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LMI-based stability and performance design of fuzzy control systems: Fuzzy models and controllers with different premises

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Abstract— **This paper presents the stability analysis of the fuzzy control systems based on the fuzzy-model-based approach. The fuzzy controller, which does not require sharing the same premises as those of the fuzzy model, is considered. The class of fuzzy controllers offers the design flexibility and robustness property to the fuzzy control systems. However, conservative stability results will usually be produced compared with the fuzzy control systems with fuzzy controllers sharing the same premises as the fuzzy models. Relaxed LMI-based stability conditions will be derived for this class of fuzzy control systems. Furthermore, LMI-based performance conditions will be given to guarantee the system performance. Numerical examples will be given to illustrate the merits of the proposed approach.**

I. Introduction

Fuzzy-model-based approach is the most common approach to analyze the system stability of the fuzzy control systems. Based on the TS-fuzzy model [1]-[2], nonlinear systems can be represented in a general and systematic form. A fuzzy controller is then proposed to deal with the nonlinear system based on the fuzzy model. In [3]-[4], basic stability conditions were derived to guarantee the system stability. The stability conditions can be expressed in linear matrix inequalities (LMIs) [5] which can be solved numerically and efficiently using convex program techniques. By sharing the same membership functions between the fuzzy plant model and the fuzzy controller, relaxed stability conditions were reported in [6]-[11].

In [3]-[4], the membership functions of the fuzzy controller can be freely designed. Furthermore, as the stability conditions do not relate to the membership functions of the fuzzy model, the fuzzy controller in [3]-[4] is suitable to deal with the nonlinear systems subject to parameter uncertainties which are represented by the fuzzy models with uncertain parameters grouped into the membership functions. Under this case, as the grades of membership are uncertainties, the stability conditions in [4]-[11] are not applicable for this class of fuzzy control systems. Hence, for this class of fuzzy control systems, it can be seen that the fuzzy controller in [3]-[4] offers better design flexibility and robustness property. However, due to the membership functions of the fuzzy model are not considered during the system analysis, conservative stability results may be produced. In this paper, the system stability of fuzzy control systems will be investigated. In order to keep the design flexibility and robustness property of the fuzzy controller, the fuzzy controller does not require sharing the same premises as those of the fuzzy model. By proper formulation of the fuzzy control systems, relaxed stability analysis approach in [6]-[11] can be partially be applied. Consequently, relaxed stability conditions can be derived. Furthermore, LMI-based performance conditions will be derived to guarantee the system performance.

This paper is organized as follows. In section II, the fuzzy model and the fuzzy controller will be presented. In section III, LMI-based stability and performance conditions will be derived. In section IV, numerical examples will be presented to illustrate effectiveness of the proposed approach. A conclusion will be drawn in section V.

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II. Fuzzy Model and Fuzzy Controller

A multivariable fuzzy-model-based control system comprising a fuzzy model and a fuzzy controller connected in closed-loop will be considered.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$; $i = 1, 2, ..., p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big) \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \; w_i(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, \ldots, \Psi$, are the grade of membership corresponding to the fuzzy term of M_α^i. It should be noted that the value of $\mu_{\mathrm{M}_\alpha^i}(x_\alpha(t))$ is uncertain when it contains the system parameter uncertainties.

B. Fuzzy Controller

A fuzzy controller with p fuzzy rules is to be designed for the nonlinear plant. The j-th rule of the fuzzy controller is of the following format:

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j AND … AND $g_\Omega(\mathbf{x}(t))$ is N_Ω^j

THEN $\mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t)$ (5)

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \ldots, \Omega$; $j = 1, 2, \ldots, p$; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j to be designed. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{6}$$

where

$$\sum_{j=1}^{p} m_j(\mathbf{x}(t)) = 1,\ m_j(\mathbf{x}(t)) \in \begin{bmatrix}0 & 1\end{bmatrix} \text{ for all } j \tag{7}$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\right)} \tag{8}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_β^j.

C. Published Stability Conditions

LMI-based stability conditions in terms of LMIs have been derived to test the system stability of the fuzzy control systems formed by the fuzzy model of (2) and the fuzzy controller of (6). The stability conditions are summarized in the following theorem.

Theorem 1 [3]-[4]*: The fuzzy control system, formed by the nonlinear plant in the form of* (2) *and the fuzzy controller of* (6) *is guaranteed to be asymptotically stable if there exists symmetric matrix* $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ *such that the following LMIs hold.*

$\mathbf{P} > 0$ *;*

$\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right) < 0$, *i = 1, 2, …, p; j = 1, 2, …, p.*

III. Stability Analysis and Performance Design

In this section, the system stability of the fuzzy control system formed by the nonlinear plant in the form of (2) and the fuzzy controller of (6) will be investigated. Furthermore, LMI-based performance conditions will be derived to design the system performance. In the following analysis, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted by w_i and m_j respectively for simplicity. The property that $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} m_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_i = 1$ will be used during the analysis.

