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LMI relaxed stability conditions for fuzzy-model-based control systems

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Abstract— **This paper investigates the system stability of the fuzzy-model-based control systems. New stability conditions in terms of linear matrix inequalities (LMIs) will be derived based on the Lyapunov-based approach. It will be shown that the proposed stability conditions offer relaxed stability result than that of some important published stability conditions. The feedback gains of the fuzzy controller will be designed based on the LMI-based approach. A numerical example will be given to show the merits of the proposed stability conditions.**

I. Introduction

Fuzzy-model-based control approach offers a systematic approach to handle nonlinear plants. In this approach, the nonlinear plant is represented by the TS-fuzzy model [1]-[2]. A fuzzy controller [3]-[4] with similar structure will be employed to close the feedback loop to form a fuzzy model-based control system. The system stability was investigated in [3]-[4] based on Lyapunov-based approach. It was shown that the fuzzy-model-based control system is guaranteed to be stable if there exists a solution to a set of linear matrix inequalities (LMIs) [5] which can be solved numerically and efficiently by some convex programming techniques. When the fuzzy controller shares the same premise as those of the fuzzy model, relaxed stability conditions can be obtained [6]. Under this design criterion of the fuzzy controller, further relaxed stability conditions were reported in [7]-[10]. In this paper, the system stability of the fuzzy-model-based control systems studied in [3]-[10] will be investigated. Based on the Lyapunov-based approach, new LMI stability conditions will be derived. It will be shown analytically and experimentally that the proposed stability conditions will offer relaxed stability results than those published in [3]-[10]. The LMI-based design of the feedback gains of the fuzzy controller will also be presented subject to the system stability.

This paper is organized as follows. In section II, the fuzzy model and the fuzzy controller will be presented. In section III, some important published stability conditions [3]-[10] will be reviewed. In section IV, new relaxed stability conditions will be derived for the fuzzy-model-based control systems. In section V, the design of the feedback gains based on the LMI-based approach will be given. In section VI, numerical example will be given to illustrate the effectiveness of the new stability conditions. A conclusion will be drawn in section VII.

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II. Fuzzy Model and Fuzzy Controller

A fuzzy-model-based control system comprises a nonlinear plant represented by the TS-fuzzy model and the fuzzy controller connected in closed loop. The details of the fuzzy model and the fuzzy controller are given as follows.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format.

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$, Ψ is a positive integer, i = 1, 2, ..., p; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t)\in\Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t)\in\Re^{m\times 1}$ is the input vector. The system behavior is described by,

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\big), \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t))=1,\ w_i(\mathbf{x}(t))\in[0\quad 1] \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t))=\frac{\mu_{\mathrm{M}_1^i}\big(f_1(\mathbf{x}(t))\big)\times\mu_{\mathrm{M}_2^i}\big(f_2(\mathbf{x}(t))\big)\times\cdots\times\mu_{\mathrm{M}_\Psi^i}\big(f_\Psi(\mathbf{x}(t))\big)}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}\big(f_1(\mathbf{x}(t))\big)\times\mu_{\mathrm{M}_2^k}\big(f_2(\mathbf{x}(t))\big)\times\cdots\times\mu_{\mathrm{M}_\Psi^k}\big(f_\Psi(\mathbf{x}(t))\big)\Big)} \tag{4}$$

is a known nonlinear function of $f_\alpha(\mathbf{x}(t))$. $\mu_{\mathrm{M}_\alpha^i}\big(f_\alpha(\mathbf{x}(t))\big)$, α = 1, 2, …, Ψ, is the grade of membership corresponding to the fuzzy term of M_α^i.

B. Fuzzy Controller

A fuzzy controller with p fuzzy rules is employed to handle the nonlinear plant. The j-th rule of the fuzzy controller is of the following format.

Rule j: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\mathbf{u}(t)=\mathbf{G}_j\mathbf{x}(t)$ (5)

where $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \quad (6)$$

C. Fuzzy-Model-Control System

The fuzzy-model-based control system is formed by connecting the fuzzy model of (2) and the fuzzy controller of (6) in closed loop. From (3), we have the following property.

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) = 1 \quad (7)$$

From (2), (6) and (7), we have,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)\right)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\mathbf{H}_{ij}\mathbf{x}(t) \quad (8)\end{aligned}$$

where

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j,\ i = 1, 2, ..., p; j = 1, 2, ..., p. \quad (9)$$

III. PUBLISHED STABILITY CONDITIONS

In this section, some important LMI stability conditions which guarantee the stability of the fuzzy-model-based control systems in form of (8) will be reviewed.

A. Wang et al's Basic Stability Conditions

Theorem 1: The equilibrium of the fuzzy-model-based control system in form of (8) *is asymptotically stable in the large if there exists a symmetric matrix* $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ *such that the following LMIs hold.*

$\mathbf{P} > 0$*;* $\mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} < 0$*, i = 1, 2, ..., p;*

$\left(\dfrac{\mathbf{H}_{ij}+\mathbf{H}_{ji}}{2}\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\dfrac{\mathbf{H}_{ij}+\mathbf{H}_{ji}}{2}\right) \le 0$*, j = 1, 2, ..., p, i < j,* $w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) \ne 0$.

Proof: see [6].

B. Tanaka et al's Relaxed Stability Conditions

Theorem 2: The equilibrium of the fuzzy-model-based control system in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ *and* $\mathbf{Q} = \mathbf{Q}^{\mathrm{T}} \in \Re^{n\times n}$ *such that the following LMIs hold.*

$\mathbf{P} > 0$*;* $\mathbf{Q} \ge 0$*;*

$\mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} + (s-1)\mathbf{Q} < 0$*, i = 1, 2, ..., p, 1 ≤ s ≤ p;*

$\left(\dfrac{\mathbf{H}_{ij}+\mathbf{H}_{ji}}{2}\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\dfrac{\mathbf{H}_{ij}+\mathbf{H}_{ji}}{2}\right) - \mathbf{Q} \le 0$*, 1 ≤ i < j ≤ p,* $w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) \ne 0$*;*

where s is an integer denoting the maximum number of fired fuzzy subsystems at an instance.

