Stability and Robustness Analysis of Uncertain Multivariable Continuous-Time Nonlinear Systems with Digital Fuzzy Controller[1]

H. K. Lam, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
F. H. F. Leung, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
P. K. S. Tam, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

Abstract

This paper presents the analyses of the stability and robustness of multivariable continuous-time nonlinear systems subject to parameter uncertainties and with digital fuzzy controllers. To proceed with the analysis, first, an uncertain multivariable nonlinear plant will be represented by a fuzzy plant model with parameter uncertainties. Second, a digital fuzzy controller is designed to close the feedback loop. Third, the stability criteria, the robust area and the largest sampling period will be derived in terms of the matrix measures of the system parameters and the norms of the parameter uncertainties. An application example on stabilizing an uncertain nonlinear mass-spring-damper system will be given to show the stabilizibility and robustness properties of the proposed digital fuzzy controller.

1. Introduction

Control of multivariable nonlinear systems is a difficult task. The problem becomes more complex when the plant parameters are uncertain. This paper is to analyze the stability and robustness of this class of systems with digital fuzzy controllers.

Due to the rapid advancement in the technology of digital circuits, high performance computers and DSPs can be obtained in low price. Consequently, digital control plays an important role in control engineering. To tackle a linear system with a digital controller, we have to transform its s-domain transfer function into a z-domain transfer function. Based on the z-domain transfer function, we can design a stable controller [2]. However, when the system is nonlinear and subject to parameter uncertainties, we cannot apply the same way to design the controller. In this paper, we tackle this class of systems using the fuzzy control approach.

To design a stable fuzzy controller, there are a number of ways [4, 6, 7]. The authors of [1] proposed a fuzzy model to describe the behavior of a nonlinear plant. Based on this model, a stability condition was derived [3, 9]. However, when the fuzzy model cannot describe the plant exactly, the stability condition becomes invalid. Some stability conditions have been derived for uncertain fuzzy model based control systems [5, 8, 10]. Nevertheless, these analyses were only carried out in solely analog systems or digital systems. The analysis on sampled data systems with uncertain continuous-time nonlinear plants and digital fuzzy controllers interfaced by sampler and zero-order-hold are rare. In this paper, we will derive the stability criterion, the robust area and the largest sampling period for this class of systems.

2. Fuzzy plant model and fuzzy controller

An uncertain multivariable fuzzy control system can be regarded as consisting of a fuzzy plant model and a digital fuzzy controller closing the feedback loop.

2.1. Fuzzy plant model with uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The i-th rule is of the following format,

Rule i: IF x_1 is M_1^i and … and x_n is M_n^i

THEN $\dot{\mathbf{x}} = (\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u}$ (1)

where M_k^i is a fuzzy term of rule i corresponding to the state x_k, $k = 1, \ldots, n$, $i = 1, \ldots, p$; $\Delta\mathbf{A}^i \in \Re^{n\times n}$ and $\Delta\mathbf{B}^i \in \Re^{n\times m}$ are the uncertainties of $\mathbf{A}^i \in \Re^{n\times n}$ and $\mathbf{B}^i \in \Re^{n\times m}$ respectively; $\mathbf{x} \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u} \in \Re^{m\times 1}$ is the input vector. The inferred system states are given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w^i(\mathbf{x})\left((\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x}(t) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u}(t)\right) \quad (2)$$

$$\sum_{i=1}^{p} w^i(\mathbf{x}) = 1, \; w^i(\mathbf{x}) \in [0, \;\; 1] \text{ for all } i \quad (3)$$

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$$w^i(\mathbf{x}) = \frac{\mu_{\mathrm{M}_1^i}(x_1) \circ \mu_{\mathrm{M}_2^i}(x_2) \circ \cdots \circ \mu_{\mathrm{M}_n^i}(x_n)}{\sum_{i=1}^{p}(\mu_{\mathrm{M}_1^i}(x_1) \circ \mu_{\mathrm{M}_2^i}(x_2) \circ \cdots \circ \mu_{\mathrm{M}_n^i}(x_n))} \quad (4)$$

$\mu_{\mathrm{M}_k^i}(x_k)$ is the grade of membership and '$\circ$' denotes the t-norm operator.