A. Stability Analysis

From (2) and (6), the fuzzy control system is as follows,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{p} m_j\mathbf{G}_j\mathbf{x}(t)\right)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\end{aligned} \tag{9}$$

To investigate the stability of (9), the following Lyapunov function candidate will be considered.

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{10}$$

where $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n} > 0$. From (9) and (10),

$$\begin{aligned}\dot{V}(t) &= \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\\
&= \left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)\\
&\quad + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\left(\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\right)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j\mathbf{x}(t)^{\mathrm{T}}\left(\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\right)\mathbf{x}(t)\\
&= \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\rho m_j\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\ + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\end{pmatrix}\mathbf{x}(t)\\
&= \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\left(w_j + \rho m_j - w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\ + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\end{pmatrix}\mathbf{x}(t)\\
&= \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\ + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\end{pmatrix}\mathbf{x}(t)\\
&\quad + \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\left(\rho m_j - w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\ + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\end{pmatrix}\mathbf{x}(t)\end{aligned} \tag{11}$$

where $\rho > 1$. It is assumed that the membership functions of the fuzzy controller are design such that $\rho m_j - w_j \geq 0$, $j = 1, 2, \ldots, p$. Based on the property that $\sum_{j=1}^{p}\left(\rho m_j - w_j\right) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\left(\rho m_j - w_j\right) = \rho - 1$, we have,

$$\dot{V}(t)=\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\mathbf{x}(t)^{\mathrm{T}}\left(\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\right)\mathbf{x}(t)$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\end{pmatrix}\mathbf{x}(t)$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{\Lambda}_i-\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{\Lambda}_i$$
$$=\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)-(\rho-1)\mathbf{\Lambda}_i\end{pmatrix}\mathbf{x}(t)$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)+\mathbf{\Lambda}_i\end{pmatrix}\mathbf{x}(t) \tag{12}$$

where $\mathbf{\Lambda}_i\in\Re^{n\times n}$, i = 1, 2, ..., p, is an arbitrary matrix. From (12), let $\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\geq 0$, i = 1, 2, ..., p; $i<j$ where $\mathbf{R}_{ij}=\mathbf{R}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$, we have,

$$\dot{V}(t)=\frac{1}{\rho}\sum_{i=1}^{p}w_i^2\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_i\end{pmatrix}\mathbf{x}(t)$$
$$+\frac{1}{\rho}\sum_{j=1}^{p}\sum_{i<j}w_iw_j\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\\-(\rho-1)\mathbf{\Lambda}_i+\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_j\\+\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\end{pmatrix}\mathbf{x}(t)$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)+\mathbf{\Lambda}_i\end{pmatrix}\mathbf{x}(t)$$
$$\leq\begin{bmatrix}w_1\mathbf{x}(t)\\w_2\mathbf{x}(t)\\\vdots\\w_p\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\overline{\mathbf{R}}\begin{bmatrix}w_1\mathbf{x}(t)\\w_2\mathbf{x}(t)\\\vdots\\w_p\mathbf{x}(t)\end{bmatrix}$$
$$+\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho m_j-w_j\right)\mathbf{x}(t)^{\mathrm{T}}\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}\\+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)+\mathbf{\Lambda}_i\end{pmatrix}\mathbf{x}(t) \tag{13}$$

where $\overline{\mathbf{R}}=\begin{bmatrix}\overline{\mathbf{R}}_{11}&\overline{\mathbf{R}}_{12}&\cdots&\overline{\mathbf{R}}_{1p}\\\overline{\mathbf{R}}_{21}&\overline{\mathbf{R}}_{22}&\cdots&\overline{\mathbf{R}}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\overline{\mathbf{R}}_{p1}&\overline{\mathbf{R}}_{p2}&\cdots&\overline{\mathbf{R}}_{pp}\end{bmatrix}$,

$\overline{\mathbf{R}}_{ii}=\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_i$, i = 1, 2, ..., p,

$\overline{\mathbf{R}}_{ij}=\dfrac{\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)-(\rho-1)\mathbf{\Lambda}_i\\+\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_j\end{pmatrix}}{2}+\mathbf{R}_{ij}$, j = 1, 2, ..., p; $i<j$. It can be seen that the asymptotically stability of the fuzzy control system of (10) is guaranteed by the stability conditions of $\overline{\mathbf{R}}<0$ and $\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)+\mathbf{\Lambda}_i<0$, i = 1, 2, ..., p; j = 1, 2, ..., p. The stability analysis results are summarized in the following Theorem.