Proof: see [7].

Remark 1: It has been shown in [7] that if the stability conditions of Theorem 1 hold, the stability conditions of Theorem 2 will also hold.

C. Kim et al's Relaxed Stability Conditions

Theorem 3: The equilibrium of the fuzzy-model-based control system in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ *and* $\mathbf{X}_{ij} = \mathbf{X}_{ij}{}^{\mathrm{T}} \in \Re^{n\times n}$ *such that the following LMIs hold.*

$\mathbf{P} > 0$*;*

$\mathbf{\Lambda}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{\Lambda}_{ii} + \mathbf{X}_{ii} < 0$*, i = 1, 2, ..., p;*

$\mathbf{\Lambda}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{\Lambda}_{ij} + \mathbf{X}_{ij} \le 0$*, 1 ≤ i < j ≤ p;*

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdots & \mathbf{X}_{1p}\\ \mathbf{X}_{12} & \mathbf{X}_{22} & \cdots & \mathbf{X}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{X}_{1p} & \mathbf{X}_{2p} & \cdots & \mathbf{X}_{pp}\end{bmatrix} > 0;$$

where $\mathbf{\Lambda}_{ij} = \dfrac{\mathbf{H}_{ij}+\mathbf{H}_{ji}}{2}$.

Proof: See [8].

Remark 2: It was pointed out in [9] that the stability conditions of Theorem 5 can be further improved by modifying the LMI conditions $\mathbf{\Lambda}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{\Lambda}_{ij} + \mathbf{X}_{ij} \le 0$, $1 \le i < j \le p$ to $2\mathbf{\Lambda}_{ij}{}^{\mathrm{T}}\mathbf{P} + 2\mathbf{P}\mathbf{\Lambda}_{ij} + \mathbf{X}_{ij} + \mathbf{X}_{ij}{}^{\mathrm{T}} \le 0$, $1 \le i < j \le p$ where $\mathbf{X}_{ij} = \mathbf{X}_{ji}{}^{\mathrm{T}} \in \Re^{n\times n}$.

Remark 3: It has been shown in [8] that if the stability conditions of Theorem 2 hold, the stability conditions of Theorem 3 will also hold.

D. Marcelo et al's Relaxed Stability Conditions

Theorem 4: The equilibrium of the fuzzy-model-based control system in form of (8) *is asymptotically stable in the large if there exist symmetric matrices* $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{T}_{ijh} = \mathbf{T}_{ijh}{}^{\mathrm{T}} \in \Re^{n\times n}$, $\hat{\mathbf{R}}_{ij} = \hat{\mathbf{R}}_{ij}{}^{\mathrm{T}} \in \Re^{n\times n}$ *and* $\mathbf{S}_{ijh} \in \Re^{n\times n}$ *such that the following LMIs hold.*

$\mathbf{P} > 0$*;*

$\mathbf{T}_{ijh} \ge 0$*, i, j, h = 1, 2, ..., p; i < j;*

$$\mathbf{Qt}_h = \begin{bmatrix} \hat{\mathbf{Q}}_1 - \mathbf{Z}_{1h} & \mathbf{Qn}_{12h} & \cdots & \mathbf{Qn}_{1ph}\\ \mathbf{Qn}_{21h} & \hat{\mathbf{Q}}_2 - \mathbf{Z}_{2h} & \cdots & \mathbf{Qn}_{2ph}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{Qn}_{p1h} & \mathbf{Qn}_{p2h} & \cdots & \hat{\mathbf{Q}}_p - \mathbf{Z}_{ph}\end{bmatrix} < 0,\ h = 1, 2, ..., p;$$

where, for i, j, h = 1, 2, ..., p,

$\hat{\mathbf{Q}}_i = \mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii}$*;*

$$\mathbf{Z}_{jh} = \begin{cases} \hat{\mathbf{R}}_{jh}, & if\, j < h \\ \hat{\mathbf{R}}_{hj}, & if\, j > h \\ \mathbf{0}, & if\, h = j \end{cases};$$

$$\mathbf{Qn}_{ijh} = \begin{cases} \hat{\mathbf{Q}}_{ij} + \mathbf{T}_{ijh} + (\mathbf{S}_{ijh} - \mathbf{S}_{ijh}^{\mathrm{T}}) + \frac{1}{2}\mathbf{W}_{ijh}, & if\, i < j \\ \hat{\mathbf{Q}}_{ji} + \mathbf{T}_{jih} + (\mathbf{S}_{jih}^{\mathrm{T}} - \mathbf{S}_{jih}) + \frac{1}{2}\mathbf{W}_{jih}, & if\, i > j \end{cases};$$

$$\hat{\mathbf{Q}}_{ij} = \left(\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\right)^{\mathrm{T}} \mathbf{P} + \mathbf{P}\left(\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\right);$$

$$\mathbf{W}_{lkh} = \begin{cases} \hat{\mathbf{R}}_{lk}, & if\, l = h \text{ or } k = h \\ \mathbf{0}, & if\, l \neq h \text{ or } k \neq h \end{cases};$$

Proof: See [10].

Remark 4: It has been shown in [10] that if the stability conditions of Theorem 3 hold, the stability conditions of Theorem 4 will also hold.

