2006 IEEE International Conference on Fuzzy Systems
Sheraton Vancouver Wall Centre Hotel, Vancouver, BC, Canada
July 16-21, 2006

LMI-based stability and performance conditions for continuous-time nonlinear systems in Takagi-Sugeno's form

H.K. Lam, *Member, IEEE* and F.H.F. Leung, *Senior Member, IEEE*

***Abstract*— This paper presents the stability analysis and performance design of the continuous-time fuzzy-model-based control systems. A nonlinear controller will be proposed to stabilize the nonlinear systems in Takagi-Sugeno's form. LMI-based stability conditions will be derived the parameter-dependent Lyapunov function to guarantee the system stability. Furthermore, based on the commonly-used performance index, LMI-based performance conditions will be derived to achieve the system performance. A numerical example will be given to illustrate the effectiveness of the proposed approach.**

I. Introduction

Fuzzy-model-based control approach offers a systematic and effective framework to investigate the system stability. In general, the stability analysis is carried out based on the TS-fuzzy models which represent the system dynamics of the nonlinear plants. In the last two decade, fruitful stability analysis results [3]-[13] were obtained to guarantee the system stability of the continuous-time or discrete-time fuzzy-model-based control systems. Basic LMI (linear matrix inequality)-based stability conditions were developed in [3]-[4] using Lyapunov-based approach for the fuzzy-model-based control systems. In [4], an efficient design technique, namely parallel distributed compensation (PDC) technique, was proposed to design the fuzzy controllers. Further relaxed stability conditions were then obtained in [5]-[11] based on the PDC-design technique. In [3]-[11], the stability analysis of the fuzzy-model-based control systems was investigated based on a parameter-independent Lyapunov function (PILF). The stability analysis was extended to parameter-dependent Lyapunov function (PDLF) for continuous-time [11] and discrete-time system [12]-[13]. Furthermore, in [12]-[13], a non-PDC nonlinear controller was proposed to stabilize the discrete-time nonlinear systems represented by TS-fuzzy models. It has been shown that the non-PDC control laws with PDLF-based approach can further relax the stability result.

In the continuous-time PDC approach with PDLY, two difficulties have to face during the system analysis: 1) unlike the discrete-time case, the continuous-time case using PDLF will generate bi-polar time derivative information of the membership functions which increases the difficulty on stability analysis, 2) the resultant stability conditions cannot be simply expressed in LMI forms. In [11], to deal with the problem 1, the bi-polar time derivative information is represented by some weighted functions. However, the number of stability condition will be increased by the multiplication property of the fuzzy-model-based approach. To deal with the problem 2, some positive-definitive terms were added to generate terms in quadratic form. Stability conditions in LMI form can be generated by using the Schur complement technique. However, stability results will be degraded by the additional positive-definitive terms added. Furthermore, the dimension of the matrices in the stability conditions will be increased by the Schur complement technique.

In this paper, the non-PDC design approach using the PDLF proposed in [12]-[13] will be extended to the continuous-time nonlinear systems. To deal with the problem 1, the property of the membership functions, which allows introducing some free matrices, will be employed during the stability analysis. Unlike the weighted-sum representation of the time derivative information in [11] which will increase the order of the multiplication, our approach converts the time derivative information into additive terms only. The difficulty in problem 1 can thus be alleviated. To deal with the problem 2, the non-PDC control laws will be employed. As some of the nonlinear terms can be compensated by the non-PDC control laws during the system analysis, the order of the multiplication can be further reduced and, more importantly, the stability conditions can be expressed in LMI forms without introducing extra positive-definite terms. LMI-based stability conditions will be derived using the PDLF-based approach to guarantee the stability of the fuzzy-model-based systems. In order to design the system performance, a commonly-used performance index will be employed to measure quantitatively the system performance. Based on this performance index, LMI-based performance conditions will be derived to aid the design of the system performance.

This paper is organized as follows. In section II, the fuzzy model and the non-PDC nonlinear controller will be presented. In section III, LMI-based stability and performance conditions will be derived. In section IV, a numerical example will be presented to illustrate effectiveness of the proposed approach. A conclusion will be drawn in section V.

II. Fuzzy Model and Non-PDC Nonlinear Controller

A multivariable fuzzy-model-based control system

This work was supported in part by the Division of Engineering, The King's College London and Centre for Multimedia Signal Processing, Department of Electronic and Information Engineering, The Hong Kong Polytechnic University.

H.K. Lam is with the Division of Engineering, The King's College London, Strand, London, WC2R 2LS, United Kingdom (e-mail: hak-keung.lam@kcl.ac.uk).

F.H.F. Leung is with Centre for Multimedia Signal Processing, Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (e-mail: enfrank@polyu.edu.hk).

comprising a nonlinear plant represented by a fuzzy model and a non-PDC nonlinear controller connected in closed-loop will be considered.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$; $i = 1, 2, ..., p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big) \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0\quad 1] \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, …, \Psi$, are the grade of membership corresponding to the fuzzy term of M_α^i.

B. Non-PDC Nonlinear Controller

The non-PDC nonlinear controller for the nonlinear plant represented by the fuzzy model of (2) is proposed as follows.