Theorem 2: The fuzzy control system of (10) *formed by the nonlinear plant in form of* (2) *and the fuzzy controller of* (6) *is guaranteed to be asymptotically stable if the membership functions of the fuzzy controller are designed such that there exists a* $\rho>1$ *leading to* $\rho m_j(\mathbf{x}(t))-w_j(\mathbf{x}(t))\geq 0$, *j = 1, 2, ..., p and there exists matrices* $\mathbf{P}=\mathbf{P}^{\mathrm{T}}\in\Re^{n\times n}$, $\mathbf{\Lambda}_i\in\Re^{n\times n}$ *and* $\mathbf{R}_{ij}=\mathbf{R}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$ *such that the following LMIs are satisfied.*

$\mathbf{P}=\mathbf{P}^{\mathrm{T}}\in\Re^{n\times n}>0$;

$\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\geq 0$, *i = 1, 2, ..., p; i < j;*

$\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)+\mathbf{\Lambda}_i<0$, *i = 1, 2, ..., p; j = 1, 2, ..., p;*

$$\overline{\mathbf{R}}=\begin{bmatrix}\overline{\mathbf{R}}_{11}&\overline{\mathbf{R}}_{12}&\cdots&\overline{\mathbf{R}}_{1p}\\\overline{\mathbf{R}}_{21}&\overline{\mathbf{R}}_{22}&\cdots&\overline{\mathbf{R}}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\overline{\mathbf{R}}_{p1}&\overline{\mathbf{R}}_{p2}&\cdots&\overline{\mathbf{R}}_{pp}\end{bmatrix}<0;$$

where $\overline{\mathbf{R}}_{ii}=\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_i$, *i = 1, 2, ..., p,*

$\overline{\mathbf{R}}_{ij}=\dfrac{\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)-(\rho-1)\mathbf{\Lambda}_i\\+\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)-(\rho-1)\mathbf{\Lambda}_j\end{pmatrix}}{2}+\mathbf{R}_{ij}$, *j = 1, 2, ..., p; i < j.*

Remark 1: The solution to the stability conditions in Theorem 1 is also the solution to the stability conditions of Theorem 2. Referring to Theorem 1, let **P** be the solution, we have $\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)<0$, i = 1, 2, ..., p; j = 1, 2, ..., p. Considering the stability conditions in Theorem 2, let $\mathbf{\Lambda}_i=\mathbf{0}$ and $\mathbf{R}_{ij}=-\dfrac{\begin{pmatrix}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\right)\\+\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i\right)\end{pmatrix}}{2}$ (which leads to $\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\geq 0$, j = 1, 2, ..., p; $i<j$), it can be seen that the stability conditions in Theorem 2 are satisfied with the **P** given by Theorem 1. However, the solution given by Theorem 2 may not be the solution of the stability conditions in Theorem 1.

B. Performance Design

In this section, LMI-based performance conditions will be derived to guarantee the system performance of the fuzzy control systems. The system performance is quantitatively measured by the following performance index which is commonly used in the optimal control techniques [13].

$$J=\int_{\tau_0}^{\tau_1}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}dt \tag{14}$$

where $\tau_1 - \tau_0 > 0$ denotes the optimization period, $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n\times n} > 0$, $\mathbf{J}_2 \in \Re^{n\times m}$, $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m\times m} > 0$ and $\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \in \Re^{(n+m)\times(n+m)} > 0$.

From (6) and (14), we have,

$$J = \int_{\tau_0}^{\tau_1} \begin{bmatrix} \mathbf{x}(t)^{\mathrm{T}} & \left(\sum_{i=1}^{p} w_i \mathbf{G}_i \mathbf{x}(t) \right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \sum_{j=1}^{p} w_j \mathbf{G}_j \mathbf{x}(t) \end{bmatrix} dt$$

$$= \int_{\tau_0}^{\tau_1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{G}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt \tag{15}$$

The system performance can be optimized by minimizing the performance index J. Let

$$J < \eta \int_{\tau_0}^{\tau_1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_1^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-2} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt \tag{16}$$

where η is a non-zero positive scalar. By minimizing the value of η, the performance index J can be minimized. From (15) and (16), we have,

$$\int_{\tau_0}^{\tau_1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{G}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j \end{bmatrix} - \eta \begin{bmatrix} \mathbf{X}_1^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-2} \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt < 0 \tag{17}$$

