Stability Analysis and Design of Fuzzy Observer-Controller for Fuzzy Systems[1]

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Abstract

Stability of a fuzzy system controlled by a fuzzy observer-controller will be investigated in this paper. In general, the stability condition is a common solution of [p(p+1)+2p)]/2 linear matrix inequalities (LMIs) where p is the number of rules of the fuzzy plant model. In this paper, the number is reduced to 2p+1. Furthermore, gains of the fuzzy observer-controller can be derived from the solution of the LMIs. Separation principle of the fuzzy observer-controller will be proved.

I. INTRODUCTION

Nonlinear system can be represented by a TSK fuzzy model [1-2]. It expresses the dynamics of the nonlinear plant as a weighted sum of some linear systems. This model can be obtained by using system identification methods [1-2, 7, 14] or direct derivation from the mathematical model. Sufficient stability conditions have been derived for the control systems comprising the fuzzy plant model and a fuzzy controller [5-6, 8-10, 13]. In order to deal with plants with immeasurable system states, fuzzy observer-controllers were proposed [3, 12]. In [3, 12], sufficient stability conditions were derived: there exists a common positive definite solution $\mathbf{P} \in \Re^{2n \times 2n}$ in $\frac{p(p+1)}{2}$ linear matrix inequalities (LMIs) [3], where n and p are numbers of system states and fuzzy rules of the fuzzy plant model respectively. Based on the analysis results in [3, 12] and let $\mathbf{P} = \begin{bmatrix} \mathbf{P}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_e \end{bmatrix}$ where $\mathbf{P}_p \in \Re^{n \times n}$ and $\mathbf{P}_e \in \Re^{n \times n}$ are positive definite matrices, the separation principle can be applied [12]. The fuzzy observer-controller control system is guaranteed to be stable if there are common positive definite solutions $\mathbf{P}_p \in \Re^{n \times n}$ in $\frac{p(p+1)}{2}$ LMIs and $\mathbf{P}_e \in \Re^{n \times n}$ in p LMIs. If the input matrix is linear, a linear closed-loop system can be obtained under some conditions [4]. The contributions of this paper are threefold. First, a fuzzy observer-controller is proposed such that the total number of the LMIs of the stability conditions is reduced to $2p+1$. The chance of finding the solution will thus be increased. Second, a methodology is derived to design the gains of the fuzzy observer-controller. Third, the separation principle will still hold.

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II. FUZZY PLANT MODEL, CONTROLLER AND OBSERVER

The nonlinear plant to be tackled is of the following form,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(\mathbf{x}(t))\mathbf{x}(t) \end{cases} \tag{1}$$

where $\mathbf{A}(\mathbf{x}(t)) \in \Re^{n \times n}$ and $\mathbf{B}(\mathbf{x}(t)) \in \Re^{n \times m}$ are known system and input matrices respectively, $\mathbf{x}(t) \in \Re^{n \times 1}$ is the fully or partially immeasurable system state vector and $\mathbf{u}(t) \in \Re^{m \times 1}$ is the input vector; $\mathbf{y}(t) \in \Re^{l \times 1}$ is the system output vector and $\mathbf{C}(\mathbf{x}(t)) \in \Re^{l \times n}$ is the output matrix. The system of (1) is represented by a fuzzy plant model, which expresses the nonlinear plant as a weighted sum of linear systems. A fuzzy observer-controller is used to close the feedback loop.

A. Fuzzy Plant Model

Let p be the number of fuzzy rules,

Rule i: IF f_1 is M_1^i and ... and f_Ψ is M_Ψ^i THEN

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_i\mathbf{x}(t) \end{cases} \tag{2}$$

where M_α^i is a fuzzy term of rule i corresponding to f_α which is a function of the measurable system states or output states of the nonlinear plant, $\alpha = 1, \ldots, \Psi$, $i = 1, \ldots, p$, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n \times n}$, $\mathbf{B}_i \in \Re^{n \times m}$ and $\mathbf{C}_i \in \Re^{l \times n}$ are the known system, input and output matrices of the i-th rule sub-system respectively. The system dynamics are described by,

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \\ \mathbf{y}(t) = \sum_{i=1}^{p} w_i\mathbf{C}_i\mathbf{x}(t) \end{cases}, \tag{3}$$

$$\text{where } \sum_{i=1}^{p} w_i = 1,\ w_i \in [0\ \ 1] \text{ for all } i \tag{4}$$

$$w_i = \frac{\mu_{\mathrm{M}_1^i}(f_1) \times \mu_{\mathrm{M}_2^i}(f_2) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi)}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1) \times \mu_{\mathrm{M}_2^k}(f_2) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi)\right)} \tag{5}$$

is a nonlinear function of the measurable system states or output states, $\mu_{\mathrm{M}_k^i}(f_k)$ is the grade of membership of the fuzzy term M_k^i. (Fuzzy modeling is discussed in [1-2, 7, 14].)