2.2. Fuzzy controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The j-th rule of the fuzzy controller is of the following format:

Rule j: IF x_1 is N_1^j and … and x_n is N_n^j

THEN $\mathbf{u}(t) = \mathbf{G}^j\mathbf{x}(k\tau) + \mathbf{r}$, $k\tau \le t < (k+1)\tau$ (5)

where N_l^j is a fuzzy term of rule j corresponding to the state x_l, $l = 1, \ldots, n$, $j = 1, \ldots, c$, k is a positive integer, τ is the sampling period, $\mathbf{G}^j \in \Re^{m\times n}$ is the feedback gain of rule j, $\mathbf{r} \in \Re^{m\times 1}$ is the input vector. The inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{c} m^j(\mathbf{x})\mathbf{G}^j(\mathbf{x}(k\tau) + \mathbf{r}) \quad (6)$$

$$\sum_{j=1}^{c} m^j(\mathbf{x}) = 1, \; m^j(\mathbf{x}) \in [0, \; 1] \quad \text{for all } j \quad (7)$$

$$m^j(\mathbf{x}) = \frac{\mu_{\mathrm{N}_1^j}(x_1) \circ \mu_{\mathrm{N}_2^j}(x_2) \circ \cdots \circ \mu_{\mathrm{N}_n^j}(x_n)}{\sum_{j=1}^{c}(\mu_{\mathrm{N}_1^j}(x_1) \circ \mu_{\mathrm{N}_2^j}(x_2) \circ \cdots \circ \mu_{\mathrm{N}_n^j}(x_n))} \quad (8)$$

$\mu_{\mathrm{N}_l^j}(x_l)$ is the grade of membership.

3. Stability and robustness analysis

The stability and robustness of an uncertain fuzzy control system are to be analyzed in this section. In order to carry out the analysis, the closed-loop fuzzy system should be obtained first. From (1) to (8), and reshuffling the terms, the closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j ((\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x}(t) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^j(\mathbf{x}(k\tau) - \mathbf{x}(t)) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}) \quad (9)$$

$$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j, \Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j \quad (10)$$

Let $\mathbf{T} \in \Re^{n\times n}$ be a transformation matrix in rank n, $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. The results can be summarized by the following three theorems.

Theorem 1. *The fuzzy control system without uncertainty, i.e.* $\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| = 0$, *and* $\tau = 0$ *is stable if the following inequalities hold.*

$\mu\left[\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right] \le -\varepsilon$ *for all i and j*

where $\mu\left[\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right] = \lim_{\Delta t\to 0^+}\frac{\left\|\mathbf{I} + \mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\Delta t\right\| - 1}{\Delta t} = \lambda_{max}\left(\frac{\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1} + (\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1})^*}{2}\right)$ *is the corresponding matrix measure of the induced norm* $\left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|$ *(or the logarithmic derivative of* $\left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|$*);* $\lambda_{max}(\cdot)$ *denotes the largest eigenvalue of the argument matrix;* $(\cdot)^*$ *denotes the conjugate transpose of the argument matrix.*

Definition 1: *The robust area of a fuzzy control system is defined as the area in the parameter space inside which uncertainties are allowed to exist without affecting the system stability.*

Theorem 2. *The robust area of the fuzzy control system with* $\tau = 0$ *is governed by,*

$\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} \le -\mu\left[\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right] - \varepsilon$ *for all i and j*

The uncertain fuzzy control system with $\tau = 0$ *is stable if the uncertainty* $\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|$, *with* $\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| \le \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\max} \le \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}}$ *for all i and j*

