, there must exist a positive definite matrix $\mathbf{Q}_m$ such that $\sum_{k=1}^{c}\mathbf{Q}_{ik}-c\mathbf{Q}_m>\mathbf{0}$. In the stability analysis, we need a stable matrix $\mathbf{H}_m$ to guarantee the system stability. The existence of $\mathbf{H}_m$ will be shown as the follows. By multiplying $\mathbf{P}^{-1}$ to both side of (13), we have $-\mathbf{P}^{-1}\mathbf{Q}_m\mathbf{P}^{-1}=\mathbf{P}^{-1}\mathbf{H}_m{}^{\mathrm{T}}+\mathbf{H}_m\mathbf{P}^{-1}$

As $\mathbf{Q}_m=\frac{\varepsilon}{c}\mathbf{I}$, $\Rightarrow-\frac{\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1}}{c}=\mathbf{P}^{-1}\mathbf{H}_m{}^{\mathrm{T}}+\mathbf{H}_m\mathbf{P}^{-1}$ $\Rightarrow-\frac{\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1}}{c}=\left(-\mathbf{P}^{-1}\right)\left(-\mathbf{H}_m{}^{\mathrm{T}}\right)+\left(-\mathbf{H}_m\right)\left(-\mathbf{P}^{-1}\right)$. Let $\overline{\mathbf{Q}}=\frac{\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1}}{c}$ which is symmetric positive definite matrix, and $\overline{\mathbf{P}}=\overline{\mathbf{P}}^{\mathrm{T}}=-\mathbf{P}^{-1}$ which is a symmetric negative definite matrix and $\overline{\mathbf{H}}_m=\overline{\mathbf{H}}_m{}^{\mathrm{T}}=-\mathbf{H}_m$, we have a Lyapunov equation $-\overline{\mathbf{Q}}=\overline{\mathbf{P}}^{\mathrm{T}}\overline{\mathbf{H}}_m+\overline{\mathbf{H}}_m\overline{\mathbf{P}}$. Once $\overline{\mathbf{P}}$ is known, a stable matrix $\mathbf{H}_m$ can be solved. **QED**

From above, we obtain $\mathbf{Q}_m=\frac{\varepsilon}{c}\mathbf{I}>\mathbf{0}$ and prove the existence of $\mathbf{H}_m$. The stable matrix $\mathbf{H}_m$ in (13) is not necessary to be known as the nonlinear controller of (16) depends on $\mathbf{Q}_m$ but not $\mathbf{H}_m$.

A sufficient condition for the stability of the fuzzy model based nonlinear control system can be summarised by the following lemma.

Lemma 1: *A fuzzy model based nonlinear control system of* (8) *is guaranteed to be stable if we choose the nonlinear gains of the nonlinear controller of* (5) *as,*

$$\begin{cases}\mu_{N^j}(\mathbf{x}(t))=\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{p}w_i\mathbf{Q}_{ij}-\frac{\varepsilon}{c}\mathbf{I}\right)\mathbf{x}(t) \text{ when } \mathbf{x}(t)\neq\mathbf{0}\\ \mu_{N^j}(\mathbf{x}(t))=\frac{1}{c} \text{ when } \mathbf{x}(t)=\mathbf{0}\end{cases} \quad \text{for } j = 1, 2, \ldots, c$$

$\min_i\lambda_{\min}\left(\sum_{k=1}^{c}\mathbf{Q}_{ik}\right)>\varepsilon>0$ *and there is a common solution of* $\mathbf{P}$ *for the following p linear matrix inequalities,*

$\sum_{k=1}^{c}\mathbf{Q}_{ik}>\mathbf{0}$ *for all* i = 1, 2, ..., p

where,

$\mathbf{Q}_{ij}=-\left(\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{H}_{ij}\right)$ *for* i = 1, 2, .., p; j = 1, 2, .., c

$\mathbf{H}_{ij}=\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j$

Lemma 1 states the way of choosing the nonlinear gains of the nonlinear controller. The number of sub-controllers is not necessarily the same as that of the TSK fuzzy plant model. This gives a flexibility of designing the nonlinear controller. With a smaller number of sub-controllers, the nonlinear controller is simpler in structure and lower in cost. The number of linear matrix inequalities is p, instead of $p(p+1)/2$ as stated in [6].