B. Fuzzy Observer

A fuzzy observer having p fuzzy rules is to be designed. The j-th rule of the fuzzy observer is of the following format:

Rule j: IF f_1 is M_1^j and ... and f_Ψ is M_Ψ^j

$$\text{THEN } \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_j\hat{\mathbf{x}}(t) + \mathbf{B}_j\mathbf{u}(t) + \mathbf{K}_j\mathbf{C}_j\mathbf{e}(t) \tag{6}$$

where $\hat{\mathbf{x}}(t) \in \Re^{n \times 1}$ is the estimate of the system state $\mathbf{x}(t)$,

$\mathbf{K}_j \in \Re^{n \times l}$, j = 1, 2, …, p, are the observer gains to be designed, $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. The fuzzy observer has the same antecedents of the fuzzy plant model of (2). The dynamics of the fuzzy observer are described by,

$$\dot{\hat{\mathbf{x}}}(t) = \sum_{j=1}^{p} w_j \left(\mathbf{A}_j \hat{\mathbf{x}}(t) + \mathbf{B}_j \mathbf{u}(t) + \mathbf{K}_j \mathbf{C}_j \mathbf{e}(t) \right) \tag{7}$$

It should be noted that although the sub-observer in each rule is a full-state observer, it can be a partial-state observer as w_j can be a function of the known states in addition to $\mathbf{y}(t)$.

C. Fuzzy Controller

A fuzzy controller having p fuzzy rules is to be designed for the plant. The k-th rule is of the following format:

Rule k: IF f_1 is M_1^k and … and f_Ψ is M_Ψ^k

THEN $\mathbf{u}(t) = m_k(\hat{\mathbf{x}}(t)) \mathbf{G}_k \hat{\mathbf{x}}(t)$ (8)

where $\mathbf{G}_k \in \Re^{m \times n}$ is the feedback gain of rule k, k = 1, 2, …, p; $m_k(\hat{\mathbf{x}}(t))$ is a scalar gain depending on the estimated states and has the following property,

$$\sum_{k=1}^{p} w_k m_k(\hat{\mathbf{x}}(t)) = 1 \tag{9}$$

Both $\mathbf{G}_k$ and $m_k(\hat{\mathbf{x}}(t))$ are to be designed. Then, the inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{k=1}^{p} w_k m_k(\hat{\mathbf{x}}(t)) \mathbf{G}_k \hat{\mathbf{x}}(t) \tag{10}$$

III. Stability Analysis and Gain Design

In the following, we shall analyze the stability and robustness of the fuzzy observer-controller control system and give the design of $m_k(\hat{\mathbf{x}}(t))$. From (3) and (10), writing $m_k(\hat{\mathbf{x}}(t))$ as m_k and making use of the properties of (4) and (9), $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} w_j m_j = 1$, the closed-loop system is given by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i \left(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \sum_{k=1}^{p} w_k m_k \mathbf{G}_k \hat{\mathbf{x}}(t) \right)$$

$$= \mathbf{H}\hat{\mathbf{x}}(t) + \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \mathbf{H}_{ik} \hat{\mathbf{x}}(t) + \sum_{i=1}^{p} w_i \mathbf{A}_i \mathbf{e}(t) \tag{11}$$

where $\mathbf{H}_{ik} = \mathbf{A}_i + \mathbf{B}_i \mathbf{G}_k - \mathbf{H}$ (12)

$\mathbf{H} \in \Re^{n \times n}$ is a stable matrix to be designed. To study the stability of (11), consider the following Lyapunov function.

$$V_p(\mathbf{x}(t)) = \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t) \tag{13}$$

where $\mathbf{P}_p \in \Re^{n \times n}$ is a symmetric positive definite matrix.