The feedback gains are designed as $\mathbf{G}_i = \mathbf{N}_i \mathbf{X}^{-1}$ where $\mathbf{N}_i = \in \Re^{m\times n}$, $i = 1, 2, ..., p$ and $\mathbf{X} = \mathbf{P}^{-1} > 0$. From (17), we have,

$$\int_{\tau_0}^{\tau_1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1} \end{bmatrix} \mathbf{W} \begin{bmatrix} \mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt < 0 \tag{18}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{N}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{N}_j \end{bmatrix} - \eta \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{19}$$

It can be seen that the inequality of (19) holds when $\mathbf{W} < 0$. From (19) and by Schur complement, $\mathbf{W} < 0$ is equivalent to the following inequality.

$$\sum_{i=1}^{p} w_i \mathbf{W}_i < 0 \tag{20}$$

where $\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0$,

$$\mathbf{W}_i = \begin{bmatrix} -\eta\mathbf{I} & \mathbf{0} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \mathbf{N}_i^{\mathrm{T}} \\ \mathbf{X}_1 & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{0} & \mathbf{N}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3 \end{bmatrix}, i = 1, 2, ..., p.$$

It can be seen that the inequality of (20) holds when $\mathbf{W}_i < 0$, $i = 1, 2, ..., p$, which are the performance conditions. The stability conditions in Theorem 2 can be expressed in terms of $\mathbf{X}$ and $\mathbf{N}_j$ by pre- and post-multiplying diag{$\mathbf{X}, \mathbf{X} \cdots, \mathbf{X}$} to $\overline{\mathbf{R}} < 0$ and $\mathbf{X}$ to other stability conditions by letting $\mathbf{X}\mathbf{\Lambda}_i = \mathbf{V}_i$ and $\mathbf{S}_{ij} = \mathbf{X}\mathbf{R}_{ij}\mathbf{X}$. The stability and performance conditions are summarized in the following theorem.

Theorem 3: The fuzzy control system of (10) *formed by the nonlinear plant in form of* (2) *and the fuzzy controller of* (6) *is guaranteed to be asymptotically stable if the membership functions of the fuzzy controller are designed such that there exists a $\rho > 1$ leading to $\rho m_j(\mathbf{x}(t)) - w_j(\mathbf{x}(t)) \geq 0$, j = 1, 2, ..., p and there exists matrices $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{V}_i \in \Re^{n\times n}$, $\mathbf{S}_{ij} = \mathbf{S}_{ji}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{J}_2 \in \Re^{n\times m}$ and $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m\times m}$ such that the following LMI-based stability and performance conditions are satisfied.*

Stability Conditions:

$\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \Re^{n\times n} > 0$;

$\mathbf{S}_{ij} + \mathbf{S}_{ij}^{\mathrm{T}} \geq 0$, *i = 1, 2, ..., p; i < j;*

$\mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{V}_i < 0$, *i = 1, 2, ..., p; j = 1, 2, ..., p;*

$$\overline{\mathbf{S}} = \begin{bmatrix} \overline{\mathbf{S}}_{11} & \overline{\mathbf{S}}_{12} & \cdots & \overline{\mathbf{S}}_{1p} \\ \overline{\mathbf{S}}_{21} & \overline{\mathbf{S}}_{22} & \cdots & \overline{\mathbf{S}}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\mathbf{S}}_{p1} & \overline{\mathbf{S}}_{p2} & \cdots & \overline{\mathbf{S}}_{pp} \end{bmatrix} < 0;$$

where $\overline{\mathbf{S}}_{ii} = \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{N}_i + \mathbf{N}_i^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} - (\rho - 1)\mathbf{V}_i$, *i = 1, 2, ..., p,*

$$\overline{\mathbf{S}}_{ij} = \frac{\begin{pmatrix} \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X} + \mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{N}_j - (\rho-1)\mathbf{V}_i \\ + \mathbf{X}\mathbf{A}_j^{\mathrm{T}} + \mathbf{A}_j\mathbf{X} + \mathbf{N}_i^{\mathrm{T}}\mathbf{B}_j^{\mathrm{T}} + \mathbf{B}_j\mathbf{N}_i - (\rho-1)\mathbf{V}_j \end{pmatrix}}{2} + \mathbf{S}_{ij},$$

j = 1, 2, ..., p; i < j and the feedback gains are designed as $\mathbf{G}_i = \mathbf{N}_i\mathbf{X}_1^{-1}$ *i = 1, 2, ..., p.*

Performance Conditions:

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0;$$

$$\mathbf{W}_i = \begin{bmatrix} -\eta\mathbf{I} & \mathbf{0} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \mathbf{N}_i^{\mathrm{T}} \\ \mathbf{X}_1 & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{0} & \mathbf{N}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3 \end{bmatrix} < 0, \textit{ i = 1, 2, ..., p.}$$

IV. Numerical Examples

Two examples will be given in this section to illustrate the merits of the stability analysis and performance design results. In the first example, a numerical example will be given to show that the stability region given by Theorem 2 is larger than that given by Theorem 1. In the second example, the LMI-based stability and performance conditions will be employed to design a stable and well-performed fuzzy controller for a cart-pole typed inverted pendulum.