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{\Gamma}(\mathbf{x}(t))^{-1}\mathbf{x}(t) + \sum_{j=1}^{p}\dot{w}_j(\mathbf{x}(t))\overline{\mathbf{G}}_j\mathbf{\Gamma}(\mathbf{x}(t))^{-1}\mathbf{x}(t) \tag{5}$$

where $\mathbf{G}_j \in \Re^{m\times n}$ and $\overline{\mathbf{G}}_j \in \Re^{m\times n}$ are the feedback gains to be designed; $\mathbf{\Gamma}(\mathbf{x}(t)) = \mathbf{\Gamma}(\mathbf{x}(t))^{\mathrm{T}} = \left(\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k\right)$; $\mathbf{P}_k = \mathbf{P}_k^{\mathrm{T}} \in \Re^{n\times n} > 0$, $k = 1, 2, ..., p$.

Remark 1: As $\mathbf{P}_k$, $k = 1, 2, ..., p$, is a positive definite matrix and with the property of $w_k(\mathbf{x}(t))$ shown in (3), $\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k$ is thus a non-singular matrix which implies the existence of $\mathbf{\Gamma}(\mathbf{x}(t))^{-1} = \left(\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k\right)^{-1}$.

III. Stability Analysis and Performance Design

In this section, the system stability and performance design of the fuzzy-model-based control system will be presented. In the following analysis, $w_i(\mathbf{x}(t))$, $\dot{w}_j(\mathbf{x}(t))$ and $\mathbf{\Gamma}(\mathbf{x}(t))$ are denoted by $w_i, \dot{w}_j$ and $\mathbf{\Gamma}$ respectively for simplicity, and the properties that $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j = 1$ and $\sum_{i=1}^{p}\dot{w}_i(\mathbf{x}(t)) = 0$ will be used. From (2) and (5), the fuzzy-model-based control system is defined as follows.

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(\sum_{j=1}^{p} w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t) + \sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\right)\right)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\big(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\mathbf{\Gamma}^{-1}\big)\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{B}_i\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t) \tag{6}$$

A. Stability Analysis

The stability of the fuzzy-model-based control system of (6) will be investigated. Considering the following Lyapunov function candidate,

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) \tag{7}$$

From (6) and (7), we have,

$$\dot{V}(t) = \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{x}}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t)$$
$$= \left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\big(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\mathbf{\Gamma}^{-1}\big)\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{B}_i\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{x}(t)$$
$$+ \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\big(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\mathbf{\Gamma}^{-1}\big)\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{B}_i\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\right)$$
$$- \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{x}(t)^{\mathrm{T}}\Big(\mathbf{A}_i^{\mathrm{T}}\mathbf{\Gamma}^{-1} + \mathbf{\Gamma}^{-1}\mathbf{A}_i + \mathbf{\Gamma}^{-1}\big(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\big)\mathbf{\Gamma}^{-1}\Big)\mathbf{x}(t)$$
$$+ \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\big(\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j\big)\mathbf{\Gamma}^{-1}\mathbf{x}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\big(\mathbf{\Gamma}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{\Gamma} + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\big)\mathbf{\Gamma}^{-1}\mathbf{x}(t)$$
$$+ \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\big(\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j\big)\mathbf{\Gamma}^{-1}\mathbf{x}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) \tag{8}$$

Let $\overline{\mathbf{x}}(t) = \mathbf{\Gamma}^{-1}\mathbf{x}(t)$, and put $\mathbf{\Gamma} = \sum_{j=1}^{p} w_j\mathbf{P}_j$, $\dot{\mathbf{\Gamma}} = \sum_{j=1}^{p}\dot{w}_j\mathbf{P}_j = \left(\sum_{i=1}^{p} w_i\right)\sum_{j=1}^{p}\dot{w}_j\mathbf{P}_j = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{P}_j$ and $\sum_{j=1}^{p}\dot{w}_j\mathbf{I} = \sum_{j=1}^{p}\dot{w}_j\left(\sum_{i=1}^{p} w_i\mathbf{\Lambda}_i\right) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_k\mathbf{\Lambda}_i = \mathbf{0}$ where $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{n\times n}$, $i = 1, 2, ..., p$, is an arbitrary matrix, into (8), we have,

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{\Gamma A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{\Gamma}+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\right)\bar{\mathbf{x}}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\dot{w}_j\bar{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j\right)\bar{\mathbf{x}}(t)$$
$$-\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\dot{w}_j\bar{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}_j\bar{\mathbf{x}}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\right)\bar{\mathbf{x}}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\dot{w}_j\bar{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\bar{\mathbf{x}}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\right)\bar{\mathbf{x}}(t)$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(w_j+\rho\dot{w}_j-w_j\right)\bar{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\bar{\mathbf{x}}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\right)\bar{\mathbf{x}}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(w_j+\rho\dot{w}_j\right)\bar{\mathbf{x}}^{\mathrm{T}}\frac{1}{\rho}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\bar{\mathbf{x}}(t) \quad (9)$$
$$-\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\bar{\mathbf{x}}(t)$$

where ρ is a non-zero positive scalar. It should be noted that $\sum_{j=1}^{p}\left(w_j+\rho\dot{w}_j\right) = \sum_{j=1}^{p}w_j+\rho\sum_{j=1}^{p}\dot{w}_j = 1$. Based on this property, from (9), we have,

