Stable Fuzzy Controller Design for Uncertain Nonlinear Systems: Genetic Algorithm Approach[1]

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Abstract: This paper addresses the stable fuzzy controller design problem of nonlinear systems. The methodology is based on a fuzzy logic approach and the genetic algorithm (GA). In order to analyze the system stability, the TSK fuzzy plant model is employed to describe the dynamics of the nonlinear plant. A fuzzy controller is then developed to close the feedback loop. The stability conditions are derived. The feedback gains of the fuzzy controller and the solution for meeting the stability conditions are determined using the GA. An application example on stabilizing an inverted pendulum system will be given. Simulation and experimental results will be presented to verify the applicability of the proposed approach.

I. INTRODUCTION

Fuzzy control is particularly useful for ill-defined nonlinear systems. Control actions of a fuzzy controller are usually described by some linguistic rules, making the control algorithm easy to understand. To facilitate a systematic tuning procedure, a fuzzy controller implemented by a neural-fuzzy network was proposed in [7-8]. Through tuning, fuzzy rules can be generated automatically. Genetic algorithm (GA), which is a powerful searching algorithm [5], has been applied to fuzzy systems to help generate the membership functions and/or the rule sets [16]. These methods make the design simple; however, they do not guarantee the system stability.

In order to investigate the system stability, the Takagi-Sugeno-Kang (TSK) fuzzy plant model approach was proposed [1-2, 10, 14, 21-24]. A nonlinear system is modeled as a weighted sum of some simple sub-systems. It gives a fixed structure to some of the nonlinear systems, and facilitates the analysis of them. There are two ways to obtain the fuzzy plant model: 1) by performing identification methods through the use of the input-output data of the plant [1-2, 10, 14], 2) by deriving directly from the mathematical model of the nonlinear plant [9]. Stability of the fuzzy system formed by a fuzzy plant model and a fuzzy controller was investigated. Different stability conditions based on the Lyapunov stability theory [3, 6, 9] and other methods [11-13, 15, 17-19, 23-24] were reported. Using these stability conditions, the closed-loop system stability can be tested after finding the fuzzy controller parameters, which are usually determined by trial and error. Furthermore, the ways to solve the stability condition are usually not considered. If the stability conditions can be formulated as some linear matrix inequalities (LMIs) [9, 23], some software can help find the solution numerically. However, formulating the stability conditions into an LMI problem will limit the realm of the stability analysis. In order to have a systematic method to obtain a fuzzy controller with guaranteed system stability, a fuzzy controller derived from GA [20, 24] is proposed. The stability conditions for fuzzy control system are first derived. Based on these conditions, the parameters of the fuzzy controller are obtained using GA.

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II. TSK FUZZY PLANT MODEL AND FUZZY CONTROLLER

We consider a fuzzy control system formed by a nonlinear plant connected with a fuzzy controller in closed loop. The TSK fuzzy plant model is employed to describe the dynamics of the nonlinear plant.

A. TSK Fuzzy Plant Model with Parameter Uncertainties

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format,

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i and … and $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \quad (1)$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ containing the parameter uncertainties of the nonlinear plant, $\alpha = 1, 2, \ldots, \Psi$, $i = 1, 2, \ldots, p$, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The inferred system is given by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big), \quad (2)$$

$$\text{where } \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0,\ 1] \text{ for all } i \quad (3)$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \quad (4)$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ is the membership function corresponding to M_α^i. The value of $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ can be known or unknown. If it is an unknown function, $f_\alpha(\mathbf{x}(t))$ reflects the parameter uncertainties of the nonlinear plant. A fuzzy controller will be obtained based on the TSK fuzzy plant model of (2).

B. Fuzzy Controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The j-th rule of the controller is of the following format:

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j and … and $g_\Omega(\mathbf{x}(t))$ is N_Ω^j

$$\text{THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) \quad (5)$$

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where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, β = 1, 2, …, Ω, j = 1, 2, …, c, Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j to be designed. The inferred output of the fuzzy controller is given by $\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)$ (6)

where $\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1$, $m_j(\mathbf{x}(t)) \in [0,\ 1]$ for all j (7)

$$m_j(\mathbf{x}(t)) = \frac{\mu_{N_1^j}(g_1(\mathbf{x}(t)))\times\mu_{N_2^j}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{N_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{c}\left(\mu_{N_1^k}(g_1(\mathbf{x}(t)))\times\mu_{N_2^k}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{N_\Omega^k}(g_\Omega(\mathbf{x}(t)))\right)} \quad (8)$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{N_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the membership function corresponding to N_β^j to be designed.