A. Example 1

A numerical example will be given to show the effectiveness of the stability conditions in Theorem 2. A fuzzy model with the following 2 fuzzy rules is considered.

Rule i: IF $x_1(t)$ is M_1^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t),\ i = 1, 2 \tag{21}$$

where $\mathbf{A}_1 = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} a & -10 \\ 1 & 1 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}$; $2 \le a \le 6$ and $1 \le b \le 25$. A 2-rule fuzzy controller of (6) is employed to close the feedback loop. It is assumed that the membership functions of fuzzy controller, which are different from the membership functions of the fuzzy plant, are designed such that $\rho m_j(\mathbf{x}(t)) - w_j(\mathbf{x}(t)) \ge 0$, $j = 1, 2$, with $\rho = 5$. The feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ are designed such that all eigenvalues of $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ are located at -10. Fig. 1 shows the stability regions of Theorem 1 and Theorem 2. It can be seen from Fig. 1 that the stability conditions in Theorem 2 provides larger stability region than those of Theorem 1. Furthermore, the published stability conditions given in [6]-[11] cannot be applied as the premises of the fuzzy model and fuzzy controller are different.

B. Example 2

An application example on stabilizing a cart-pole typed inverted pendulum [14] will be given.

Step I) The dynamic equations of the inverted pendulum on the cart [14] is given by,

$$\dot{x}_1(t) = x_2(t) \tag{22}$$

$$\dot{x}_2(t) = \frac{\begin{pmatrix} -F_1(M+m)x_2(t) - m^2l^2x_2(t)^2\sin x_1(t)\cos x_1(t) \\ +F_0 mlx_4(t)\cos x_1(t) + (M+m)mgl\sin x_1(t) \\ -ml\cos x_1(t)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{23}$$

$$\dot{x}_3(t) = x_4(t) \tag{24}$$

$$\dot{x}_4(t) = \frac{\begin{pmatrix} F_1 mlx_2(t)\cos x_1(t) + (J+ml^2)mlx_2(t)^2\sin x_1(t) \\ -F_0(J+ml^2)x_4(t) - m^2gl^2\sin x_1(t)\cos x_1(t) \\ +(J+ml^2)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{25}$$

where $x_1(t)$ and $x_2(t)$ denote the angular displacement (rad) and the angular velocity (rad/s) of the pendulum from vertical respectively, $x_3(t)$ and $x_4(t)$ denote the displacement (m) and the velocity (m/s) of the cart respectively, $g = 9.8$ m/s^2 is the acceleration due to gravity, $m = 0.22$ kg is the mass of the pendulum, $M = 1.3282$ kg is the mass of the cart, $l = 0.304$ m is the length from the center of mass of the pendulum to the shaft axis, $J = ml^2/3$ kgm^2 is the moment of inertia of the pendulum around the center of mass, $F_0 = 22.915$ N/m/s and $F_1 = 0.007056$ N/rad/s are the friction factors of the cart and the pendulum respectively, and $u(t)$ is the force (N) applied to the cart. The objective of this application example is to employed the proposed fuzzy controller to control the nonlinear plant such that $x_1(t) = x_3(t) = 0$ at steady state. The nonlinear plant can be represented by a fuzzy plant model with two fuzzy rules [14]. The i-th rule is given by,

Rule i: IF $x_1(t)$ is M_1^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t) \text{ for } i = 1, 2 \tag{26}$$

The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\big) \tag{27}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^{\mathrm{T}}$;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0 ml/a_1 \\ 0 & 0 & 1 & 0 \\ -m^2gl^2/a_1 & F_1 ml/a_1 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0 ml\cos(\pi/3)/a_2 \\ 0 & 0 & 1 & 0 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos(\pi/3)/a_2 & F_1 ml\cos(\pi/3)/a_2 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix}$$

; $\mathbf{B}_1 = \begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J+ml^2)/a_1 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 0 \\ -ml\cos(\pi/3)/a_2 \\ 0 \\ (J+ml^2)/a_2 \end{bmatrix}$;

$a_1 = (M+m)(J+ml^2) - m^2l^2$,

$a_2 = (M+m)(J+ml^2) - m^2l^2\cos(\pi/3)^2$. The membership functions are defined as $w_1(x_1(t)) = \mu_{\mathrm{M}_1^1}(x_1(t)) = \left(1 - \frac{1}{1+e^{-7(x_1(t)-\pi/6)}}\right)\frac{1}{1+e^{-7(x_1(t)+\pi/6)}}$ and $w_2(x_1(t)) = 1 - \mu_{\mathrm{M}_1^1}(x_1(t))$.