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}w_iw_j\left(w_k+\rho\dot{w}_k\right)$$
$$\times\bar{\mathbf{x}}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\\+\frac{1}{\rho}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_k^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_k-\mathbf{P}_k\right)\end{pmatrix}\bar{\mathbf{x}}(t)$$
$$-\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\bar{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\bar{\mathbf{x}}(t)$$
$$=\sum_{i=1}^{p}\sum_{k=1}^{p}w_i^2\left(w_k+\rho\dot{w}_k\right)\bar{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{Q}_{iik}\bar{\mathbf{x}}(t)$$
$$+\sum_{j=1}^{p}\sum_{i<j}\sum_{k=1}^{p}w_iw_j\left(w_k+\rho\dot{w}_k\right)\bar{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ijk}+\mathbf{Q}_{jik}\right)\bar{\mathbf{x}}(t) \quad (10)$$
$$-\frac{1}{\rho}\sum_{i=1}^{p}w_i^2\bar{\mathbf{x}}^{\mathrm{T}}\overline{\mathbf{Q}}_{ii}\bar{\mathbf{x}}(t)-\frac{1}{\rho}\sum_{j=1}^{p}\sum_{i<j}w_iw_j\bar{\mathbf{x}}^{\mathrm{T}}\left(\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji}\right)\bar{\mathbf{x}}(t)$$

where

$$\mathbf{Q}_{ijk}=\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j+\frac{1}{\rho}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_k^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_k-\mathbf{P}_k\right),\ i,\ j,\ k = 1,\ 2,\ ...,\ p;$$

$\overline{\mathbf{Q}}_{ij}=\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j$, i, j = 1, 2, ..., p. Let $\mathbf{R}_{ijk}+\mathbf{R}_{ijk}^{\mathrm{T}}\geq 0$, j, k = 1, 2, ..., p; $i<j$ and $\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\leq 0$, j = 1, 2, ..., p; i < j where $\mathbf{R}_{ijk}=\mathbf{R}_{jik}^{\mathrm{T}}\in\Re^{n\times n}$ and $\mathbf{R}_{ij}=\overline{\mathbf{R}}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$. From (10), we have,

$$\dot{V}(t)\leq\sum_{i=1}^{p}\sum_{k=1}^{p}w_i^2\left(w_k+\rho\dot{w}_k\right)\bar{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{Q}_{iik}\bar{\mathbf{x}}(t)$$
$$+\sum_{j=1}^{p}\sum_{i<j}\sum_{k=1}^{p}w_iw_j\left(w_k+\rho\dot{w}_k\right)\bar{\mathbf{x}}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{Q}_{ijk}+\mathbf{Q}_{jik}\\+\mathbf{R}_{ijk}+\mathbf{R}_{ijk}\end{pmatrix}\bar{\mathbf{x}}(t)$$
$$-\frac{1}{\rho}\sum_{i=1}^{p}w_i^2\bar{\mathbf{x}}^{\mathrm{T}}\overline{\mathbf{Q}}_{ii}\bar{\mathbf{x}}(t)-\frac{1}{\rho}\sum_{j=1}^{p}\sum_{i<j}w_iw_j\bar{\mathbf{x}}^{\mathrm{T}}\begin{pmatrix}\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji}\\+\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\end{pmatrix}\bar{\mathbf{x}}(t)$$
$$=\sum_{k=1}^{p}\left(w_k+\rho\dot{w}_k\right)\begin{bmatrix}w_1\bar{\mathbf{x}}(t)\\w_p\bar{\mathbf{x}}(t)\\\vdots\\w_p\bar{\mathbf{x}}(t)\end{bmatrix}^{\mathrm{T}}\mathbf{S}_k\begin{bmatrix}w_1\bar{\mathbf{x}}(t)\\w_p\bar{\mathbf{x}}(t)\\\vdots\\w_p\bar{\mathbf{x}}(t)\end{bmatrix}-\frac{1}{\rho}\begin{bmatrix}w_1\bar{\mathbf{x}}(t)\\w_p\bar{\mathbf{x}}(t)\\\vdots\\w_p\bar{\mathbf{x}}(t)\end{bmatrix}^{\mathrm{T}}\overline{\mathbf{S}}\begin{bmatrix}w_1\bar{\mathbf{x}}(t)\\w_p\bar{\mathbf{x}}(t)\\\vdots\\w_p\bar{\mathbf{x}}(t)\end{bmatrix} \quad (11)$$

where $\mathbf{S}_k=\begin{bmatrix}\mathbf{Q}_{11k}&\mathbf{S}_{12k}&\cdots&\mathbf{S}_{1pk}\\\mathbf{S}_{21k}&\mathbf{Q}_{22k}&\cdots&\mathbf{S}_{2pk}\\\vdots&\vdots&\vdots&\vdots\\\mathbf{S}_{p1k}&\mathbf{S}_{p2k}&\cdots&\mathbf{Q}_{ppk}\end{bmatrix}$, k = 1, 2, ..., p,

$\mathbf{S}_{ijk}=\frac{\mathbf{Q}_{ijk}+\mathbf{Q}_{jik}}{2}+\mathbf{R}_{ijk}$, j, k = 1, 2, ..., p; i < j,

$\overline{\mathbf{S}}=\begin{bmatrix}\mathbf{Q}_{11}&\overline{\mathbf{S}}_{12}&\cdots&\overline{\mathbf{S}}_{1p}\\\overline{\mathbf{S}}_{21}&\mathbf{Q}_{22}&\cdots&\overline{\mathbf{S}}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\overline{\mathbf{S}}_{p1}&\overline{\mathbf{S}}_{p2}&\cdots&\mathbf{Q}_{pp}\end{bmatrix}$, $\overline{\mathbf{S}}_{ij}=\frac{\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji}}{2}+\overline{\mathbf{R}}_{ij}$, j = 1, 2, ..., p; i < j. Let the value of ρ be designed such that $w_k+\rho\dot{w}_k>0$, k = 1, 2, ..., p. It can be seen from (11) that if $\mathbf{S}_k<0$, k = 1, 2, ..., p and $\overline{\mathbf{S}}>0$, we obtain $\dot{V}(t)\leq 0$ (equality holds when $\bar{\mathbf{x}}(t)$ = $\mathbf{x}(t)$ = $\mathbf{0}$) which implies the asymptotically stability of the fuzzy-mode-based control system of (6).