$$\dot{V}_p(\mathbf{x}(t)) = \frac{1}{2} \left(\dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \dot{\mathbf{x}}(t) \right) \tag{14}$$

From (11) and (14), we have,

$$\dot{V}_p(\mathbf{x}(t)) = \frac{1}{2} \left(\mathbf{H}\hat{\mathbf{x}}(t) + \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \mathbf{H}_{ik} \hat{\mathbf{x}}(t) + \sum_{i=1}^{p} w_i \mathbf{A}_i \mathbf{e}(t) \right)^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t)$$
$$+ \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \left(\mathbf{H}\hat{\mathbf{x}}(t) + \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \mathbf{H}_{ik} \hat{\mathbf{x}}(t) + \sum_{i=1}^{p} w_i \mathbf{A}_i \mathbf{e}(t) \right)$$

$$= \frac{1}{2} \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t) + \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{H} \hat{\mathbf{x}}(t) + \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{H}_{ik}{}^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t) + \tag{15}$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{H}_{ik} \hat{\mathbf{x}}(t) + \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{A}_i \mathbf{e}(t)$$

From (15) and $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, we have,

$$\dot{V}_p(\mathbf{x}(t)) = \frac{1}{2} (\mathbf{x}(t) - \mathbf{e}(t))^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{P}_p \mathbf{x}(t) + \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{H} (\mathbf{x}(t) - \mathbf{e}(t))$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{H}_{ik}{}^{\mathrm{T}} \mathbf{P}_p (\mathbf{e}(t) + \hat{\mathbf{x}}(t)) +$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k (\mathbf{e}(t) + \hat{\mathbf{x}}(t))^{\mathrm{T}} \mathbf{P}_p \mathbf{H}_{ik} \hat{\mathbf{x}}(t) + \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{A}_i \mathbf{e}(t)$$

$$= -\frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{Q}_p \mathbf{x}(t) - \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{H} \mathbf{e}(t) - \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{Q}_{ik} \hat{\mathbf{x}}(t) \tag{16}$$
$$+ \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \mathbf{e}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{H}_{ik} (\mathbf{x}(t) - \mathbf{e}(t)) + \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}_p \mathbf{A}_i \mathbf{e}(t)$$

where $\mathbf{Q}_p \in \Re^{n \times n}$ and $\mathbf{Q}_{ij} \in \Re^{n \times n}$, i = 1, 2, …, p; k = 1, 2, …, p, are symmetric positive definite matrix and symmetric matrices respectively. They are defined as follows,

$$\mathbf{Q}_p = -\left(\mathbf{H}^{\mathrm{T}} \mathbf{P}_p + \mathbf{P}_p \mathbf{H} \right) \tag{17}$$

$$\mathbf{Q}_{ik} = -\left(\mathbf{H}_{ik}{}^{\mathrm{T}} \mathbf{P}_p + \mathbf{P}_p \mathbf{H}_{ik} \right), \; i = 1, 2, \ldots, p; \; k = 1, 2, \ldots, p \tag{18}$$

Based on the Rayleigh principle, the following property holds for a symmetric positive definite matrix $\mathbf{P} \in \Re^{n \times n}$,

$$0 < \lambda_{\min}(\mathbf{P}) \|\mathbf{x}(t)\|^2 \le \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \le \lambda_{\max}(\mathbf{P}) \|\mathbf{x}(t)\|^2 \tag{19}$$

where $\lambda_{\max}(\mathbf{P})$ and $\lambda_{\min}(\mathbf{P})$ are the maximum and minimum eigenvalues of $\mathbf{P}$ respectively, $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. Then, (16) implies,

$$\dot{V}_p(\mathbf{x}(t)) \le -\frac{1}{2} \lambda_{\min}(\mathbf{Q}_p) \|\mathbf{x}(t)\|^2 - \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{Q}_{ik} \hat{\mathbf{x}}(t)$$
$$+ \|\mathbf{P}_p \mathbf{H}\| \|\mathbf{x}(t)\| \|\mathbf{e}(t)\| + \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \|\mathbf{P}_p \mathbf{H}_{ik}\| \|\mathbf{x}(t)\| \|\mathbf{e}(t)\| \tag{20}$$
$$+ \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \|\mathbf{P}_p \mathbf{H}_{ik}\| \|\mathbf{e}(t)\|^2 + \sum_{i=1}^{p} w_i \|\mathbf{P}_p \mathbf{A}_i\| \|\mathbf{x}(t)\| \|\mathbf{e}(t)\|$$