C. Fuzzy Control System

In order to carry out the analysis, the closed-loop fuzzy system should be obtained. From (2) and (6),

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\mathbf{H}_{ij}\mathbf{x}(t) \quad (9)$$

where $\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j$ (10)

III. Stability Analysis

To analyze the stability of the fuzzy control system of (9), consider the Taylor series,

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \dot{\mathbf{x}}(t)\Delta t + \mathbf{o}(\Delta t) \quad (11)$$

where $\mathbf{o}(\Delta t) = -\mathbf{x}(t) - \dot{\mathbf{x}}(t)\Delta t + \mathbf{x}(t+\Delta t)$ is the error term and $\Delta t > 0$,

$$\lim_{\Delta t\to 0^+}\frac{\|\mathbf{o}(\Delta t)\|}{\Delta t} = 0 \quad (12)$$

From (9) and (11), writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, and multiplying a transformation matrix $\mathbf{T} \in \Re^{n\times n}$ of rank n to both sides, we have

$$\mathbf{Tx}(t+\Delta t) = \mathbf{Tx}(t) + \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{TH}_{ij}\mathbf{x}(t)\Delta t + \mathbf{To}(\Delta t)$$
$$= \left(\mathbf{I} + \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right)\mathbf{Tx}(t) + \mathbf{To}(\Delta t)$$

The reason for introducing $\mathbf{T}$ will be given at the end of this section. Taking norm on both sides of the above equation,

$$\|\mathbf{Tx}(t+\Delta t)\| \le \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right)\right\|\|\mathbf{Tx}(t)\| + \|\mathbf{To}(\Delta t)\| \quad (13)$$

where $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. From (13),

$$\|\mathbf{Tx}(t+\Delta t)\| \le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\|\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\|\|\mathbf{Tx}(t)\| + \|\mathbf{To}(\Delta t)\|$$

$$\Rightarrow \|\mathbf{Tx}(t+\Delta t)\| - \|\mathbf{Tx}(t)\| \le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\|\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\| - 1\right)\|\mathbf{Tx}(t)\| + \|\mathbf{To}(\Delta t)\|$$

$$\Rightarrow \lim_{\Delta t\to 0^+}\frac{\|\mathbf{Tx}(t+\Delta t)\| - \|\mathbf{Tx}(t)\|}{\Delta t} \le$$
$$\lim_{\Delta t\to 0^+}\left[\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\|\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\| - 1\right)\|\mathbf{Tx}(t)\| + \|\mathbf{To}(\Delta t)\|\right]/\Delta t \quad (14)$$

From (12) and (14),

$$\frac{d\|\mathbf{Tx}(t)\|}{dt} \le \lim_{\Delta t\to 0^+}\frac{\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\|\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\| - 1\right)}{\Delta t}\|\mathbf{Tx}(t)\|$$
$$\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right]\|\mathbf{Tx}(t)\| \quad (15)$$

where

$$\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right] = \lim_{\Delta t\to 0^+}\frac{\|\mathbf{I} + \mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\| - 1}{\Delta t} = \lambda_{\max}\left(\frac{\mathbf{TH}_{ij}\mathbf{T}^{-1} + (\mathbf{TH}_{ij}\mathbf{T}^{-1})^*}{2}\right) \quad (16)$$

is the corresponding matrix measure [4] of the induced matrix norm of $\|\mathbf{TH}_{ij}\mathbf{T}^{-1}\|$ (or the logarithmic derivative of $\|\mathbf{TH}_{ij}\mathbf{T}^{-1}\|$); $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue, * denotes the conjugate transpose. From (15), if $\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right]$ satisfies the following inequality,

$$\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right] \le -\varepsilon \text{ for all } i \text{ and } j. \quad (17)$$

where ε is a nonzero positive constant, it can be proved (15) implies a stable system of (9). Before conducting this proof, consider the following inequality obtained from (15) and (17).

$$\frac{d\|\mathbf{Tx}(t)\|}{dt} \le -\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \varepsilon\|\mathbf{Tx}(t)\| = -\varepsilon\|\mathbf{Tx}(t)\| \quad (18)$$

where $t_o < t$ is an arbitrary initial time. We shall show that (19) implies an exponentially stable closed-loop system of (9), and $\mathbf{x}(t) \to 0$ as $t \to \infty$.

Proof From (18),

$$\left(\frac{d\|\mathbf{Tx}(t)\|}{dt} + \varepsilon\|\mathbf{Tx}(t)\|\right)e^{\varepsilon(t-t_o)} \le 0 \quad (19)$$

$$\frac{d}{dt}\left(\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)}\right) \le 0 \Rightarrow \|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\|$$

$$\Rightarrow \|\mathbf{Tx}(t)\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)} \quad (20)$$

Since ε is a positive value, $\|\mathbf{Tx}(t)\| \to 0$ as $t \to \infty$. In order to show $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, considering the following property and $\mathbf{T}$ has rank n,

$$\|\mathbf{Tx}(t)\|^2 = \mathbf{x}(t)^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{Tx}(t) \quad (21)$$

As $\mathbf{T}^{\mathrm{T}}\mathbf{T}$ is symmetric positive definite ($\mathbf{T}$ has rank n), from (21), $\|\mathbf{Tx}(t)\| \to 0$ only when $\mathbf{x}(t) \to 0$. **QED**

The stability conditions of the closed-loop fuzzy system can be summarized by the following lemma:

Lemma 1. *The fuzzy control system as given by* (9), *which may have parameter uncertainties, is exponentially stable if*

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$\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}$ *is designed such that,*

$\mu[\mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}] \le -\varepsilon$ *for all i and j*

where ε *is a nonzero positive constant scalar.*

It should be noted that with the use of a suitable transformation matrix **T**, any Hurwitz matrix having a positive or zero matrix measure can be transformed into another matrix having a negative matrix measure (see (17)). The stability conditions derived can then be applied. The problem left is to find such a matrix **T** for a given system. This will be discussed later. From the above derivation and Lemma 1, we also see the system stability is not affected by the membership functions of the fuzzy controller.