IV. Solving the Stability Conditions and Obtaining the Feedback Gains

In this section, the problems of solving the stability conditions derived in the previous section and obtaining the feedback gains of the proposed nonlinear controller will be tackled using the GA with arithmetic crossover and non-uniform mutation [27]. From Lemma 1, the closed-loop control system formed by (2) and (5) is stable if there exist a transformation matrix $\mathbf{P}$ and $\mathbf{G}_j$, j =1, 2, .., p, satisfying the following condition,

$$-\sum_{j=1}^{c}\left[\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\right]>\mathbf{0} \text{ for } i = 1, 2, .., p \tag{21}$$

Using GA, we can find $\mathbf{P}=\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$ and $\mathbf{G}_j=\begin{bmatrix} G_{11}^j & G_{11}^j & \cdots & G_{1n}^j \\ G_{21}^j & G_{22}^j & \cdots & G_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}^j & G_{m2}^j & \cdots & G_{mn}^j \end{bmatrix}$ such that the conditions of (21) are satisfied. The fitness function is defined as follows,

$$fitness=\sum_{i=1}^{p} n_i\lambda_{\max}\left(\sum_{j=1}^{c}\left[\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\right]\right) \tag{22}$$

where $n_i \geq 0$, $i = 1, 2, .., p$, is a variable to be tuned, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of the argument. The problems of finding $\mathbf{P}$ and $\mathbf{G}_j$ are now formulated into a minimization problem. The aim is to minimize the fitness function of (22) with $\mathbf{P}$, $\mathbf{G}_j$ and n_i using the GA with arithmetic crossover and non-uniform mutation [27]. As $\mathbf{P}$, $\mathbf{G}_j$ and n_i are the variables of the fitness function of (22), they are used to form the genes of the chromosomes. The finding of the solution to this minimization problem, however, does not imply that the conditions of (21) are satisfied. Hence, different n_i, $i = 1, 2, .., p$, may need to be used to weight the terms of (22) in order to change the significance of different terms on the right hand side of (22). For instance, one of the terms in (22) is very positive, which returns a very large fitness value. Under this case, the conditions of (21) are not satisfied. A large value of n_i corresponding to that term can be used to attenuate the effect of that term in the fitness function.

The procedure for finding the nonlinear controller can be summarised as follows.

Step I) Obtain the mathematical model of the nonlinear plant to be controlled.

Step II) Obtain the TSK fuzzy plant model for the system stated in step I) by means of a fuzzy modeling method. For example, the method proposed in [3, 7].

Step III) Determine the number of sub-controllers of the nonlinear controller. Solve $\mathbf{P}$, $\mathbf{G}_j$ and n_i, j = 1, 2, .., c; i = 1 ,2, .., p, with the fitness function defined in (22) using GA.

Step IV) Design the nonlinear gains of the nonlinear controller based on Lemma 1.

V. Application Example

An application example on stabilising a cart-pole type inverted pendulum system [6] is given in this section. A nonlinear controller will be used to control the plant. Simulation results will be given. We shall see that the number of LMIs involved is p. The nonlinear controllers will be designed based on the procedure given in section IV.

Step I) Fig. 1 shows the diagram of the cart-pole type inverted pendulum system. The dynamic equation of the cart-pole type inverted pendulum system is given by,

$$\ddot{\theta}(t)=\frac{g\sin(\theta(t))-aml\dot{\theta}(t)^2\sin(2\theta(t))/2-a\cos(\theta(t))u(t)}{4l/3-aml\cos^2(\theta(t))} \tag{23}$$

where θ is the angular displacement of the pendulum, g = 9.8m/s^2 is the acceleration due to gravity, m = 2kg is the mass of the pendulum, $a = 1/(m + M)$, M = 8kg is the mass of the cart, $2l$ = 1m is the length of the pendulum, and u is the force applied to the cart. The objective of this application example is to design a fuzzy controller to close the feedback loop of (23) such that $\theta = 0$ at steady state.