IV. Stability Analysis

The system stability of the fuzzy-model-based control system of (8) will be analyzed. In the following analysis, $w_i(\mathbf{x}(t))$ is denoted by w_i for simplicity. From (6), we have the following property which will be used later.

$$\sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{10}$$

To investigate the system stability of (8), the following Lyapunov function candidate is considered,

$$V(t) = \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_1 \mathbf{x}(t) \tag{11}$$

where $\mathbf{P}_1 = \mathbf{P}_1^{\mathrm{T}} \in \Re^{n \times n} > 0$. From (2), (10) and (11), we have,

$$\begin{aligned} \dot{V}(t) &= \dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{P}_1 \mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_1 \dot{\mathbf{x}}(t) \\ &= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\ &= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\ &\quad + \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \end{aligned}$$

$$= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \mathbf{P} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \tag{12}$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix} \in \Re^{(n+m)\times(n+m)}$, $\mathbf{P}_2 \in \Re^{m \times n}$ and $\mathbf{P}_3 \in \Re^{m \times m}$. It can be seen that $\dot{V}(t) \le 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{u}(t) = \mathbf{0}$) if

$$\mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \mathbf{P} < 0,\ i = 1, 2, ..., p \tag{13}$$

which implies the asymptotically stability of the fuzzy-model-based control system of (8). The analysis results are summarized by the following theorem.

Theorem 5: The equilibrium of the fuzzy-model-based control systems in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{P}_1 = \mathbf{P}_1^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{P}_2 \in \Re^{m \times n}$ *and* $\mathbf{P}_3 \in \Re^{m \times m}$ *such that the following LMIs hold.*

$$\mathbf{P}_1 > 0;\ \mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \mathbf{P} < 0,\ i = 1, 2, ..., p;$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix}$.

Remark 5: Referring to Theorem 5, it can be seen the number of LMIs is reduced to p only compared with that in Theorem 2 to Theorem 4.

Furthermore, let $\mathbf{P}_2 = \sum_{j=1}^{p} w_j \mathbf{P}_{2j}$ and $\mathbf{P}_3 = \sum_{j=1}^{p} w_j \mathbf{P}_{3j}$ where $\mathbf{P}_{2j} \in \Re^{m \times n}$ and $\mathbf{P}_{3j} \in \Re^{m \times m}$, j = 1, 2, ..., p, (13) can be written as follows.

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{p} \sum_{j=1}^{p} w_i w_j \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \\ &\times \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A}_i + \mathbf{G}_i^{\mathrm{T}} \mathbf{P}_{2j} + \mathbf{P}_{2j}^{\mathrm{T}} \mathbf{G}_i & \mathbf{G}_i^{\mathrm{T}} \mathbf{P}_{3j} + \mathbf{P}_1 \mathbf{B}_i - \mathbf{P}_{2j}^{\mathrm{T}} \\ \mathbf{P}_{3j}^{\mathrm{T}} \mathbf{G}_i + \mathbf{B}_i^{\mathrm{T}} \mathbf{P}_1 - \mathbf{P}_{2j} & -\mathbf{P}_{3j} - \mathbf{P}_{3j}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{p} w_i w_j \mathbf{z}(t)^{\mathrm{T}} \mathbf{Q}_{ij} \mathbf{z}(t) \end{aligned} \tag{14}$$

where $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}$ and

$$\mathbf{Q}_{ij} = \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A}_i + \mathbf{G}_i^{\mathrm{T}} \mathbf{P}_{2j} + \mathbf{P}_{2j}^{\mathrm{T}} \mathbf{G}_i & \mathbf{G}_i^{\mathrm{T}} \mathbf{P}_{3j} + \mathbf{P}_1 \mathbf{B}_i - \mathbf{P}_{2j}^{\mathrm{T}} \\ \mathbf{P}_{3j}^{\mathrm{T}} \mathbf{G}_i + \mathbf{B}_i^{\mathrm{T}} \mathbf{P}_1 - \mathbf{P}_{2j} & -\mathbf{P}_{3j} - \mathbf{P}_{3j}^{\mathrm{T}} \end{bmatrix} \tag{15}$$

Let $\mathbf{R}_{ij} + \mathbf{R}_{ij}^{\mathrm{T}} \ge 0$ where $\mathbf{R}_{ij} = \mathbf{R}_{ji}^{\mathrm{T}} \in \Re^{(n+m)\times(n+m)}$. From (14), we have,

$$\dot{V}(t) \le \sum_{i=1}^{p} w_i^2 \mathbf{z}(t)^{\mathrm{T}} \mathbf{Q}_{ii} \mathbf{z}(t) + \sum_{j=1}^{p}\sum_{i<j} w_i w_j \mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{Q}_{ji} + \mathbf{R}_{ij} + \mathbf{R}_{ij}^{\mathrm{T}}\right)\mathbf{z}(t) = \begin{bmatrix} w_1\mathbf{z}(t) \\ w_2\mathbf{z}(t) \\ \vdots \\ w_p\mathbf{z}(t) \end{bmatrix}^{\mathrm{T}} \mathbf{S} \begin{bmatrix} w_1\mathbf{z}(t) \\ w_2\mathbf{z}(t) \\ \vdots \\ w_p\mathbf{z}(t) \end{bmatrix} \quad (16)$$

where $\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \cdots & \mathbf{S}_{1p} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \cdots & \mathbf{S}_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{S}_{p1} & \mathbf{S}_{p2} & \cdots & \mathbf{S}_{pp} \end{bmatrix}$, $\mathbf{S}_{ii} = \mathbf{Q}_{ii}$, i = 1, 2, ..., p and $\mathbf{S}_{ij} = \left(\frac{\mathbf{Q}_{ij} + \mathbf{Q}_{ji}}{2}\right) + \mathbf{R}_{ij}$, j = 1, 2, ..., p; $i < j$. It can be seen that $\dot{V}(t) \le 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{u}(t) = \mathbf{0}$) if $\mathbf{S} < 0$ which implies the asymptotically stability of the fuzzy-model-based control system of (8). The analysis results are summarized in the following theorem.