Theorem 3: *If the fuzzy control systems with and without parameter uncertainties are guaranteed stable by Theorem 1 and Theorem 2 respectively, the sampling period,* τ, *can be chosen from the range:*

$$0 \le \tau \le \ln\frac{\left(\frac{\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} - \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\max}}{\left\|\mathbf{T}\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\right\|_{\max}} - 1\right)}{\varepsilon}$$

Proof:

Consider the Taylor's series,

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \dot{\mathbf{x}}(t)\Delta t + \mathbf{o}(\Delta t) \quad (11)$$

where $\mathbf{o}(\Delta t)$ represents the higher order terms and $\Delta t > 0$,

$$\lim_{\Delta t\to 0^+}\frac{\|\mathbf{o}(\Delta t)\|}{\Delta t} = 0 \quad (12)$$

From (9) and (11), and multiplying a $\mathbf{T}$ to both sides of (9) (the reason for introducing the transformation matrix will be discussed at the end of this section),

$$\mathbf{Tx}(t+\Delta t) = (\mathbf{I} + \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{TH}^{ij}\mathbf{T}^{-1}\Delta t)\mathbf{Tx}(t) + \sum_{i=1}^{p}\sum_{j=1}^{c}(\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{x}(t)$$

$$+\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^i(\mathbf{x}(k\tau) - \mathbf{x}(t)) + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})\Delta t + \mathbf{To}(\Delta t)$$

$$\Rightarrow \|\mathbf{Tx}(t+\Delta t)\| \le \left\|(\mathbf{I} + \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{TH}^{ij}\mathbf{T}^{-1}\Delta t)\mathbf{Tx}(t)\right\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{x}(t) \right. \quad (13)$$

$$\left. + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^i(\mathbf{x}(k\tau) - \mathbf{x}(t)) + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})\Delta t\right\| + \|\mathbf{T}\|\|\mathbf{o}(\Delta t)\|$$

From (13),

$$\lim_{\Delta t \to 0^+} \frac{\|\mathbf{Tx}(t+\Delta t)\| - \|\mathbf{Tx}(t)\|}{\Delta t} \le \lim_{\Delta t \to 0^+} \{(\left\|\mathbf{I} + \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{TH}^{ij}\mathbf{T}^{-1}\Delta t\right\| - 1)\|\mathbf{Tx}(t)\|$$

$$+\left\|[\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{x}(t) + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^i(\mathbf{x}(k\tau) - \mathbf{x}(t)) \right. \quad (14)$$

$$\left. + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})]\Delta t\right\| + \|\mathbf{T}\|\|\mathbf{o}(\Delta t)\|\} / \Delta t$$

From (12) and (14),

$$\frac{\mathrm{d}\|\mathbf{Tx}(t)\|}{\mathrm{d}t} \le \lim_{\Delta t \to 0^+} \frac{\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\|\mathbf{I} + \mathbf{TH}^{ij}\mathbf{T}^{-1}\Delta t\| - 1)}{\Delta t}\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{x}(t)\right.$$

$$\left. + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^i(\mathbf{x}(k\tau) - \mathbf{x}(t)) + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})\right\|$$

$$\le \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}]\|\mathbf{Tx}(t)\|$$

$$+\left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{x}(t) + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{G}^i(\mathbf{x}(k\tau) - \mathbf{x}(t)) \right. \quad (15)$$

$$\left. + \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})\right\|$$

Hence,

$$\frac{\mathrm{d}\|\mathbf{Tx}(t)\|}{\mathrm{d}t} \le \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}]\|\mathbf{Tx}(t)\| + \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|\|\mathbf{Tx}(t)\|$$

$$+ \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|)\|\mathbf{T}(\mathbf{x}(k\tau) - \mathbf{x}(t))\|$$

$$+\left\|\sum_{i=1}^{p} w^i \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|$$