Step II) The nonlinear plant can be represented by a fuzzy model with four fuzzy rules. The i-th rule is given by,

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND $f_2(\mathbf{x}(t))$ is M_2^i

THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i u(t)$ for $i = 1, 2, 3, 4$ (24)

so that the system dynamics is described by,

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{4} w_i\left(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i u(t)\right) \tag{25}$$

where $\mathbf{x}(t)=[x_1(t)\quad x_2(t)]^{\mathrm{T}}=[\theta(t)\quad \dot{\theta}(t)]^{\mathrm{T}}$, $\theta(t)\in\left[\theta_{\min}\quad \theta_{\max}\right]=\left[-\frac{22\pi}{45}\quad \frac{22\pi}{45}\right]$ and $\dot{\theta}(t)\in[\dot{\theta}_{\min}\quad \dot{\theta}_{\max}]=[-5\quad 5]$; $f_1(\mathbf{x}(t))=\frac{g-amlx_2(t)^2\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right)$ and $f_2(\mathbf{x}(t))=-\frac{a\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}$; $\mathbf{A}_1=\mathbf{A}_2=\begin{bmatrix} 0 & 1 \\ f_{1_{\min}} & 0 \end{bmatrix}$ and $\mathbf{A}_3=\mathbf{A}_4=\begin{bmatrix} 0 & 1 \\ f_{1_{\max}} & 0 \end{bmatrix}$; $\mathbf{B}_1=\mathbf{B}_3=\begin{bmatrix} 0 \\ f_{2_{\min}} \end{bmatrix}$ and $\mathbf{B}_2=\mathbf{B}_4=\begin{bmatrix} 0 \\ f_{2_{\max}} \end{bmatrix}$; $f_{1_{\min}}=9$ and $f_{1_{\max}}=18$, $f_{2_{\min}}=-0.1765$ and $f_{2_{\max}}=-0.0052$; $w_i=\frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))}{\sum_{l=1}^{4}\left(\mu_{\mathrm{M}_1^l}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^l}(f_2(\mathbf{x}(t)))\right)}$; $\mu_{\mathrm{M}_1^\beta}(f_1(\mathbf{x}(t)))=\frac{-f_1(\mathbf{x}(t))+f_{1_{\max}}}{f_{1_{\max}}-f_{1_{\min}}}$ for β = 1, 2; $\mu_{\mathrm{M}_1^\delta}(f_1(\mathbf{x}(t)))=1-\mu_{\mathrm{M}_1^1}(f_1(\mathbf{x}(t)))$ for δ = 3, 4;

$$\mu_{M_2^\varepsilon}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}} \quad \text{for} \quad \varepsilon = 1, 3$$

and $\mu_{M_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{M_2^1}(f_2(\mathbf{x}(t)))$ for ϕ = 2, 4 are the membership functions as shown in Fig. 2. (Details about the derivation of the TSK fuzzy plant model for the cart-pole type inverted pendulum system can be found in [5].)

Step III) When a nonlinear controller having 4 sub-controllers is designed for the plant of (25), we have,

$$u(t) = \sum_{j=1}^{4} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{26}$$

In order to guarantee the closed-loop system stability and obtain the feedback gains of the nonlinear controller of (26), from (22), we have to solve the $\mathbf{P}$, $\mathbf{G}_j$ and n_i, j = 1, 2, 3, 4; i = 1, 2, 3, 4, using GA with the following fitness function,

$$fitness = \sum_{i=1}^{4} n_i \lambda_{\max}\left[\sum_{j=1}^{4}\left[(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\right]\right] \tag{27}$$

The minimum and maximum values of each element $\mathbf{P}$ are chosen to be −1 and 1 respectively. The minimum and maximum values of each element of $\mathbf{G}_1$ to $\mathbf{G}_4$ are chosen to be 0 and 4500 respectively. The minimum and maximum values of n_i are chosen to be 0 and 10 respectively. The population size is 10 and the initial values of $\mathbf{P}$, $\mathbf{G}_j$ and n_i are randomly generated. After applying the GA process, we obtain $\mathbf{P} = \begin{bmatrix} 15.0698 & -2.2784 \\ -17.9689 & -3.3757 \end{bmatrix}$ and $\mathbf{G}_1 = [4176.4868 \quad 4438.2388]$, $\mathbf{G}_2 = [4200.1314 \quad 3710.2710]$, $\mathbf{G}_3 = [4223.6645 \quad 3631.4680]$ and $\mathbf{G}_4 = [4308.6079 \quad 4053.5941]$.