Theorem 6: The equilibrium of the fuzzy-model-based control systems in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{P}_1 = \mathbf{P}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{P}_{2j} \in \Re^{m\times n}$, $\mathbf{P}_{3j} \in \Re^{m\times m}$ *and* $\mathbf{R}_{ij} = \mathbf{R}_{ji}^{\mathrm{T}} \in \Re^{(n+m)\times(n+m)}$ *such that the following LMIs hold.*

$\mathbf{P}_1 > 0$; $\mathbf{R}_{ij} + \mathbf{R}_{ij}^{\mathrm{T}} \ge 0$, *j = 1, 2, ..., p; i < j;*

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \cdots & \mathbf{S}_{1p} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \cdots & \mathbf{S}_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{S}_{p1} & \mathbf{S}_{p2} & \cdots & \mathbf{S}_{pp} \end{bmatrix} < 0;$$

where $\mathbf{S}_{ii} = \mathbf{Q}_{ii}$, *i = 1, 2, ..., p,* $\mathbf{S}_{ij} = \left(\frac{\mathbf{Q}_{ij} + \mathbf{Q}_{ji}}{2}\right) + \mathbf{R}_{ij}$, *j = 1, 2, ..., p; i < j; and*

$$\mathbf{Q}_{ij} = \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{A}_i + \mathbf{G}_i^{\mathrm{T}}\mathbf{P}_{2j} + \mathbf{P}_{2j}^{\mathrm{T}}\mathbf{G}_i & \mathbf{G}_i^{\mathrm{T}}\mathbf{P}_{3j} + \mathbf{P}_1\mathbf{B}_i - \mathbf{P}_{2j}^{\mathrm{T}} \\ \mathbf{P}_{3j}^{\mathrm{T}}\mathbf{G}_i + \mathbf{B}_i^{\mathrm{T}}\mathbf{P}_1 - \mathbf{P}_{2j} & -\mathbf{P}_{3j} - \mathbf{P}_{3j}^{\mathrm{T}} \end{bmatrix}$$

, j = 1, 2, ..., p; i < j;

V. Design of Feedback Gains based on LMI Approach

In this section, the design of the feedback gains $\mathbf{G}_j$ will be obtained based on the LMI-based approach. In the following analysis, let $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix} = \mathbf{P}^{-1} \in \Re^{(n+m)\times(n+m)}$, $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \in \Re^{n\times n} > 0$, $\mathbf{X}_2 \in \Re^{m\times n}$, $\mathbf{X}_3 \in \Re^{m\times m}$, $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}_1^{-1}$ and $\mathbf{N}_j \in \Re^{m\times n}$, j = 1, 2, ..., p. The existence of $\mathbf{X}$ will be discussed later. From (12),

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \mathbf{X}^{-1\mathrm{T}}\mathbf{X}^{\mathrm{T}} \left(\mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \mathbf{P} \right) \mathbf{X}\mathbf{X}^{-1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \mathbf{X}^{-1\mathrm{T}} \left(\begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}\mathbf{X} + \mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \right) \mathbf{X}^{-1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&= \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \mathbf{X}^{-1\mathrm{T}} \left(\begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{G}_i & -\mathbf{I}_m \end{bmatrix}^{\mathrm{T}} \right) \mathbf{X}^{-1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&= \sum_{i=1}^{p} w_i \bar{\mathbf{z}}(t)^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_1\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X}_1 + \mathbf{B}_i\mathbf{X}_2 + \mathbf{X}_2^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & \mathbf{B}_i\mathbf{X}_3 + \mathbf{N}_i^{\mathrm{T}} - \mathbf{X}_2^{\mathrm{T}} \\ \mathbf{X}_3^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{N}_i - \mathbf{X}_2 & -\mathbf{X}_3 - \mathbf{X}_3^{\mathrm{T}} \end{bmatrix} \bar{\mathbf{z}}(t)
\end{aligned} \quad (17)$$

where $\bar{\mathbf{z}}(t) = \mathbf{X}^{-1}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}$. It can be seen that $\dot{V}(t) \le 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{u}(t) = \mathbf{0}$) if

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{X}_1\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X}_1 + \mathbf{B}_i\mathbf{X}_2 + \mathbf{X}_2^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & \mathbf{B}_i\mathbf{X}_3 + \mathbf{N}_i^{\mathrm{T}} - \mathbf{X}_2^{\mathrm{T}} \\ \mathbf{X}_3^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{N}_i - \mathbf{X}_2 & -\mathbf{X}_3 - \mathbf{X}_3^{\mathrm{T}} \end{bmatrix} < 0$$

, i = 1, 2, ..., p (18)

which implies the asymptotically stability of the fuzzy-model-based control system of (8). The analysis results are summarized in the following theorem.

Theorem 7: The equilibrium of the fuzzy-model-based control systems in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{X}_2 \in \Re^{m\times n}$, $\mathbf{X}_3 \in \Re^{m\times m}$ *and* $\mathbf{N}_j \in \Re^{m\times n}$ *such that the following LMIs hold.*

$\mathbf{X}_1 > 0$;

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{X}_1\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X}_1 + \mathbf{B}_i\mathbf{X}_2 + \mathbf{X}_2^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & \mathbf{B}_i\mathbf{X}_3 + \mathbf{N}_i^{\mathrm{T}} - \mathbf{X}_2^{\mathrm{T}} \\ \mathbf{X}_3^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{N}_i - \mathbf{X}_2 & -\mathbf{X}_3 - \mathbf{X}_3^{\mathrm{T}} \end{bmatrix} < 0$$

, i = 1, 2, ..., p

where the feedback gain is design as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}_1^{-1}$*, j = 1, 2, ..., p.*

Furthermore, let $\mathbf{X}_2 = \sum_{j=1}^{p} w_j \mathbf{X}_{2j}$ and $\mathbf{X}_3 = \sum_{j=1}^{p} w_j \mathbf{X}_{3j}$ where $\mathbf{X}_{2j} \in \Re^{m\times n}$ and $\mathbf{X}_{3j} \in \Re^{m\times m}$, j = 1, 2, ..., p, (17) can be written as follows.