Let $\|\mathbf{T}(\mathbf{x}(k\tau) - \mathbf{x}(t))\| \le \gamma\|\mathbf{Tx}(t))\|$ (16)

where γ is a positive constant. Then,

$$\frac{\mathrm{d}\|\mathbf{Tx}(t)\|}{\mathrm{d}t} \le \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}] + \|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\| \quad (17)$$

$$+ (\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|)\gamma)\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p} w^i \mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|$$

Let $\mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}]$ for all i and j be designed such that

$$\mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|_{\max} - (\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max})\gamma) - \varepsilon \quad (18)$$

where $\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|_{\max} \ge \|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|$, $\|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max} \ge \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|$ and ε is a designed positive constant. Increasing the value of ε will usually result in a system with improved performance but degraded robustness. From (17) and (18).

$$\frac{\mathrm{d}\|\mathbf{Tx}(t)\|}{\mathrm{d}t} \le -\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \varepsilon\|\mathbf{Tx}(t)\| + \sum_{i=1}^{p} w^j \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|$$

$$= -\varepsilon\|\mathbf{Tx}(t)\| + \sum_{i=1}^{p} w^i \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)}) \le \sum_{i=1}^{p} w^i \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|e^{\varepsilon(t-t_o)} \quad (19)$$

where t_o is an arbitrary initial time. Based on (19), there are two cases to investigate the system behavior: $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \ne \mathbf{0}$. For the former case, it can be shown that if the condition of (18) is satisfied, the closed-loop system of (9) is stable, so $\|\mathbf{Tx}(t)\| \to 0$ which implies $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

Proof. For $\mathbf{r} = \mathbf{0}$, from (19),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)}) \le 0$$

$$\Rightarrow \|\mathbf{Tx}(t)\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} \quad (20)$$

Since ε is a positive value, $\|\mathbf{Tx}(t)\| \to 0$ as $t \to \infty$.

Consider,

$$\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\|\mathbf{x}(t)\|^2 \le \|\mathbf{Tx}(t)\|^2 = \mathbf{x}(t)^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{Tx}(t) \le \sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\|\mathbf{x}(t)\|^2 \quad (21)$$

where $\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ and $\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ denote the maximum and minimum singular value of $\mathbf{T}^{\mathrm{T}}\mathbf{T}$ respectively. As $\mathbf{T}^{\mathrm{T}}\mathbf{T}$ must be symmetric positive definite, from (21), $\|\mathbf{Tx}(t)\| \to 0$ only when $\|\mathbf{x}(t)\| \to 0$.

For the latter case of $\mathbf{r} \ne \mathbf{0}$, the system states are bounded if the condition of (18) is satisfied and $\mathbf{r}$ is bounded.

Proof. For $\mathbf{r} \ne \mathbf{0}$, from (19),

$$\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\| + \int_{t_o}^{t}\sum_{i=1}^{p} w^i \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|e^{\varepsilon(\tau-t_o)}\mathrm{d}\tau$$

$$\Rightarrow \|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|\int_{t_o}^{t} e^{\varepsilon(\tau-t_o)}\mathrm{d}\tau$$

where $\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\| \ge \max_i \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|_{\max} \ge \|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|$

then $\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\| + \dfrac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(e^{\varepsilon(t-t_o)} - 1)$

$$\Rightarrow \|\mathbf{Tx}(t)\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}} + \Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1 - e^{-\varepsilon(t-t_o)}) \quad (22)$$

Since the right hand side of (22) is finite if $\mathbf{r}$ is bounded, $\|\mathbf{Tx}\|$ is also bounded. From (21), the system states are also bounded, i.e. $\|\mathbf{x}\|$ is bounded. Consequently, the condition of (18) is a sufficient stability condition of the fuzzy control system. With (18), Theorem 1 and Theorem 2 can readily be obtained. This ends the proofs of Theorem 1 and Theorem 2. **QED**