Step II) A two-rule fuzzy controller is employed to control the nonlinear plant. From (6), the output of the fuzzy controller is defined as follows.

$$u(t) = \sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{28}$$

The membership functions of the fuzzy control are designed as $m_1(x_1(t)) = \mu_{\mathrm{N}_1^1}(x_1(t)) = 0.9e^{-\frac{x_1(t)^2}{2\times 1.5^2}}$ and $m_2(x_1(t)) = \mu_{\mathrm{N}_1^2}(x_1(t)) = 1 - 0.9e^{-\frac{x_1(t)^2}{2\times 1.5^2}}$. Fig. 2 shows the membership functions of the fuzzy model and fuzzy controller. It can be seen that the membership functions of the fuzzy controller satisfied the conditions of $\rho m_j(x_1(t)) + w_j(x_1(t)) \ge 0$, $j = 1, 2$, with $\rho = 20$.

Step III) Based on the stability conditions in Theorem 3, with η = 10^{-3}, four fuzzy controllers with feedback gains designed based on different $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ can be obtained. Table I tabulates the feedback gains of the four fuzzy controllers which are denoted by fuzzy controllers 1 to 4. It can be seen that different $\mathbf{J}_1$ and $\mathbf{J}_3$ put different weights on the system states and control signal respectively which lead to different feedback gains to satisfy the performance index.

Fig. 3 shows the system state responses and the control signals with the fuzzy controllers using various feedback gains under the initial state conditions of $\mathbf{x}(0)=\begin{bmatrix}\frac{\pi}{3} & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Referring to this figure, it can be seen that the nonlinear plant can be stabilized successfully by the fuzzy controllers 1 to 4. The system stability of the fuzzy control systems are guaranteed by the stability conditions in Theorem 3. The minimum and maximum magnitudes of the control signals produced by the four fuzzy controllers are tabulated in Table 1. Referring to Fig. 3, the fuzzy controllers 3, which put heaviest weight on $x_3(t)$ and the least weight on $u(t)$, offer the best state response in terms of raise time and settling time on $x_3(t)$ at the cost of large magnitude of control signal. Referring to Fig. 3 and Table 1, it can be seen that the fuzzy controllers designed under $\mathbf{J}_1=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 100 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ offer faster state responses on $x_3(t)$ while the fuzzy controllers with $\mathbf{J}_3$ = 100 offer smaller magnitude of control signals. Hence, it can be shown that the performance conditions offer an effective way to design the system performance subject to the performance index of (14).

V. Conclusion

The system stability and performance design have been investigated. A fuzzy controller, which does not require sharing the same premises as those of the fuzzy model, has been proposed to control the nonlinear systems. Relaxed stability conditions have been derived for this class of fuzzy control systems. The stability results are applicable to fuzzy control systems with uncertain grades of membership. Furthermore, LMI-based performance conditions have been derived to guarantee the system performance. Numerical examples have been given to illustrate the effectiveness of the proposed approach.

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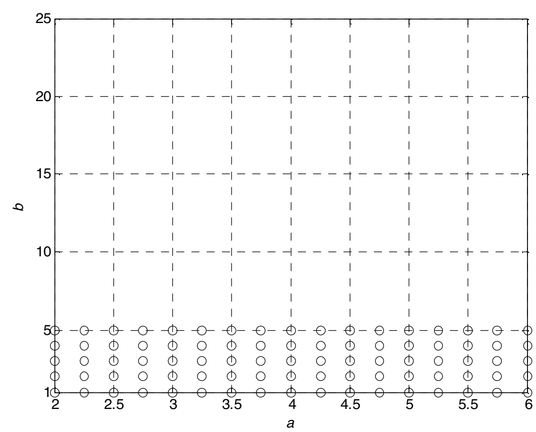

Fig. 1(a). Theorem 1.

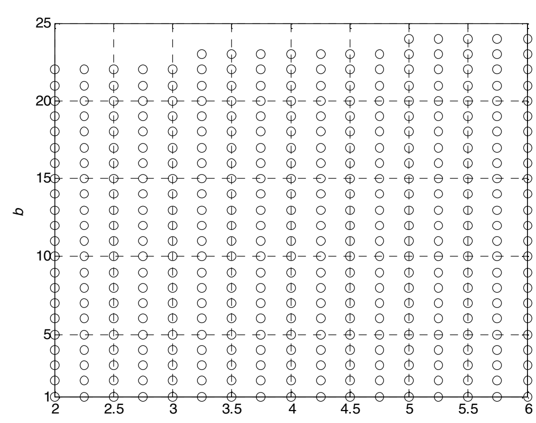

Fig. 1(b). Theorem 2.
Fig. 1. Stability regions of Theorem 1 and Theorem 2.