B. Performance Design

In this section, LMI-based performance conditions will be derived to guarantee the system performance of the fuzzy-model-based control systems. The system performance is quantitatively measured by the following performance index which is commonly used in the optimal control techniques [14].

$$J=\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}dt \quad (12)$$

where $\tau_1-\tau_0>0$ denotes the optimization period, $\mathbf{J}_1=\mathbf{J}_1^{\mathrm{T}}\in\Re^{n\times n}>0$, $\mathbf{J}_2\in\Re^{n\times m}$, $\mathbf{J}_3=\mathbf{J}_3^{\mathrm{T}}\in\Re^{m\times m}>0$ and $\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}=\begin{bmatrix}\mathbf{K}_1&\mathbf{K}_2\\\mathbf{K}_2^{\mathrm{T}}&\mathbf{K}_3\end{bmatrix}^{-1}\in\Re^{(n+m)\times(n+m)}>0$. From (5) and (12), we have,

$$J=\int_{\tau_0}^{\tau_1}\left[\mathbf{x}(t)^{\mathrm{T}}\quad\left(\sum_{i=1}^{p}w_i\mathbf{G}_i\mathbf{\Gamma}^{-1}\mathbf{x}(t)+\sum_{i=1}^{p}\dot{w}_i\overline{\mathbf{G}}_i\mathbf{\Gamma}^{-1}\mathbf{x}(t)\right)^{\mathrm{T}}\right]$$

$$\times\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \sum_{j=1}^{p}w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\end{bmatrix}dt$$

$$=\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{\Gamma}^{-1}\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\mathbf{\Gamma}^{-1}\overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix}$$

$$\times\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p}w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}dt \tag{13}$$

Let

$$J<\eta\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{\Gamma}^{-2} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-2}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}dt \tag{14}$$

where η is a non-zero positive scalar. By minimizing the value of η, the performance index J can be minimized. From (13) and (14), we have,

$$\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\left(\begin{matrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{\Gamma}^{-1^{\mathrm{T}}}\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\mathbf{\Gamma}^{-1^{\mathrm{T}}}\overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix}\\ \times\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p}w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\end{bmatrix}\\ -\eta\begin{bmatrix}\mathbf{\Gamma}^{-2} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-2}\end{bmatrix}\end{matrix}\right)\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}dt \tag{15}$$

From (15), we have,

$$\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{\Gamma}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-1}\end{bmatrix}\mathbf{W}\begin{bmatrix}\mathbf{\Gamma}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}dt<0 \tag{16}$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix}$$
$$\times\begin{bmatrix}\mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p}w_j\mathbf{G}_j+\sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\end{bmatrix}-\eta\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} \tag{17}$$

It can be seen that the inequality of (17) holds when $\mathbf{W}<0$. From (17), by Schur complement, $\mathbf{W}<0$ is equivalent to the following conditions.

$$\overline{\mathbf{W}}=\begin{bmatrix}-\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\overline{\mathbf{G}}_i^{\mathrm{T}}\\ \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{0} & \sum_{j=1}^{p}w_i\mathbf{G}_i+\sum_{i=1}^{p}\dot{w}_i\overline{\mathbf{G}}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3\end{bmatrix}<0$$

$$=\begin{bmatrix}-\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\left(\mathbf{F}+\overline{\mathbf{G}}_i\right)^{\mathrm{T}}\\ \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{0} & \sum_{j=1}^{p}w_i\mathbf{G}_i+\sum_{i=1}^{p}\dot{w}_i\left(\mathbf{F}+\overline{\mathbf{G}}_i\right) & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3\end{bmatrix}<0$$

$$=\begin{bmatrix}-\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \sum_{i=1}^{p}w_i\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\frac{1}{\rho}\left(w_i+\rho\dot{w}_i-w_i\right)\left(\mathbf{F}+\overline{\mathbf{G}}_i\right)^{\mathrm{T}}\\ \sum_{i=1}^{p}w_i\mathbf{P}_i & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{0} & \sum_{j=1}^{p}w_i\mathbf{G}_i+\sum_{i=1}^{p}\frac{1}{\rho}\left(w_i+\rho\dot{w}_i-w_i\right)\left(\mathbf{F}+\overline{\mathbf{G}}_i\right) & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3\end{bmatrix}<0$$