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \bar{\mathbf{z}}(t)^{\mathrm{T}} \overline{\mathbf{Q}}_{ij} \bar{\mathbf{z}}(t) \quad (19)$$

where

$$\overline{\mathbf{Q}}_{ij} = \begin{bmatrix} \mathbf{X}_1\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X}_1 + \mathbf{B}_i\mathbf{X}_{2j} + \mathbf{X}_{2j}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & \mathbf{B}_i\mathbf{X}_{3j} + \mathbf{N}_j^{\mathrm{T}} - \mathbf{X}_{2j}^{\mathrm{T}} \\ \mathbf{X}_{3j}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{N}_j - \mathbf{X}_{2j} & -\mathbf{X}_{3j} - \mathbf{X}_{3j}^{\mathrm{T}} \end{bmatrix} \tag{20}$$

Let $\overline{\mathbf{R}}_{ij} + \overline{\mathbf{R}}_{ij}^{\mathrm{T}} \geq 0$ where $\overline{\mathbf{R}}_{ij} = \overline{\mathbf{R}}_{ji}^{\mathrm{T}} \in \Re^{(n+m)\times(n+m)}$. From (20), we have,

$$\dot{V}(t) = \begin{bmatrix} w_1\overline{\mathbf{z}}(t) \\ w_2\overline{\mathbf{z}}(t) \\ \vdots \\ w_p\overline{\mathbf{z}}(t) \end{bmatrix}^{\mathrm{T}} \overline{\mathbf{S}} \begin{bmatrix} w_1\overline{\mathbf{z}}(t) \\ w_2\overline{\mathbf{z}}(t) \\ \vdots \\ w_p\overline{\mathbf{z}}(t) \end{bmatrix} \tag{21}$$

where $\overline{\mathbf{S}} = \begin{bmatrix} \overline{\mathbf{S}}_{11} & \overline{\mathbf{S}}_{12} & \cdots & \overline{\mathbf{S}}_{1p} \\ \overline{\mathbf{S}}_{21} & \overline{\mathbf{S}}_{22} & \cdots & \overline{\mathbf{S}}_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\mathbf{S}}_{p1} & \overline{\mathbf{S}}_{p2} & \cdots & \overline{\mathbf{S}}_{pp} \end{bmatrix}$, $\overline{\mathbf{S}}_{ii} = \overline{\mathbf{Q}}_{ii}$, i = 1, 2, ..., p and $\overline{\mathbf{S}}_{ij} = \left(\dfrac{\overline{\mathbf{Q}}_{ij} + \overline{\mathbf{Q}}_{ji}}{2}\right) + \overline{\mathbf{R}}_{ij}$, j = 1, 2, ..., p; $i < j$. It can be seen that $\dot{V}(t) \leq 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{u}(t) = \mathbf{0}$) if $\overline{\mathbf{S}} < 0$, which implies the asymptotically stability of the fuzzy-model-based control system of (8). The analysis results are summarized by the following theorem.

Theorem 8: The equilibrium of the fuzzy-model-based control systems in form of (8) *is asymptotically stable in the large if there exist matrices* $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{X}_{2j} \in \Re^{m\times n}$, $\mathbf{X}_{3j} \in \Re^{m\times m}$, $\overline{\mathbf{R}}_{ij} = \overline{\mathbf{R}}_{ji}^{\mathrm{T}} \in \Re^{(n+m)\times(n+m)}$ *and* $\mathbf{N}_j \in \Re^{m\times n}$ *such that the following LMIs hold.*

$\mathbf{X}_1 > 0$*;* $\mathbf{R}_{ij} + \mathbf{R}_{ij}^{\mathrm{T}} \geq 0$*, j = 1, 2, ..., p; i < j;*

$$\overline{\mathbf{S}} = \begin{bmatrix} \overline{\mathbf{S}}_{11} & \overline{\mathbf{S}}_{12} & \cdots & \overline{\mathbf{S}}_{1p} \\ \overline{\mathbf{S}}_{21} & \overline{\mathbf{S}}_{22} & \cdots & \overline{\mathbf{S}}_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\mathbf{S}}_{p1} & \overline{\mathbf{S}}_{p2} & \cdots & \overline{\mathbf{S}}_{pp} \end{bmatrix} < 0\,;$$

where $\overline{\mathbf{S}}_{ii} = \overline{\mathbf{Q}}_{ii}$*, i = 1, 2, ..., p,* $\overline{\mathbf{S}}_{ij} = \left(\dfrac{\overline{\mathbf{Q}}_{ij} + \overline{\mathbf{Q}}_{ji}}{2}\right) + \overline{\mathbf{R}}_{ij}$*, j = 1, 2, ..., p; i < j,*

$$\overline{\mathbf{Q}}_{ij} = \begin{bmatrix} \mathbf{X}_1\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X}_1 + \mathbf{B}_i\mathbf{X}_{2j} + \mathbf{X}_{2j}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & \mathbf{B}_i\mathbf{X}_{3j} + \mathbf{N}_j^{\mathrm{T}} - \mathbf{X}_{2j}^{\mathrm{T}} \\ \mathbf{X}_{3j}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{N}_j - \mathbf{X}_{2j} & -\mathbf{X}_{3j} - \mathbf{X}_{3j}^{\mathrm{T}} \end{bmatrix}$$

, j = 1, 2, ..., p; i < j and the feedback gain is design as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}_1^{-1}$*, j = 1, 2, ..., p.*

Remark 6: It can be seen that if the stability conditions in Theorem 7 to Theorem 8 are satisfied, $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} > 0$, and $-\mathbf{X}_3 - \mathbf{X}_3^{\mathrm{T}} < 0$ for Theorem 7 and $-\mathbf{X}_{3j} - \mathbf{X}_{3j}^{\mathrm{T}} < 0$ for Theorem 8 are required. As a result, $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix}$ is a non-singular matrix. Hence, there must exist $\mathbf{P}^{-1} = \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix}$ if there exists solution to the stability conditions in Theorem 7 or Theorem 8.