The above analysis gives an upper bound to the norm of the solution in each case ((20) or (22)). Similarly, a lower bound can be obtained by following the same analysis procedures with the Taylor's series, $\mathbf{x}(t - \Delta t) = \mathbf{x}(t) - \dot{\mathbf{x}}(t)\Delta t + \mathbf{o}(\Delta t)$. Hence, the norm of the system states satisfies the following conditions,

$$\frac{\|\mathbf{Tx}(t_o)\|e^{-\eta(t-t_o)}}{\left|\sqrt{\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}\right|} \le \|\mathbf{x}(t)\| \le \frac{\|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)}}{\left|\sqrt{\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}\right|} \text{ for } \mathbf{r}=\mathbf{0} \tag{23}$$

or

$$\max\left[\frac{\|\mathbf{Tx}(t_o)\|e^{-\eta(t-t_o)} - \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\eta(t-t_o)})}{\left|\sqrt{\sigma_{\max}(\mathbf{T}^{T}\mathbf{T})}\right|}, 0\right] \text{ for } \mathbf{r}\neq\mathbf{0} \tag{24}$$

$$\le \|\mathbf{x}(t)\| \le \frac{\|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-t_o)})}{\left|\sqrt{\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})}\right|}$$

where $\mu[-\mathbf{TH}^{ij}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|_{\max} - (\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max})\gamma) + \eta$ for all i and j (25)

and η is a designed positive constant. From (23) or (24), the performance of the closed-loop system can be predicted. Condition (18) provides a sufficient criterion of stability for the system of (9).

Next, we prove Theorem 3 and derive the largest sampling period allowed. Consider,

$$\|\mathbf{Tx}(t)\| = \|\mathbf{Tx}(t) - \mathbf{Tx}(k\tau) + \mathbf{Tx}(k\tau)\|$$
$$\ge \|\mathbf{T}(\mathbf{x}(k\tau) - \mathbf{x}(t))\| - \|\mathbf{Tx}(k\tau)\| \tag{26}$$

From (22) and (26),

$$\|\mathbf{T}(\mathbf{x}(k\tau)-\mathbf{x}(t))\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-t_o)}) + \|\mathbf{T}(\mathbf{x}(k\tau)\|$$
$$= \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-t_o)})$$
$$+ \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(k\tau-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(k\tau-t_o)})$$

Let $k\tau = t-\psi$ where $k\tau \le t < (k+1)\tau$, k is a positive integer, ψ is a positive real number. Then,

$$\|\mathbf{T}(\mathbf{x}(k\tau)-\mathbf{x}(t))\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-t_o)})$$
$$+ \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-\psi-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-\psi-t_o)})$$
$$= \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)}(1+e^{\varepsilon\psi}) + 2\frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}$$
$$- \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}e^{-\varepsilon(t-t_o)}(1+e^{\varepsilon\psi})$$
$$= (\|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} + \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1-e^{-\varepsilon(t-t_o)}))(1+e^{\varepsilon\psi})$$
$$+ 2\frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon} - \frac{\|\mathbf{T}(\hat{\mathbf{B}}+\Delta\hat{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1+e^{\varepsilon\psi})$$
$$\le (1+e^{\varepsilon\psi})\|\mathbf{Tx}(t)\| \tag{27}$$

From (16) and(27), we can conclude that

$$\gamma = (1+e^{\varepsilon\psi}) \tag{28}$$

From (18) and (28),

$$\mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}] \le -\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|_{\max}$$
$$-(\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max})(1+e^{\varepsilon\psi})) - \varepsilon$$
$$\Rightarrow 0 \le \tau \le \psi \le \psi^{ij}$$
$$= \ln\frac{\left(\frac{-\varepsilon-\mu[\mathbf{TH}^{ij}\mathbf{T}^{-1}] - \|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\|_{\max}}{\|\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max}} - 1\right)}{\varepsilon} \text{ for all } i \text{ and } j. \tag{29}$$

This ends the proof of Theorem 3. **QED**

With the use of a transformation matrix **T**, any Hurwitz matrix having a positive or zero matrix measures (e.g. a matrix in phase variable canonical form has a zero matrix measure) can be transformed into another matrix having a negative matrix measure. The stability criteria derived can then be applied.