$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(w_j+\rho\dot{w}_j\right)\mathbf{T}_{ij}-\sum_{i=1}^{p}w_i\overline{\mathbf{T}}_i<0 \tag{18}$$

where $\mathbf{F}\in\Re^{m\times n}$ is an arbitrary matrix,

$$\mathbf{T}_{ij}=\begin{bmatrix}-\eta\left(1-\frac{1}{\sigma}\right)\mathbf{I} & \mathbf{0} & \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{I} & \mathbf{0} & \mathbf{G}_i^{\mathrm{T}}+\frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_j\right)^{\mathrm{T}}\\ \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_1 & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2\\ \mathbf{0} & \mathbf{G}_i+\frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_j\right) & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2^{\mathrm{T}} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_3\end{bmatrix}$$

and $\overline{\mathbf{T}}_i=\begin{bmatrix}\frac{\eta}{\sigma}\mathbf{I} & \mathbf{0} & -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & \frac{\eta}{\sigma}\mathbf{I} & \mathbf{0} & \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_i\right)^{\mathrm{T}}\\ -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\sigma}\mathbf{K}_1 & \frac{1}{\sigma}\mathbf{K}_2\\ \mathbf{0} & \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_i\right) & \frac{1}{\sigma}\mathbf{K}_2^{\mathrm{T}} & \frac{1}{\sigma}\mathbf{K}_3\end{bmatrix}$, $\sigma>$ 1. From (18), it can be seen that $\overline{\mathbf{W}}<0$ if $\mathbf{T}_{ij}<0$, i, j = 1, 2, ..., p and $\overline{\mathbf{T}}_i>0$, i = 1, 2, ..., p which are the performance conditions. The LMI-based stability and performance conditions are summarized in the following theorem.

Theorem 1: The fuzzy model-based control system of (6) *formed by the nonlinear system in form of* (2) *and the non-PDC nonlinear controller of* (5) *is guaranteed to be asymptotically stable if there exist non-zero positive scalars* ρ, η *and* $\sigma>1$ *such that* $w_k(\mathbf{x}(t))+\rho\dot{w}_k(\mathbf{x}(t))>0$, *k = 1, 2, ..., p, and matrices* $\mathbf{F}\in\Re^{m\times n}$, $\mathbf{G}_j\in\Re^{m\times n}$, $\overline{\mathbf{G}}_j\in\Re^{m\times n}$, $\mathbf{J}_1=\mathbf{J}_1^{\mathrm{T}}\in\Re^{n\times n}$, $\mathbf{J}_2\in\Re^{n\times m}$, $\mathbf{J}_3=\mathbf{J}_3^{\mathrm{T}}\in\Re^{m\times m}$, $\mathbf{P}_k=\mathbf{P}_k^{\mathrm{T}}$, $\mathbf{R}_{ijk}=\mathbf{R}_{jik}^{\mathrm{T}}\in\Re^{n\times n}$, $\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\le 0$ *and* $\mathbf{\Lambda}_i=\mathbf{\Lambda}_i^{\mathrm{T}}\in\Re^{n\times n}$ *such that the following LMI-based stability and performance conditions are satisfied.*

LMI-Based Stability Conditions:

$\mathbf{P}_k>0$, *k = 1, 2, ..., p;*

$\mathbf{R}_{ijk}+\mathbf{R}_{ijk}^{\mathrm{T}}\ge 0$, *j, k = 1, 2, ..., p; i < j;*

$\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\le 0$, *j = 1, 2, ..., p; i < j;*

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{Q}_{11k} & \mathbf{S}_{12k} & \cdots & \mathbf{S}_{1pk} \\ \mathbf{S}_{21k} & \mathbf{Q}_{22k} & \cdots & \mathbf{S}_{2pk} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{S}_{p1k} & \mathbf{S}_{p2k} & \cdots & \mathbf{Q}_{ppk} \end{bmatrix} < 0, \; k = 1, 2, \ldots, p;$$

$$\overline{\mathbf{S}} = \begin{bmatrix} \mathbf{Q}_{11} & \overline{\mathbf{S}}_{12} & \cdots & \overline{\mathbf{S}}_{1p} \\ \overline{\mathbf{S}}_{21} & \mathbf{Q}_{22} & \cdots & \overline{\mathbf{S}}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\mathbf{S}}_{p1} & \overline{\mathbf{S}}_{p2} & \cdots & \mathbf{Q}_{pp} \end{bmatrix} > 0;$$

where $\mathbf{S}_{ijk} = \frac{\mathbf{Q}_{ijk} + \mathbf{Q}_{jik}}{2} + \mathbf{R}_{ijk}$, *j, k = 1, 2, ..., p; i < j;* $\overline{\mathbf{S}}_{ij} = \frac{\overline{\mathbf{Q}}_{ij} + \overline{\mathbf{Q}}_{ji}}{2} + \overline{\mathbf{R}}_{ij}$, *j = 1, 2, ..., p; i < j.*

LMI-Based Performance Conditions:

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0;$$

$$\mathbf{T}_{ij} = \begin{bmatrix} -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{I} & \mathbf{0} & \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{I} & \mathbf{0} & \mathbf{G}_i^{\mathrm{T}} + \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_j\right)^{\mathrm{T}} \\ \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_1 & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2 \\ \mathbf{0} & \mathbf{G}_i + \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_j\right) & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2^{\mathrm{T}} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_3 \end{bmatrix} < 0,$$

i, j = 1, 2, ..., p;

$$\overline{\mathbf{T}}_i = \begin{bmatrix} \frac{\eta}{\sigma}\mathbf{I} & \mathbf{0} & -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & \frac{\eta}{\sigma}\mathbf{I} & \mathbf{0} & \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_i\right)^{\mathrm{T}} \\ -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\sigma}\mathbf{K}_1 & \frac{1}{\sigma}\mathbf{K}_2 \\ \mathbf{0} & \frac{1}{\rho}\left(\mathbf{F}+\overline{\mathbf{G}}_i\right) & \frac{1}{\sigma}\mathbf{K}_2^{\mathrm{T}} & \frac{1}{\sigma}\mathbf{K}_3 \end{bmatrix} > 0, \; i = 1, 2, \ldots, p.$$