Remark 7: The solution of the stability conditions in Theorem 4 is also a solution of those in Theorem 6. Referring to Theorem 4 and considering that there exist symmetric matrices $\mathbf{P} > 0$, $\mathbf{T}_{ijk} \geq 0$, $\mathbf{R}_{ij}$ and matrices $\mathbf{S}_{ijk}$ for all *i, j, k* = 1, 2, ..., *p; i < j* such that the stability conditions in Theorem 4 are satisfied. Referring to Theorem 4, the following LMIs hold.

$$\mathbf{Qt}_i = \begin{bmatrix} \hat{\mathbf{Q}}_1 - \mathbf{Z}_{1i} & \mathbf{Qn}_{12i} & \cdots & \mathbf{Qn}_{1pi} \\ \mathbf{Qn}_{21i} & \hat{\mathbf{Q}}_2 - \mathbf{Z}_{2i} & \cdots & \mathbf{Qn}_{2pi} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Qn}_{p1i} & \mathbf{Qn}_{p2i} & \cdots & \hat{\mathbf{Q}}_p - \mathbf{Z}_{pi} \end{bmatrix} < 0\,,\ i = 1, 2, ..., p$$

$$\Rightarrow \sum_{i=1}^{p} w_i \mathbf{Qt}_i < 0 \ \Rightarrow \sum_{i=1}^{p} w_i \widetilde{\mathbf{Q}}\mathbf{t}_i + \mathbf{H} < 0 \tag{22}$$

where

$$\widetilde{\mathbf{Q}}\mathbf{t}_i = \begin{bmatrix} \hat{\mathbf{Q}}_1 & \hat{\mathbf{Q}}_{12} + \mathbf{T}_{12i} + \left(\mathbf{S}_{12i} - \mathbf{S}_{12i}^{\mathrm{T}}\right) & \cdots & \hat{\mathbf{Q}}_{1p} + \mathbf{T}_{1pi} + \left(\mathbf{S}_{1pi} - \mathbf{S}_{1pi}^{\mathrm{T}}\right) \\ \hat{\mathbf{Q}}_{12} + \mathbf{T}_{12i} + \left(\mathbf{S}_{12i}^{\mathrm{T}} - \mathbf{S}_{12i}\right) & \hat{\mathbf{Q}}_2 & \cdots & \hat{\mathbf{Q}}_{2p} + \mathbf{T}_{2pi} + \left(\mathbf{S}_{2pi}^{\mathrm{T}} - \mathbf{S}_{2pi}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{Q}}_{1p} + \mathbf{T}_{1pi} + \left(\mathbf{S}_{1pi}^{\mathrm{T}} - \mathbf{S}_{1pi}\right) & \hat{\mathbf{Q}}_{2p} + \mathbf{T}_{2p1} + \left(\mathbf{S}_{2pi}^{\mathrm{T}} - \mathbf{S}_{2pi}\right) & \cdots & \hat{\mathbf{Q}}_p \end{bmatrix} \tag{23}$$

$$\mathbf{H} = \begin{bmatrix} -\sum_{\substack{j=1 \\ j>1}}^{p} w_j\mathbf{R}_{1j} - \sum_{\substack{j=1 \\ j<1}}^{p} w_j\mathbf{R}_{j1} & (w_1 + w_2)\frac{1}{2}\mathbf{R}_{12} & \cdots & (w_1 + w_p)\frac{1}{2}\mathbf{R}_{1p} \\ (w_1 + w_2)\frac{1}{2}\mathbf{R}_{12} & -\sum_{\substack{j=1 \\ j>2}}^{p} w_j\mathbf{R}_{2j} - \sum_{\substack{j=1 \\ j<2}}^{p} w_j\mathbf{R}_{j2} & \cdots & (w_2 + w_p)\frac{1}{2}\mathbf{R}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (w_1 + w_p)\frac{1}{2}\mathbf{R}_{1p} & (w_2 + w_p)\frac{1}{2}\mathbf{R}_{2p} & \cdots & -\sum_{\substack{j=1 \\ j>p}}^{p} w_j\mathbf{R}_{pj} - \sum_{\substack{j=1 \\ j<p}}^{p} w_j\mathbf{R}_{jp} \end{bmatrix} \tag{24}$$

It was shown in [10] that $\begin{bmatrix} w_1\mathbf{I} & w_2\mathbf{I} & \cdots & w_p\mathbf{I} \end{bmatrix}\mathbf{H}\begin{bmatrix} w_1\mathbf{I} & w_2\mathbf{I} & \cdots & w_p\mathbf{I} \end{bmatrix}^{\mathrm{T}} = \mathbf{0}$. Hence, based on this property, from (22), $\widetilde{\mathbf{Q}}\mathbf{t}_i < 0$ for all i imply $\sum_{i=1}^{p} w_i\widetilde{\mathbf{Q}}\mathbf{t}_i < 0$. Let $\max_i\left(\widetilde{\mathbf{Q}}\mathbf{t}_i\right) = \widetilde{\mathbf{Q}}\mathbf{t}_\varsigma < 0$ where $\varsigma = \arg\left(\max_i\left(\widetilde{\mathbf{Q}}\mathbf{t}_i\right)\right)$, which implies that $\sum_{i=1}^{p} w_i\widetilde{\mathbf{Q}}\mathbf{t}_i \leq \sum_{i=1}^{p} w_i\widetilde{\mathbf{Q}}\mathbf{t}_\varsigma = \widetilde{\mathbf{Q}}\mathbf{t}_\varsigma < 0$.