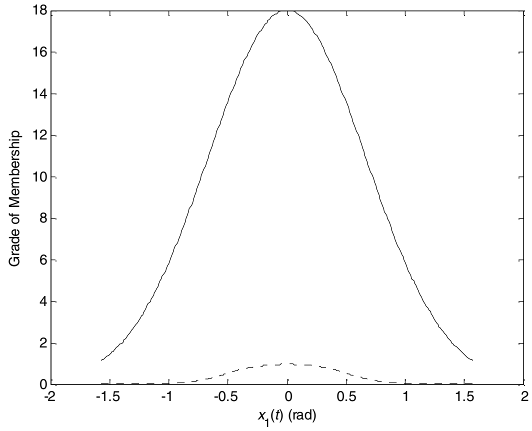

Fig. 2(a). $m_1(x_1(t)) = \mu_{N_1^1}(x_1(t)) = 0.9e^{-\frac{x_1(t)^2}{2\times 1.5^2}}$ (Solid line) and $w_1(x_1(t))$ = $\mu_{M_1^1}(x_1(t))$ = $\left(1-\frac{1}{1+e^{-7(x_1(t)-\pi/6)}}\right)\frac{1}{1+e^{-7(x_1(t)+\pi/6)}}$ (dotted line).

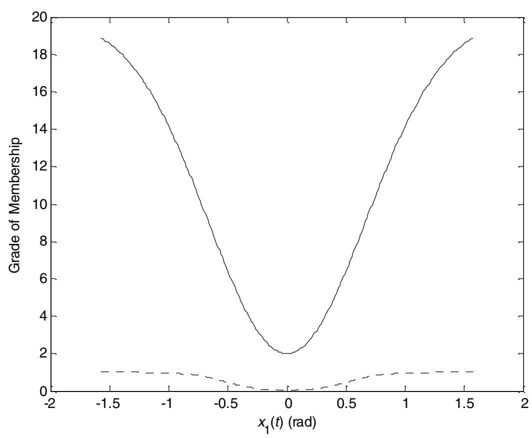

Fig. 2(b). $m_2(x_1(t)) = \mu_{N_1^2}(x_1(t)) = 1-0.9e^{-\frac{x_1(t)^2}{2\times 1.5^2}}$ (solid lines) and $w_2(x_1(t))$ = $1-\mu_{M_1^1}(x_1(t))$ (dotted).

Fig. 2. Membership functions of the fuzzy model (dotted lines) and the fuzzy controller (solid lines).

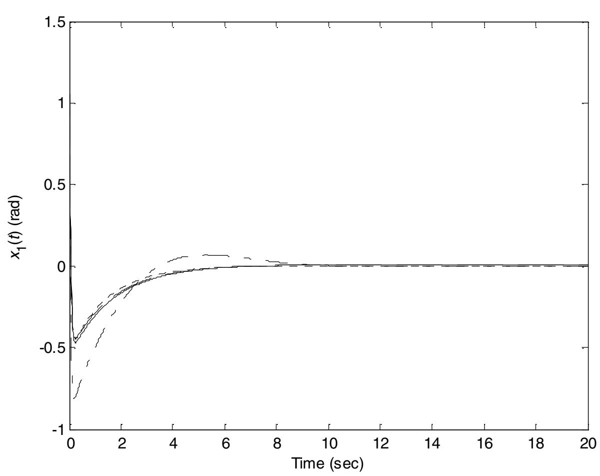

Fig. 3(a). $x_1(t)$.

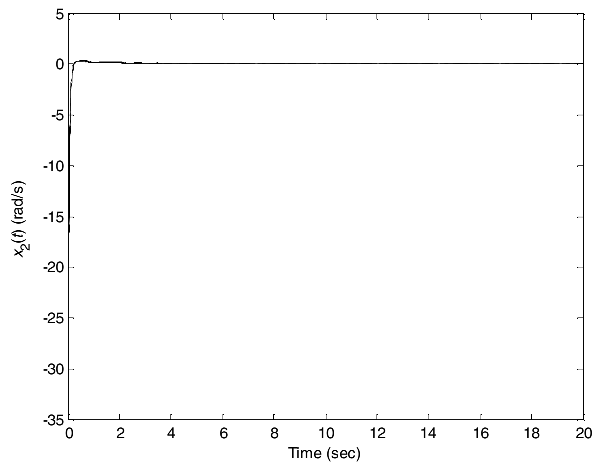

Fig. 3(b). $x_2(t)$.