From (20), (4) and (9), $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} w_j m_j = 1$, (20) becomes,

$$\dot{V}_p(\mathbf{x}(t)) \le -\frac{1}{2} \lambda_{\min}(\mathbf{Q}_p) \|\mathbf{x}(t)\|^2 - \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{Q}_{ik} \hat{\mathbf{x}}(t)$$
$$+ \left(\|\mathbf{P}_p \mathbf{H}\| + \|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} + \|\mathbf{P}_p \mathbf{A}_i\|_{\max} \right) \|\mathbf{x}(t)\| \|\mathbf{e}(t)\| + \|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} \|\mathbf{e}(t)\|^2$$

$$= -\frac{1}{2} \lambda_{\min}(\mathbf{Q}_p) \|\mathbf{x}(t)\|^2 - \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} w_i w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{Q}_{ik} \hat{\mathbf{x}}(t) \tag{21}$$
$$+ \gamma \|\mathbf{x}(t)\| \|\mathbf{e}(t)\| + \|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} \|\mathbf{e}(t)\|^2$$

where $\|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} = \max_{i,k} \|\mathbf{P}_p \mathbf{H}_{ik}\| \ge 0$ (22)

$$\|\mathbf{P}_p \mathbf{A}_i\|_{\max} = \max_{i} \|\mathbf{P}_p \mathbf{A}_i\| \ge 0 \tag{23}$$

$$\gamma = \|\mathbf{P}_p \mathbf{H}\| + \|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} + \|\mathbf{P}_p \mathbf{A}_i\|_{\max} \ge 0 \tag{24}$$

From (21),

$$\dot{V}_p(\mathbf{x}(t)) \le -\frac{1}{4} \lambda_{\min}(\mathbf{Q}_p) \|\mathbf{x}(t)\|^2 - \frac{1}{2} \sum_{k=1}^{p} w_k m_k \hat{\mathbf{x}}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{ik} \hat{\mathbf{x}}(t) \tag{25}$$
$$- \frac{1}{4} \lambda_{\min}(\mathbf{Q}_p) \|\mathbf{x}(t)\|^2 + \gamma \|\mathbf{x}(t)\| \|\mathbf{e}(t)\| + \|\mathbf{P}_p \mathbf{H}_{ik}\|_{\max} \|\mathbf{e}(t)\|^2$$

Consider the following inequality,

$$\left(\sqrt{\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)}\|\mathbf{x}(t)\|-\frac{\gamma}{2\sqrt{\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)}}\|\mathbf{e}(t)\|\right)^2\geq 0$$

$$\Rightarrow \frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2-\gamma\|\mathbf{x}(t)\|\|\mathbf{e}(t)\|+\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}\|\mathbf{e}(t)\|^2\geq 0$$

$$\Rightarrow \frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}\|\mathbf{e}(t)\|^2\geq -\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2+\gamma\|\mathbf{x}(t)\|\|\mathbf{e}(t)\| \quad (26)$$

From (25) and (26), we have,

$$\dot{V}_p(\mathbf{x}(t))\leq -\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2 -\frac{1}{2}\sum_{k=1}^{p}w_k m_k\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}\hat{\mathbf{x}}(t)+\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\|\mathbf{e}(t)\|^2 \quad (27)$$

Let m_k, $k = 1, 2, \ldots, p$, be designed as follows,

$$m_k=\begin{cases}1 & \text{if } w_k=0\\ \dfrac{n_s}{w_k\sum_{s=1}^{p}n_s} & \text{otherwise}\end{cases} \quad (28)$$

$$n_s=\begin{cases}\dfrac{1}{\sum_{k=1}^{p}sign(w_k)} & w_k\neq 0 \text{ and } \mathbf{x}(t)=\mathbf{0}\\ 0 & \text{if } w_k=0\\ \hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{is}\hat{\mathbf{x}}(t) & \text{otherwise}\end{cases}, s=1,2,\ldots p \quad (29)$$

$$sign(z)=\begin{cases}1 & \text{if } z>0\\ 0 & \text{otherwise}\end{cases} \quad (30)$$

and $\sum_{s=1}^{p}\mathbf{Q}_{is}>\mathbf{0}$ (positive definite) for $i = 1, 2, \ldots, p$ (31)