Under the solution of Theorem 4, the stability conditions in Theorem 6 will be considered. Considering $\mathbf{Q}_{ij}$ in Theorem 6, let $\mathbf{P}_1 = \mathbf{P}$, $\mathbf{P}_{2i} = \mathbf{B}_i^{\mathrm{T}}\mathbf{P}$, $\mathbf{P}_{3i} = \varepsilon\mathbf{I}$, i = 1, 2, ..., p, ε is a non-zero positive scalar and $\mathbf{R}_{ij} = \begin{bmatrix} \mathbf{T}_{ij\varsigma} + \left(\mathbf{S}_{ij\varsigma}^{\mathrm{T}} - \mathbf{S}_{ij\varsigma}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ j = 1, 2, ..., p; $i < j$, where ε is a non-zero positive scalar, we have,

$$\mathbf{S}_{ii} = \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{G}_i^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{P} + \left(\mathbf{B}_i^{\mathrm{T}}\mathbf{P}\right)^{\mathrm{T}}\mathbf{G}_i & \varepsilon\mathbf{G}_i^{\mathrm{T}} \\ \varepsilon\mathbf{G}_i & -2\varepsilon\mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i) & \varepsilon\mathbf{G}_i^{\mathrm{T}} \\ \varepsilon\mathbf{G}_i & -2\varepsilon\mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\mathbf{Q}}_i & \varepsilon\mathbf{G}_i^{\mathrm{T}} \\ \varepsilon\mathbf{G}_i & -2\varepsilon\mathbf{I} \end{bmatrix}, i = 1, 2, ..., p \tag{25}$$

$$\mathbf{S}_{ij} = \begin{bmatrix} \hat{\mathbf{Q}}_{ij} + \mathbf{T}_{ij\varsigma} + \left(\mathbf{S}_{ij\varsigma}^{\mathrm{T}} - \mathbf{S}_{ij\varsigma}\right) & \varepsilon\left(\dfrac{\mathbf{G}_i + \mathbf{G}_j}{2}\right)^{\mathrm{T}} \\ \varepsilon\left(\dfrac{\mathbf{G}_i + \mathbf{G}_j}{2}\right) & -2\varepsilon\mathbf{I} \end{bmatrix}, j = 1, 2, ..., p;$$

$$i < j \tag{26}$$

where $\hat{\mathbf{Q}}_i = \mathbf{H}_{ii}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii}$, $\hat{\mathbf{Q}}_{ij} = \left(\dfrac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\dfrac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\right)$ and $\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j$.

In Theorem 6, it is required that $\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \cdots & \mathbf{S}_{1p} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \cdots & \mathbf{S}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{S}_{p1} & \mathbf{S}_{p2} & \cdots & \mathbf{S}_{pp} \end{bmatrix} < 0$ to guarantee the system stability. From (25) and (26), with proper shifting of rows and columns of the matrix **S**, the matrix **S** can be written as $\begin{bmatrix} \mathbf{Qt}_\varsigma & \varepsilon\mathbf{\Theta}^{\mathrm{T}} \\ \varepsilon\mathbf{\Theta} & -2\varepsilon\mathbf{I} \end{bmatrix}$ where $\mathbf{\Theta}$ is a matrix related to $\left(\dfrac{\mathbf{G}_i + \mathbf{G}_j}{2}\right)$ for all i and j. As $\widetilde{\mathbf{Q}}\mathbf{t}_\varsigma < 0$, there must exists a non-zero positive ε such that $\begin{bmatrix} \mathbf{Qt}_\varsigma & \varepsilon\mathbf{\Theta}^{\mathrm{T}} \\ \varepsilon\mathbf{\Theta} & -2\varepsilon\mathbf{I} \end{bmatrix} < 0$. Hence, it can be seen that the solution of the stability conditions in Theorem 4 is also the solution of those in Theorem 6. However, the solution of the stability conditions in Theorem 6 may not be the solution of those in Theorem 4.

VI. Numerical Example

A numerical example will be given to illustrate the effectiveness of the derived stability conditions. Considering the following fuzzy model with $p = s = 2$,

Rule i: IF $x_1(t)$ is M_1^i THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2$ (27)

where $\mathbf{A}_1 = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} a & -10 \\ 1 & 3 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}$, is considered. The feedback gains, $\mathbf{G}_1$ and $\mathbf{G}_2$, of the fuzzy controller is designed such that the eignevalues of $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ are all located at −2 respectively for any values of parameters a and b. Fig. 1 to Fig. 5 show the stability regions for the stability conditions in Theorem 1 to Theorem 6 respectively for parameters $a \in [-10 \quad 4]$ and $b \in [1 \quad 15]$. Referring to these figures, it can be seen that the stability region produced by Theorem 4 is the same as that produced by the modified stability conditions in Theorem 3. However, the number of LMI stability conditions is reduced to p only which can reduce the computational demand on solving the solution. Furthermore, it can be seen that the stability conditions in Theorem 6 provide the largest stability region.

VII. Conclusion

A new set of LMI stability conditions has been derived to guarantee the system stability of the fuzzy-model-based control systems. It has been shown that the proposed stability conditions have provided relaxed stability results than those of some important published stability conditions. The feedback gain design of the fuzzy controller using LMI-based approach has been provided. A numerical example has been given to illustrate the effectiveness of the proposed approach.