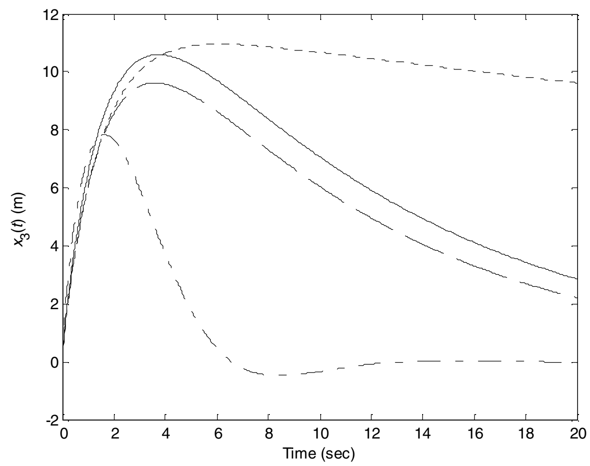

Fig. 3(c). $x_3(t)$.

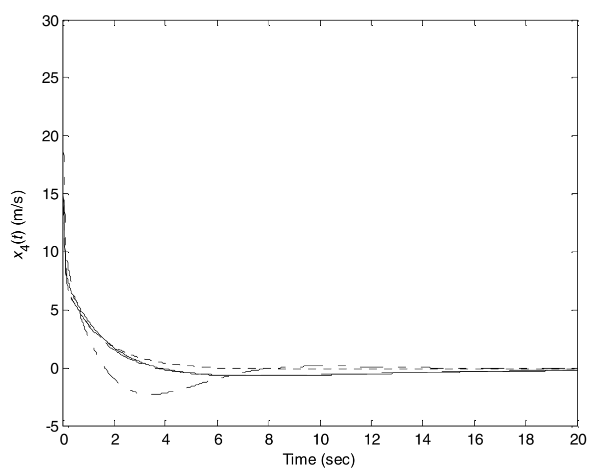

Fig. 3(d). $x_4(t)$.

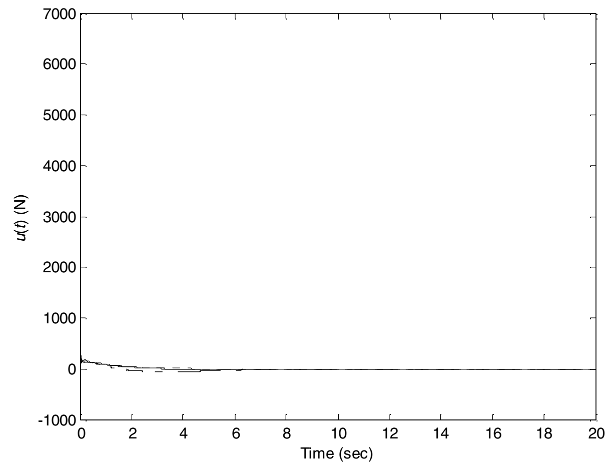

Fig. 3(e). $u(t)$.

Fig. 3. System state response and control signals under fuzzy controller 1 (solid lines), fuzzy controller 2 (dotted lines), fuzzy controller 3 (dash-dot), and fuzzy controller 4 (dashed lines).

Fuzzy Controller	Performance Index Parameters	Feedback Gains	Min. $u(t)$ (N)	Max. $u(t)$ (N)
1	$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$	$\mathbf{G}_1 = [4404.1989 \quad 341.9473 \quad 22.2183 \quad 306.7273]$ $\mathbf{G}_2 = [1938.9552 \quad 148.3632 \quad 8.5644 \quad 129.1503]$	−15.6335	2760.9899
2	$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 100$	$\mathbf{G}_1 = [2430.5433 \quad 193.9088 \quad 1.6889 \quad 186.34186]$ $\mathbf{G}_2 = [1625.5779 \quad 128.3212 \quad 1.0259 \quad 121.0424]$	−2.4707	1940.8350
3	$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$	$\mathbf{G}_1 = [11059.3228 \quad 911.9826 \quad 404.4426 \quad 916.1438]$ $\mathbf{G}_2 = [3973.9950 \quad 323.6919 \quad 133.9778 \quad 318.2300]$	−53.1304	6261.1390
4	$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 100$	$\mathbf{G}_1 = [2564.4280 \quad 201.6470 \quad 14.4104 \quad 190.6409]$ $\mathbf{G}_2 = [1617.5625 \quad 125.6765 \quad 8.1229 \quad 116.2206]$	−15.4341	1974.4903

Table I. Feedback gains under different values of performance index parameters, and the minimum and maximum amplitudes of the control signals.