4. Application Example

An uncertain nonlinear mass-spring-damper system is given as an application example [8]. In the following, we will design a stable and robust digital fuzzy controller for this system based on the derived theorems. The behavior of the uncertain nonlinear mass-spring-damper system is described by,

$$M\ddot{x} + g(x,\dot{x}) + f(x) = \phi(\dot{x})u \tag{30}$$

where M is the mass and u is the force, $f(x)$ describes the spring nonlinearity and uncertainty, $g(x,\dot{x})$ describes the damper nonlinearity and uncertainty, and $\phi(\dot{x})$ describes the input nonlinearity and uncertainty. Let

$$g(x,\dot{x}) = D(c_1x + c_2\dot{x} + c_3(t)\dot{x})$$
$$f(x) = K(c_4x + c_5x^3 + c_6(t)x) \tag{31}$$
$$\phi(\dot{x}) = 11 + c_7\dot{x}^2 + c_8(t)\cos(5\dot{x})$$

The operating range of the states is $x \in [-1.5, 1.5]$ and $\dot{x} \in [-4, 4]$. The parameters are chosen as follows: $M = D = K = 1.0$, $c_1 = 40$, $c_2 = 18$, $c_3(t) = \frac{c_3^U + c_3^L}{2} + (c_3^L - \frac{c_3^U + c_3^L}{2})\sin(10t)$ so that $c_3(t) \in [c_3^L, c_3^U]$, $c_4 = 37.99$, $c_5 = -0.1$, $c_6(t) = \frac{c_6^U + c_6^L}{2} + (c_6^L - \frac{c_6^U + c_6^L}{2})\cos(5t)$ so that $c_6(t) \in [c_6^L, c_6^U]$, $c_7 = -0.125$, $c_8(t) = \frac{c_8^U + c_8^L}{2} + (c_8^L - \frac{c_8^U + c_8^L}{2})\cos(5t)$ so that $c_8(\dot{x}) \in [c_8^L, c_8^U]$, $c_3^L = -0.05$, $c_6^L = -1.2$, $c_8^L = -0.35$, $c_3^U = 0.05$, $c_6^U = 1.2$, $c_8^U = 0.35$. It should be noted that although the parameter uncertainties c_3, c_6 and c_8 are modeled as functions of time t in order to illustrate the performance of the designed controller, the behavior of the parameter uncertainties are unknown practically. The equations used to describe the uncertainties are just for the illustrating the robustness property of the controller. Now, the nonlinear system becomes

$$\ddot{x} = -18\dot{x} - 77.99x + 0.1\dot{x}^3 - c_3\dot{x} - c_6(t)x + (11 - \frac{8}{27}\dot{x}^2 + c_7(t))u \quad (32)$$

which can exactly be represented by the following rules.

Rule i: IF x is M_1^i and $\dot{x}$ is M_2^i

THEN $\dot{\mathbf{x}} = (\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)u$, i = 1, 2, 3, 4 (33)

with the t-norm operation to be taken as multiplication. The rules of the fuzzy controller are defined as follows,

Rule j: IF x is N_1^j and $\dot{x}$ is N_2^j

THEN $u = \mathbf{G}^1\mathbf{x}(k\tau)$, j = 1, 2, 3, 4 (34)