IV. Numerical Example

A numerical example will be given to demonstrate the effectiveness of the LMI-based stability and performance conditions. Considering the fuzzy model with the following rules [11],

Rule i: IF $x_1(t)$ is M_1^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2$ (19)

where $\mathbf{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. The membership functions are defined as $w_1(x_1(t)) = \mu_{\mathrm{M}_1^1}(x_1(t)) = \frac{1+\sin(x_1(t))}{2}$ and $w_2(x_1(t)) = \mu_{\mathrm{M}_1^2}(x_1(t)) = \frac{1-\sin(x_1(t))}{2}$. The time derivative of the membership functions are obtained as $\dot{w}_1(x_1(t)) = \frac{\cos(x_1(t))\dot{x}_1(t)}{2}$ and $\dot{w}_2(x_1(t)) = -\frac{\cos(x_1(t))\dot{x}_1(t)}{2}$. The system dynamics are described as,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right) \quad (20)$$

where $\mathbf{x} = [x_1(t) \;\; x_2(t)]^{\mathrm{T}}$. We follow the assumptions in [11] that $x_1(t) \in \left[-\frac{\pi}{2} \;\; \frac{\pi}{2}\right]$ and $x_2(t) \in \left[-\frac{\pi}{2} \;\; \frac{\pi}{2}\right]$. It is also assumed that $\dot{x}_1(t) \in [-2 \;\; 2]$ which leads to $\dot{w}_1(x_1(t)) \geq -|\cos(x_1(t))|$ and $\dot{w}_2(x_1(t)) \geq -|\cos(x_1(t))|$. By choosing $\rho = \frac{1}{100}$, referring to Fig. 1, it can be seen that the conditions $w_i(x_1(t)) + \rho\dot{w}_i(x_1(t)) > 0$, $i = 1, 2$, are satisfied.

Based on Theorem 1, with $\eta = 10^{-5}$ and $\sigma = \frac{1}{\rho}$, $\mathbf{G}_i$, $\overline{\mathbf{G}}_i$ and $\mathbf{P}_i$, $i = 1, 2$ are obtained under different weighting matrices $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ and tabulated in Table 1. The nonlinear controller is in the following form,

$$u(t) = \sum_{j=1}^{2} w_j(x_1(t))\mathbf{G}_j\left(\sum_{k=1}^{2} w_k(x_1(t))\mathbf{P}_k\right)^{-1}\mathbf{x}(t) + \sum_{j=1}^{p} \dot{w}_j(x_1(t))\overline{\mathbf{G}}_j\left(\sum_{k=1}^{2} w_k(x_1(t))\mathbf{P}_k\right)^{-1}\mathbf{x}(t) \quad (21)$$

Referring to Table 1, it can be seen that different weighting matrices place different weights on $x_1(t)$ and $u(t)$ to specify the system performance. Fig. 2 shows the system state responses and control signals of the nonlinear plant with the nonlinear controller of (21) under different feedback gains. It can be seen that all nonlinear controllers can stabilize the nonlinear system. The nonlinear controller with $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 100$ offers the fastest rising time on $x_1(t)$ and the minimum magnitude on $u(t)$ as the heaviest weights are placed on $x_1(t)$ and $u(t)$ among the four sets of weighting matrices. In general, the nonlinear controllers with the heaviest weights on $x_1(t)$ offer faster rising time on $x_1(t)$ while those with the heaviest weights on $u(t)$ offer smaller range of $u(t)$. Hence, it can be seen that the LMI-based stability and performance conditions can be served as an effective tool to design a stable and well-performed nonlinear controller for nonlinear systems.

V. Conclusion

A nonlinear controller has been proposed to control nonlinear plants represented by fuzzy models. LMI-based stability conditions have been derived based on the parameter-dependent Lyapunov function. Based on the commonly-used performance index, LMI-based performance conditions have been derived to design the system performance. A numerical example has been given to illustrate the effectiveness of the proposed approach.

REFERENCES

[1] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Trans. Sys., Man., Cybern*., vol. smc-15 no. 1, pp. 116-132, Jan., 1985.

[2] M. Sugeno and G.T. Kang, “Structure identification of fuzzy model,” *Fuzzy sets and systems*, vol. 28, pp. 15-33, 1988.

[3] C.L. Chen, P.C. Chen, C.K. Chen, “Analysis and design of fuzzy control system,” *Fuzzy Sets and Systems,* vol. 57, no 2, 26, pp. 125-140, Jul. 1993.

[4] H.O. Wang, K. Tanaka, and M.F. Griffin, “An approach to fuzzy control of nonlinear systems: stability and the design issues,” *IEEE Trans. Fuzzy Syst*., vol. 4, no. 1, pp. 14-23, Feb., 1996.

[5] K. Tanaka, T. Ikeda and H.O. Wang, “Fuzzy regulator and fuzzy observer: Relaxed stability conditions and LMI-based designs,” *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250-265, 1998.

[6] W.J. Wang, S.F. Yan and C.H. Chiu, “Flexible stability criteria for a linguistic fuzzy dynamic system,” *Fuzzy Sets and Systems*, vol. 105, no. 1, pp. 63-80, Jul. 1999.