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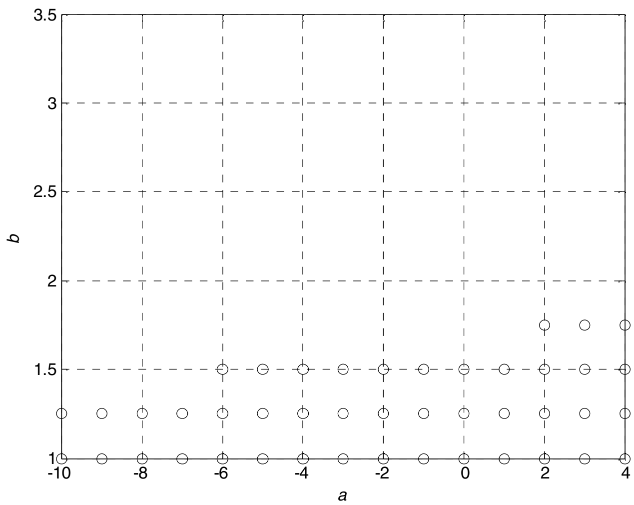

Fig. 1. Stability region based on Theorem 1.

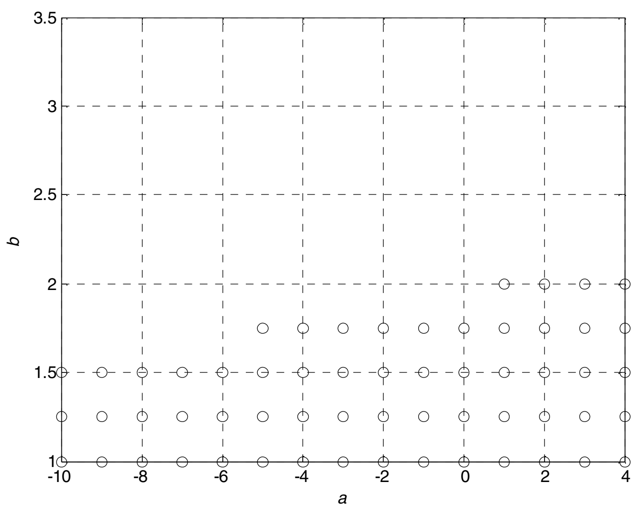

Fig. 2. Stability region based on Theorem 2.

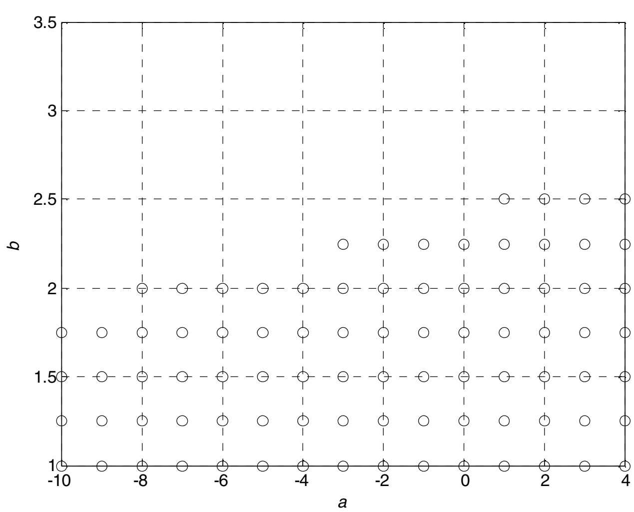

Fig. 3. Stability regions based on Theorem 3 with modification in Remark 2 and Theorem 5.

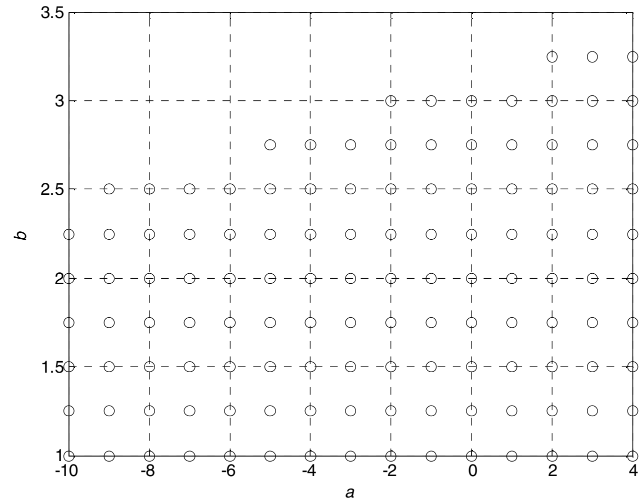

Fig. 4. Stability region based on Theorem 4.

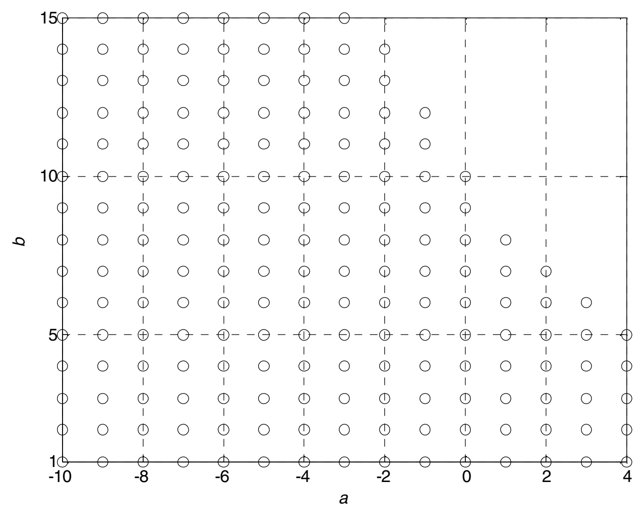

Fig. 5. Stability region based on Theorem 6.