The membership functions of M_k^i, i = 1, 2, 3, 4, k = 1, 2, are $\mu_{\mathrm{M}_1^1}(x) = \mu_{\mathrm{M}_1^2}(x) = 1 - \frac{x^2}{2.25}$, $\mu_{\mathrm{M}_1^3}(x) = \mu_{\mathrm{M}_1^4}(x) = \frac{x^2}{2.25}$, $\mu_{\mathrm{M}_2^1}(\dot{x}) = \mu_{\mathrm{M}_2^3}(\dot{x}) = 1 - \frac{\dot{x}^2}{16}$, $\mu_{\mathrm{M}_2^2}(\dot{x}) = \mu_{\mathrm{M}_2^4}(\dot{x}) = \frac{\dot{x}^2}{16}$; the membership functions of N_l^j, j = 1,2,3,4, l = 1,2, are $\mu_{\mathrm{N}_1^1}(x) = \mu_{\mathrm{N}_1^2}(x) = 1 - \frac{|x|}{15}$, $\mathrm{N}_1^3(x) = \mathrm{N}_1^4(x) = \frac{|x|}{15}$, $\mu_{\mathrm{N}_2^1}(\dot{x}) = \mu_{\mathrm{N}_2^3}(\dot{x}) = 1 - \frac{|\dot{x}|}{15}$, $\mu_{\mathrm{N}_2^2}(\dot{x}) = \mu_{\mathrm{N}_2^4}(\dot{x}) = \frac{|\dot{x}|}{15}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 = x$, $x_2 = \dot{x} + 10x_1$; $\mathbf{A}^1 = \begin{bmatrix} -10 & 1 \\ 2.01 & -8 \end{bmatrix}$, $\mathbf{A}^2 = \begin{bmatrix} -10 & 0 \\ 2.01 & -8 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} -10 & 0 \\ 2.235 & -8 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -10 & 0 \\ 2.235 & -8 \end{bmatrix}$; $\mathbf{B}^1 = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, $\mathbf{B}^2 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$, $\mathbf{B}^3 = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, $\mathbf{B}^4 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$; $\Delta\mathbf{A}^1 = \Delta\mathbf{A}^2 = \Delta\mathbf{A}^3 = \Delta\mathbf{A}^4 = \begin{bmatrix} 0 & 1 \\ 10c_3(t) - c_4(t) & -c_3(t) \end{bmatrix}$, $\Delta\mathbf{B}^1 = \Delta\mathbf{B}^2 = \Delta\mathbf{B}^3 = \Delta\mathbf{B}^4 = \begin{bmatrix} 0 \\ c_8(t)\cos(5\dot{x}) \end{bmatrix}$. The feedback gains are designed as $\mathbf{G}^{11}$=[−0.1827 −0.1818], $\mathbf{G}^{12}$=[−0.2233 −0.2222], $\mathbf{G}^{13}$=[−0.2032 −0.1818] and $\mathbf{G}^{14}$=[−0.2483 −0.2222] so that $\mathbf{H}^{11} = \mathbf{H}^{22} = \mathbf{H}^{22} = \mathbf{H}^{22} = \begin{bmatrix} -10 & 1 \\ 0 & -10 \end{bmatrix}$. The stability and robustness analysis results are tabulated in Table I. From this table, we can see that $\min(\psi^{ij})$ = 0.1153. From Theorem 3, $\tau \leq$ 0.1153. Let τ = 0.1, $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Figure 1 and Figure 2 show the responses of $x(t)$ and $\dot{x}(t)$ with initial states of $\mathbf{x}(0) = [-1,\ -1]$. Moreover, from (29), the system performance can be predicted to lie inside the range: $\sqrt{2}e^{-\eta t} \leq \|\mathbf{x}(t)\| \leq \sqrt{2}e^{-\varepsilon t}$ (35)

where ε and η are chosen as 0.1 (greater than the smallest absolute value of $\mu[\mathbf{H}^{ij}] + \|\Delta\mathbf{H}^{ij}\|_{\max} + \gamma(\|\mathbf{T}\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max})$ in Table I) and 19.9523 (the largest value of $\mu[-\mathbf{H}^{ij}] + \|\Delta\mathbf{H}^{ij}\|_{\max} + \gamma(\|\mathbf{T}\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\| + \|\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\|_{\max})$ in Table I) respectively, $\gamma = (1 + e^{\varepsilon\tau}) = 2.0101$.