[7] M. Johansson, A. Rantzer and K.E. Årzén, “Piecewise quadratic stability of fuzzy systems,” *IEEE trans. Fuzzy Systems*, vol. 7, no. 6, pp. 713-722, Dec. 1999.

[8] E. Kim, “New approaches to relaxed quadratic stability conditions of fuzzy control systems,” *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 5, pp. 523-534, 2000.

[9] Xiaodong Liu and Qingling Zhang, “New approaches to H_∞ controller designs based on fuzzy observers for T-S fuzzy systems via LMI,” *Automatica*, vol. 39, no. 9, pp. 1571-1582, Sep. 2003.

[10] M.C.M. Teixeira, E. Assunção and R.G. Avellar, “On relaxed LMI-based designs for fuzzy regulators and fuzzy observers,” *IEEE Trans. on Fuzzy Systems*, vol. 11, no. 5, pp. 613-623, Oct. 2003.

[11] K. Tanaka, T.Hori and H.O. Wang, “A multiple Lyapunov function approach to stabilization of fuzzy control systems,” *IEEE Trans. Fuzzy Systems*, vol. 11, no. 4, pp. 582-589, Oct. 2003.

[12] T.M. Guerra and L. Vermeiren, “LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno’s form,” *Automatica*, vol. 40, pp. 823-829, 2004.

[13] B.C. Ding, H.X. Sun and P. Yang, “Further study on LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno’s form,” *Automatica*, vol. 42, pp. 503-508, 2006.

[14] B.D.O. Anderson and J.B. Moore, *Optimal Control: Linear Quadratic Methods*. Prentice Hall, 1990.

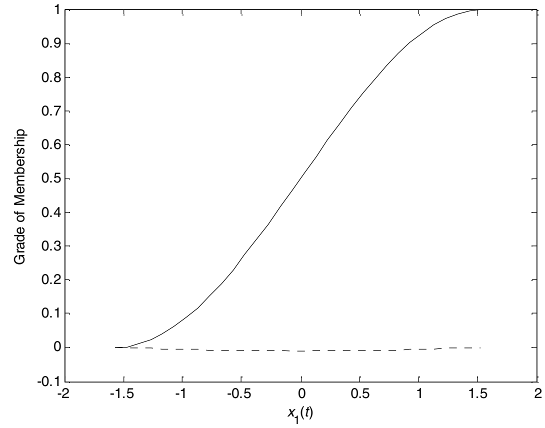

Fig. 1(a). $w_1(x_1(t))$ (solid line) and $-\frac{1}{100}\left|\cos(x_1(t))\right|$ (dotted line).

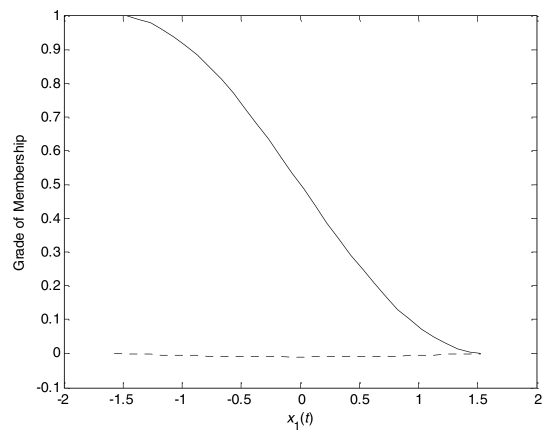

Fig. 1(b). $w_2(x_1(t))$ (solid line) and $-\frac{1}{100}\left|\cos(x_1(t))\right|$ (dotted line).

Fig. 1. Membership function $w_i(x_1(t))$ and its lower bound of the time derivative $\frac{1}{100}\dot{w}_i(x_1(t))$, i =1, 2.

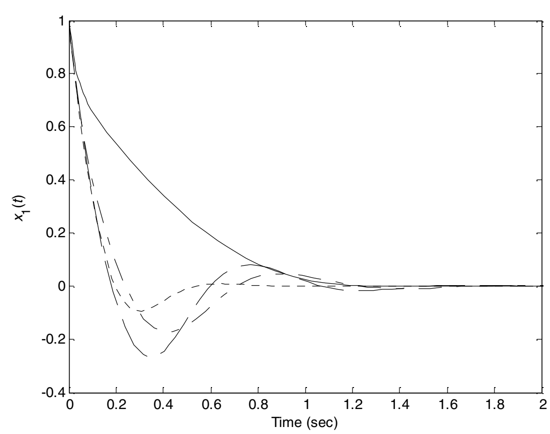

Fig. 2(a). $x_1(t)$.

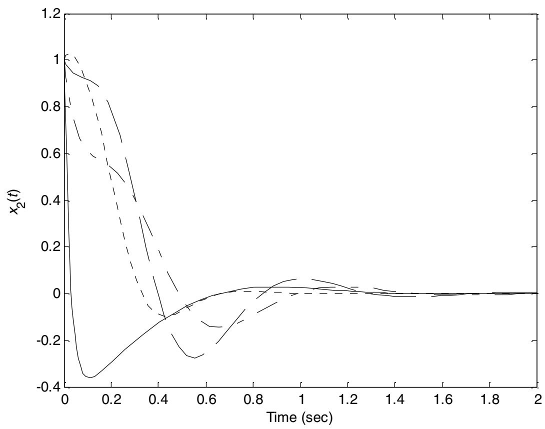

Fig. 2(b). $x_2(t)$.