Step V) According to Lemma 1, the nonlinear gains are designed as,

$$\begin{cases} \mu_{N^j}(\mathbf{x}(t)) = \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ij} - \frac{\varepsilon}{4}\mathbf{I}\right)\mathbf{x}(t) \text{ when } \mathbf{x}(t) \neq \mathbf{0} \\ \mu_{N^j}(\mathbf{x}(t)) = \frac{1}{4} \text{ when } \mathbf{x}(t) = \mathbf{0} \end{cases} \text{for } j = 1, 2, 3, 4 \tag{28}$$

As $\min_i \lambda_{\min}\left(\sum_{k=1}^{c}\mathbf{Q}_{ik}\right) = 1.3982 > \varepsilon > 0$, we choose $\varepsilon = 0.1$.

Fig. 3 and Fig. 4 show the responses of the system states under the initial conditions of $\mathbf{x}(0) = \begin{bmatrix} \frac{22\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}(0) = \begin{bmatrix} \frac{11\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}(0) = \begin{bmatrix} -\frac{11\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{x}(0) = \begin{bmatrix} -\frac{22\pi}{45} & 0 \end{bmatrix}^{\mathrm{T}}$.

From this example, it can be seen that the number of LMIs is fixed to be 4 (the number of rules of the TSK fuzzy plant model), which will not be affected, by the number of sub-controllers of the nonlinear controller.

VI. Conclusion

The stability analysis and design of TSK fuzzy model based nonlinear control systems have been discussed. A stability criterion has been derived. This criterion involves p linear matrix inequalities irrespective of the number of the sub-controllers, where p is the number of rules of the TSK fuzzy plant model. The number of sub-controllers of the nonlinear controller need not be the same as that of the TSK fuzzy plant model. A design on the nonlinear gains of the nonlinear controller has been presented. Genetic algorithm has been used to find the solution to the stability conditions and determine the feedback gains of the sub-controllers. An application example has been used to illustrate the stabilizability of the proposed nonlinear controllers and the design procedure.

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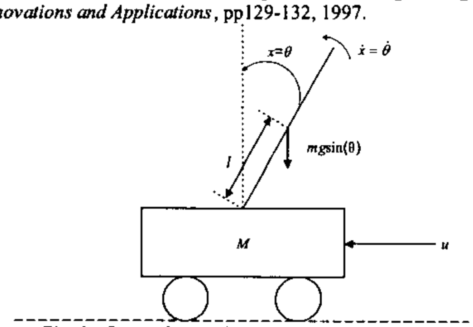

Fig. 1. Cart-pole type inverted pendulum system.

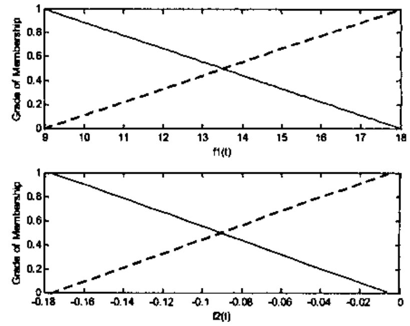

Fig. 2. Membership functions of the cart-pole type inverted pendulum.

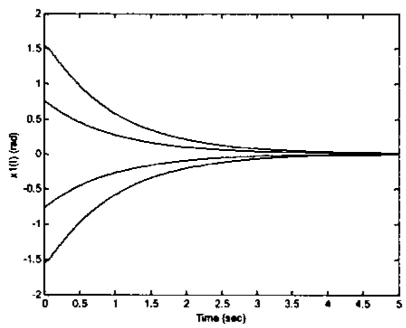

Fig. 3. Responses of $x_1(t)$ of the cart-pole type inverted pendulum system.

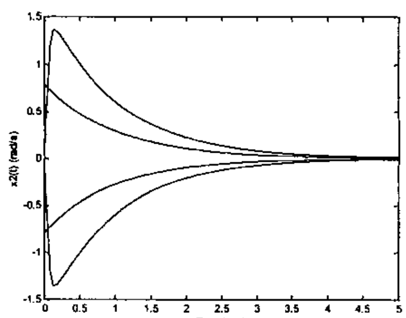

Fig. 4. Responses of $x_2(t)$ of the cart-pole type inverted pendulum system.