It should be noted that the denominator $\sum_{k=1}^{p}sign(w_k)$ in (29) will never be equal to zero as at least one of the w_k is greater than zero (a property of the fuzzy plant model). By choosing m_k, $k = 1, 2, \ldots, p$, according to (28) and (29), we can see that the property of (9) is satisfied. From (27) to (31) and considering the case that all $w_k\neq 0$, $k = 1, 2, \ldots, p$, such that

$$m_k=\frac{\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}\hat{\mathbf{x}}(t)}{w_k\sum_{s=1}^{p}\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{is}\hat{\mathbf{x}}(t)}, \text{ then (27) becomes,}$$

$$\dot{V}_p(\mathbf{x}(t))\leq -\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2-\frac{1}{2}\sum_{k=1}^{p}w_k\frac{\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}\hat{\mathbf{x}}(t)}{w_k\sum_{s=1}^{p}\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{is}\hat{\mathbf{x}}(t)}\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}\hat{\mathbf{x}}(t) +\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\|\mathbf{e}(t)\|^2$$

$$\leq -\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2-\frac{1}{2}\frac{\sum_{k=1}^{p}\left(\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{ik}\hat{\mathbf{x}}(t)\right)^2}{\sum_{s=1}^{p}\hat{\mathbf{x}}(t)^{\mathrm{T}}\sum_{i=1}^{p}w_i\mathbf{Q}_{is}\hat{\mathbf{x}}(t)}+\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\|\mathbf{e}(t)\|^2 \quad (32)$$

From (31), as $\sum_{s=1}^{p}\mathbf{Q}_{is}$, $i = 1, 2, \ldots, p$, are positive definite matrices, $\dfrac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p}w_i\hat{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\hat{\mathbf{x}}(t)\right)^2}{\sum_{i=1}^{p}\hat{\mathbf{x}}(t)^{\mathrm{T}}w_i\sum_{k=1}^{p}\mathbf{Q}_{ik}\hat{\mathbf{x}}(t)}$ in (32) is semi-positive definite. Thus, from (32), we have,

$$\dot{V}_p(\mathbf{x}(t))\leq -\frac{1}{4}\lambda_{\min}(\mathbf{Q}_p)\|\mathbf{x}(t)\|^2+\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\|\mathbf{e}(t)\|^2 \quad (33)$$

(33) will be used later. Then, we consider the state convergence ability of the fuzzy observer (3). We shall show that $\mathbf{e}(t)=\mathbf{x}(t)-\hat{\mathbf{x}}(t)\to\mathbf{0}$ as $t\to\infty$. Form (3) and (7),

$$\dot{\mathbf{e}}(t)=\dot{\mathbf{x}}(t)-\dot{\hat{\mathbf{x}}}(t)=\sum_{i=1}^{p}w_i\left(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\right)-\sum_{j=1}^{p}w_j\left(\mathbf{A}_j\hat{\mathbf{x}}(t)+\mathbf{B}_j\mathbf{u}(t)+\mathbf{K}_j\mathbf{C}_j\mathbf{e}(t)\right)$$

$$=\sum_{i=1}^{p}w_i\left(\mathbf{A}_i-\mathbf{K}_i\mathbf{C}_i\right)\mathbf{e}(t)=\sum_{i=1}^{p}w_i\mathbf{J}_i\mathbf{e}(t)$$

where $\mathbf{J}_i=\mathbf{A}_i-\mathbf{K}_i\mathbf{C}_i$, $i = 1, 2, \ldots, p$ (35)

To study the stability of (34), consider the Lyapunov function:

$$V_e(\mathbf{e}(t))=\frac{1}{2}\mathbf{e}(t)^{\mathrm{T}}\mathbf{P}_e\mathbf{e}(t) \quad (36)$$

where $\mathbf{P}_e\in\Re^{n\times n}$ is a symmetric positive definite matrix. From (36),

$$\dot{V}_e(\mathbf{e}(t))=\frac{1}{2}\left(\dot{\mathbf{e}}(t)^{\mathrm{T}}\mathbf{P}_e\mathbf{e}(t)+\mathbf{e}(t)^{\mathrm{T}}\mathbf{P}_e\dot{\mathbf{e}}(t)\right) \quad (37)$$