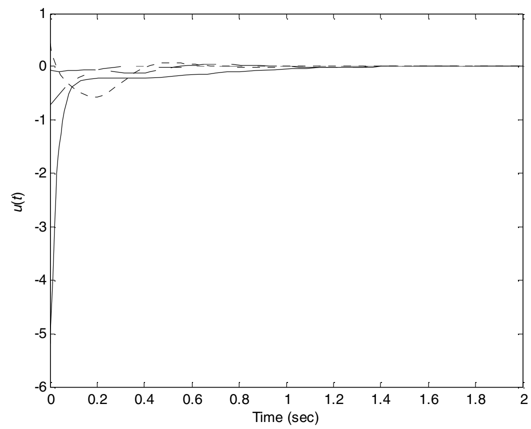

Fig. 2(c). $u(t)$.

Fig. 2. System state response and control signals under $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 1$ (solid lines), $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 1$ (dotted lines), $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 100$ (dash-dot), and $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 100$ (dashed lines).

Performance Index Parameters	$\mathbf{G}_i$, $\overline{\mathbf{G}}_i$ and $\mathbf{P}_i$, $i = 1, 2$	Min. $u(t)$	Max. $u(t)$
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$	$\mathbf{G}_1 = [7.4985\times10^{-4} \quad -2.4298\times10^{-3}]$; $\mathbf{G}_2 = [-1.9991\times10^{-3} \quad -2.3689\times10^{-3}]$ $\overline{\mathbf{G}}_1 = [-4.1720\times10^{-6} \quad 7.7924\times10^{-6}]$; $\overline{\mathbf{G}}_2 = [-7.9056\times10^{-6} \quad -1.5128\times10^{-5}]$ $\mathbf{P}_1 = \begin{bmatrix} 1.2834\times10^{-3} & 1.4194\times10^{-4} \\ 1.4194\times10^{-4} & 2.7613\times10^{-3} \end{bmatrix}$; $\mathbf{P}_2 = \begin{bmatrix} 1.2543\times10^{-3} & 9.1731\times10^{-5} \\ 9.1731\times10^{-5} & 2.5435\times10^{-3} \end{bmatrix}$	–5.1994	0.0008
$\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$	$\mathbf{G}_1 = [2.2625\times10^{-3} \quad -9.3098\times10^{-3}]$; $\mathbf{G}_2 = [1.8540\times10^{-3} \quad -8.3328\times10^{-3}]$ $\overline{\mathbf{G}}_1 = [6.1067\times10^{-6} \quad 2.5998\times10^{-5}]$; $\overline{\mathbf{G}}_2 = [1.6111\times10^{-5} \quad -7.1409\times10^{-5}]$ $\mathbf{P}_1 = \begin{bmatrix} 9.4210\times10^{-4} & 8.0632\times10^{-6} \\ 8.0632\times10^{-6} & 7.7031\times10^{-3} \end{bmatrix}$; $\mathbf{P}_2 = \begin{bmatrix} 9.3833\times10^{-4} & 2.1379\times10^{-5} \\ 2.1379\times10^{-5} & 6.4850\times10^{-3} \end{bmatrix}$	–0.5696	0.4387
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 100$	$\mathbf{G}_1 = [1.7648\times10^{-4} \quad -2.0651\times10^{-4}]$; $\mathbf{G}_2 = [-1.7264\times10^{-4} \quad -1.9972\text{e-}004]$ $\overline{\mathbf{G}}_1 = [4.7857\times10^{-8} \quad 3.2661\times10^{-6}]$; $\overline{\mathbf{G}}_2 = [2.1268\times10^{-7} \quad 1.2876\times10^{-6}]$ $\mathbf{P}_1 = \begin{bmatrix} 1.0388\times10^{-3} & 2.4079\times10^{-5} \\ 2.4079\times10^{-5} & 3.1025\times10^{-3} \end{bmatrix}$; $\mathbf{P}_2 = \begin{bmatrix} 1.0048\times10^{-3} & -1.0105\times10^{-5} \\ -1.0105\times10^{-5} & 3.1275\times10^{-3} \end{bmatrix}$	–0.7095	0.0094
$\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 100$	$\mathbf{G}_1 = [4.7815\times10^{-4} \quad -5.8949\times10^{-4}]$; $\mathbf{G}_2 = [2.8717\times10^{-4} \quad -4.9685\times10^{-4}]$ $\overline{\mathbf{G}}_1 = [8.8067\times10^{-6} \quad 3.6847\times10^{-6}]$; $\overline{\mathbf{G}}_2 = [1.3074\times10^{-5} \quad -2.6565\times10^{-6}]$ $\mathbf{P}_1 = \begin{bmatrix} 9.3715\times10^{-4} & 1.0196\times10^{-5} \\ 1.0196\times10^{-5} & 5.4030\times10^{-3} \end{bmatrix}$; $\mathbf{P}_2 = \begin{bmatrix} 9.2778\times10^{-4} & -5.3483\times10^{-6} \\ -5.3483\times10^{-6} & 5.3106\times10^{-3} \end{bmatrix}$	–0.1245	0.0472

Performance index parameters, $\mathbf{G}_i$, $\overline{\mathbf{G}}_i$ and $\mathbf{P}_i$, and the minimum and maximum control signals.