5. Conclusions

Digital fuzzy controlled continuous-time systems subject to parameter uncertainties have been analyzed. The stability criterion, the robust area and the largest sampling period of the digital fuzzy controller have been derived. Based of the analysis results, a digital fuzzy controller has been designed to stabilize an uncertain nonlinear mass-spring-damper system.

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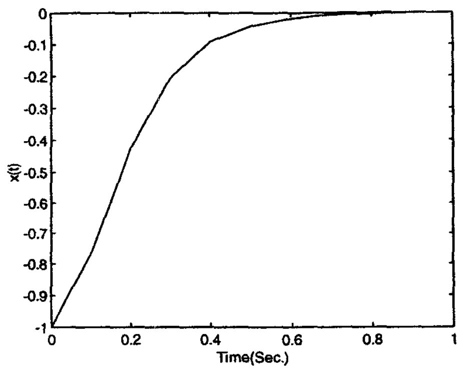

Figure 1. Response of $x(t)$

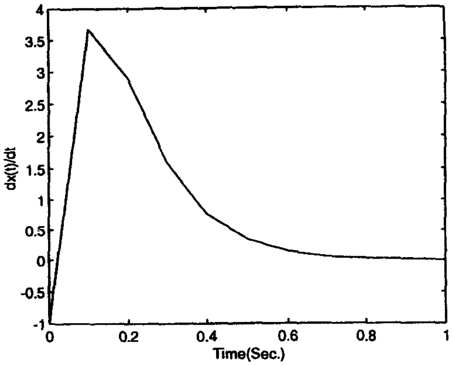

Figure 2. Response of $\dot{x}(t)$

i,j	$\mu[\mathbf{H}^{ij}]$	$\mu[-\mathbf{H}^{ij}]$	$\lVert\Delta\mathbf{H}^{ij}\rVert_{\text{Robust area}}$	$\mu[\mathbf{H}^{ij}]+\lVert\Delta\mathbf{H}^{ij}\rVert_{\max}+\gamma(\lVert\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\rVert+\lVert\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\rVert_{\max})$	$\mu[-\mathbf{H}^{ij}]+\lVert\Delta\mathbf{H}^{ij}\rVert_{\max}+\gamma(\lVert\mathbf{TB}^i\mathbf{G}^i\mathbf{T}^{-1}\rVert+\lVert\mathbf{T}\Delta\mathbf{B}^i\mathbf{G}^i\mathbf{T}^{-1}\rVert_{\max})$	ψ^{ij}
1, 1	−9.5000	10.5000	9.4000	−1.8514	18.1486	4.7544
1, 2	−9.8674	10.5771	9.7674	−0.8967	19.5477	2.0937
1, 3	−9.6125	10.3875	9.5125	−1.6172	18.3828	4.0572
1, 4	−9.9600	10.4844	9.8600	−0.5657	19.8788	1.2506
2, 1	−9.1117	10.5247	9.0117	−1.4630	18.1733	3.8932
2, 2	−9.5000	10.5000	9.4000	−0.5294	19.4706	1.2237
2, 3	−9.2002	10.4362	9.1002	−1.2049	18.4315	3.1271
2, 4	−9.6125	10.3875	9.5125	−0.2182	19.7818	0.4051
3, 1	−9.3875	10.6125	9.2875	−1.7389	18.2611	4.5124
3, 2	−9.7741	10.6704	9.6741	−0.8034	19.6410	1.8798
3, 3	−9.5000	10.5000	9.4000	−1.5047	18.4953	3.8119
3, 4	−9.8865	10.5580	9.7865	−0.4921	19.9523	1.0776
4, 1	−9.0024	10.6339	8.9024	−1.3538	18.2825	3.6370
4, 2	−9.3875	10.6125	9.2875	−0.4169	19.5831	0.9414
4, 3	−9.0919	10.5445	8.9919	−1.0966	18.5398	2.8679
4, 4	−9.5000	10.5000	9.4000	−0.1057	19.8943	0.1153

Table I. The stability and robustness analysis results.