From (34) and (37),

$$\dot{V}_e(\mathbf{e}(t))=\frac{1}{2}\left[\left(\sum_{i=1}^{p}w_i\mathbf{J}_i\mathbf{e}(t)\right)^{\mathrm{T}}\mathbf{P}_e\mathbf{e}(t)+\mathbf{e}(t)^{\mathrm{T}}\mathbf{P}_e\left(\sum_{i=1}^{p}w_i\mathbf{J}_i\mathbf{e}(t)\right)\right]$$

$$=-\frac{1}{2}\sum_{i=1}^{p}w_i\mathbf{e}(t)^{\mathrm{T}}\mathbf{Q}_{e_i}\mathbf{e}(t)\leq -\frac{1}{2}\max_i\lambda_{\min}\left(\mathbf{Q}_{e_i}\right)\|\mathbf{e}(t)\|^2 \quad (38)$$

where $\mathbf{Q}_{e_i}\in\Re^{n\times n}$, $i = 1, 2, \ldots, p$, are a symmetric positive matrices,

$$\mathbf{Q}_{e_i}=-\left(\mathbf{J}_i^{\mathrm{T}}\mathbf{P}_e+\mathbf{P}_e\mathbf{J}_i\right),\ i=1,2,\ldots,p \quad (39)$$

$$\max_i\lambda_{\min}\left(\mathbf{Q}_{e_i}\right)\geq\lambda_{\min}\left(\mathbf{Q}_{e_i}\right)>0,\ i=1,2,\ldots,p \quad (40)$$

From (13), (36) and the property of (19), we have,

$$V_p(\mathbf{x}(t))=\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}_p\mathbf{x}(t)\geq\frac{1}{2}\lambda_{\min}\left(\mathbf{P}_p\right)\|\mathbf{x}(t)\|^2\Rightarrow\frac{2V_p(\mathbf{x}(t))}{\lambda_{\min}\left(\mathbf{P}_p\right)}\geq\|\mathbf{x}(t)\|^2 \quad (41)$$

$$V_e(\mathbf{e}(t))=\frac{1}{2}\mathbf{e}(t)^{\mathrm{T}}\mathbf{P}_e\mathbf{e}(t)\geq\frac{1}{2}\lambda_{\min}\left(\mathbf{P}_e\right)\|\mathbf{x}(t)\|^2\Rightarrow\frac{2V_e(\mathbf{e}(t))}{\lambda_{\min}\left(\mathbf{P}_e\right)}\geq\|\mathbf{e}(t)\|^2 \quad (42)$$

From (33), (38), (41) and (42), we have,

$$\begin{bmatrix}\dot{V}_p(\mathbf{x}(t))\\ \dot{V}_e(\mathbf{e}(t))\end{bmatrix}\leq\begin{bmatrix}-\frac{1}{2}\frac{\lambda_{\min}(\mathbf{Q}_p)}{\lambda_{\min}(\mathbf{P}_p)} & \frac{2}{\lambda_{\min}(\mathbf{P}_e)}\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\\ 0 & -\frac{\max_i\lambda_{\min}(\mathbf{Q}_{e_i})}{\lambda_{\min}(\mathbf{P}_e)}\end{bmatrix}\begin{bmatrix}V_p(\mathbf{x}(t))\\ V_e(\mathbf{e}(t))\end{bmatrix} \quad (43)$$

As $-\dfrac{1}{2}\dfrac{\lambda_{\min}(\mathbf{Q}_p)}{\lambda_{\min}(\mathbf{P}_p)}$ and $-\dfrac{\max_i\lambda_{\min}(\mathbf{Q}_{e_i})}{\lambda_{\min}(\mathbf{P}_e)}$ in (43) are both negative values and are eigenvalues of $\begin{bmatrix}-\frac{1}{2}\frac{\lambda_{\min}(\mathbf{Q}_p)}{\lambda_{\min}(\mathbf{P}_p)} & \frac{2}{\lambda_{\min}(\mathbf{P}_e)}\left(\frac{\gamma^2}{\lambda_{\min}(\mathbf{Q}_p)}+\|\mathbf{P}_p\mathbf{H}_{ik}\|_{\max}\right)\\ 0 & -\frac{\max_i\lambda_{\min}(\mathbf{Q}_{e_i})}{\lambda_{\min}(\mathbf{P}_e)}\end{bmatrix}$, it implies

$\begin{bmatrix} \dot{V}_p(\mathbf{x}(t)) \\ \dot{V}_e(\mathbf{e}(t)) \end{bmatrix} \leq \mathbf{0}$ [11]. Hence, we can conclude that the fuzzy observer-controller system of (11) is asymptotically stable, i.e. $\mathbf{x}(t) \to \mathbf{0}$ and $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$. The same result is obtained under different cases of m_k, $k = 1, 2, \ldots, p$. It can be seen that in (43), the separation property holds, as the design of the fuzzy controller and observer can be achieved separately by finding $\mathbf{P}_p$ and $\mathbf{P}_e$ independently. The result and the design of the m_k are summarized into the following Lemma:

Lemma 1: *A fuzzy observer-controller system formed by* (3), (7) *and* (10) *is guaranteed to be asymptotically stable if the following conditions are satisfied,*

(i). There exists a solution $\mathbf{P}_p$ *for the following (p+1) LMIs,*

$$\begin{cases} \mathbf{Q} > \mathbf{0} \\ \sum_{s=1}^{p} \mathbf{Q}_{is} > \mathbf{0}, \; i = 1, 2, \ldots, p \end{cases}$$

where $\mathbf{Q} = -\mathbf{H}^{\mathrm{T}}\mathbf{P}_p - \mathbf{P}_p\mathbf{H}$, $\mathbf{H}$ *is a stable matrix,* $\mathbf{Q}_{ik} = -\left(\mathbf{H}_{ik}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ik}\right)$, $\mathbf{H}_{ik} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_k - \mathbf{H}$.

(ii). The gains of the fuzzy controller sub-controllers, m_k, *k = 1, 2, ..., p, are designed as follows,*

$$m_k = \begin{cases} 1 & \text{if } w_k = 0 \\ \dfrac{n_s}{w_k \sum_{s=1}^{p} n_s} & \text{otherwise} \end{cases}$$

$$n_s = \begin{cases} \dfrac{1}{\sum_{k=1}^{p} sign(w_k)} & w_k \neq 0 \text{ and } \mathbf{x}(t) = \mathbf{0} \\ 0 & \text{if } w_k = 0 \\ \hat{\mathbf{x}}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{is} \hat{\mathbf{x}}(t) & \text{otherwise} \end{cases}$$

$$sign(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

(iii) There exists a positive definite matrix $\mathbf{P}_e$ *such that* $\mathbf{Q}_{e_i} > \mathbf{0}$, *i = 1, 2, ..., p, where* $\mathbf{Q}_{e_i} = -\left(\mathbf{J}_i{}^{\mathrm{T}}\mathbf{P}_e + \mathbf{P}_e\mathbf{J}_i\right)$, $\mathbf{J}_i = \mathbf{A}_i - \mathbf{K}_i\mathbf{C}_i$, *i = 1, 2, ..., p, are stable matrices.*

From Lemma 1, the total number of LMIs is $2p+1$. The steps for finding the fuzzy observer-controller are as follows.

Step I) Obtain the fuzzy plant model of the system.

Step II) Choose $\mathbf{K}_i$, $i = 1, 2, \ldots, p$, for the fuzzy observer. Find $\mathbf{P}_e$ by solving the p LMIs as stated in (iii) of Lemma 1. If $\mathbf{P}_e$ cannot be found, choose other $\mathbf{K}_i$, $i = 1, 2, \ldots, p$.

Step III) Choose a stable matrix $\mathbf{H}$ and the $\mathbf{G}_k$, $k = 1, 2, \ldots, p$ for the fuzzy controller. Find $\mathbf{P}_p$ by solving the $p+1$ LMIs as stated in (i) of Lemma 1. If $\mathbf{P}_p$ cannot be found, choose other $\mathbf{H}$ and $\mathbf{G}_k$, $k = 1, 2, \ldots, p$, for the fuzzy controller.

Step IV). Design m_k, $k = 1, \ldots, p$, as stated in (ii) of Lemma 1.

IV. CONCLUSION

The stability of fuzzy observer-controller control systems has been analyzed. The stability conditions, which involve $2p+1$ LMIs instead of $\left(p(p+1)+2p\right)/2$ LMIs, have been derived. A design methodology for the gains of the fuzzy controller has been given. The separation principle holds during the design of the observer and the controller.

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