VI. Stability Conditions and Feedback Gains

In this section, the problems of solving the stability conditions derived in the previous section and obtaining the feedback gains of the fuzzy controller will be tackled using the GA with arithmetic crossover and non-uniform mutation [5]. From Lemma 1, the uncertain fuzzy control system is stable if there exists a transformation matrix **T** satisfying the following conditions,

$$\mu[\mathbf{T}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{T}^{-1}] \le -\varepsilon,\ i = 1, 2, \ldots, p, j = 1, 2, \ldots, c \quad (22)$$

The objectives are to find $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$ and $\mathbf{G}_j = \begin{bmatrix} G_{11}^j & G_{12}^j & \cdots & G_{1n}^j \\ G_{21}^j & G_{22}^j & \cdots & G_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}^j & G_{m2}^j & \cdots & G_{mn}^j \end{bmatrix}$ such that the above conditions are satisfied. Let a fitness function be defined as follows,

$$fitness = \sum_{i=1}^{p}\sum_{j=1}^{c} n_{ij}\,\mu[\mathbf{T}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{T}^{-1}] \quad (23)$$

where $n_{ij} \ge 0$, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, c$, are constant scalars. The problems of finding **T** and $\mathbf{G}_j$ are now formulated into a minimization problem. The aim is to minimize the fitness function of (23) using the GA. As **T** and $\mathbf{G}_j$ are the variables of the fitness function of (23), they will be used to form the genes of the chromosomes. The finding of the solution to this minimization problem, however, does not imply that the conditions of (22) are satisfied. Hence, different n_{ij}, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, c$, may need to be tried to weight the conditions of (22) in order to change the significance of different terms on the right hand side of (23). For instance, one of the terms in (23) is very negative, which returns a very small fitness value. However, under this case, the conditions of (22) may not be satisfied. A small value of n_{ij} corresponding to that term can be used to attenuate the effect of that term in the fitness function. This may help the GA process to find a solution that satisfies the conditions of (22) during the minimizing process.

The procedure to obtain the fuzzy controller using the GA can be summarized into the following steps.

1) Obtain the fuzzy model of the nonlinear plant.
2) Determine the number of rules and the membership functions of the fuzzy controller.
3) Solve **T** and $\mathbf{G}_j$ with the fitness function defined in (23) and $n_{ij} = 1$, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, c$ using the GA. If **T** and $\mathbf{G}_j$ cannot be found, adjust n_{ij} accordingly.

VI. Application Example

An application example on stabilizing an inverted pendulum (Model 505 inverted pendulum) [25] as shown in Fig. 1 will be given in this section. Referring to Fig. 1, the plant consists of a pendulum rod that supports a sliding balance rod. The balance rod is driven via a toothed belt and a pulley, which in turn is driven by a drive shaft connected to a dc servomotor below the pendulum rod. The inverted pendulum can stand upright by steering the sliding rod in the presence of gravity. The balance weight in the bottom may be adjusted to alter the center of the gravity of the pendulum rod and hence the system dynamics (a source of parameter uncertainties). The plant has 2 sensors for measuring the system states in real time. The first one is a high-resolution encoder to measure the angle of the pendulum rod. Another one is a shaft encoder to sense the position of the sliding rod. The objective of this application example is to design a fuzzy controller to balance the inverted pendulum based on the design procedures mentioned in the previous section such that $\theta(t) = 0$ and $x(t) = 0$ (Fig. 2).

1). Referring to Fig. 2, the state space equations representing this inverted pendulum are as follows,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_1(t) & 0 & f_2(t) & 0 \\ 0 & 0 & 0 & 1 \\ f_3(t) & 0 & f_4(t) & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ f_5(t) \\ 0 \\ f_6(t) \end{bmatrix}u(t) \quad (24)$$

where $\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)\ \ x_4(t)]^T = [\theta(t)\ \ \dot{\theta}(t)\ \ x(t)\ \ \dot{x}(t)]^T$; $\theta(t)$ is the angular displacement of the pendulum rod (in rad), $\dot{\theta}(t)$ is the angular velocity of the pendulum rod (in rad/s), $x(t)$ is the position of the sliding rod (in m) and $\dot{x}(t)$ is the velocity of the sliding rod (in m/s). The plant is operating in the region that $\theta(t) \in [\theta_{\min}\ \ \theta_{\max}] = [x_{1\min}\ \ x_{1\max}] = \left[-\frac{\pi}{7}\ \ \frac{\pi}{7}\right]$, $\dot{\theta}(t) \in [\dot{\theta}_{min}\ \ \dot{\theta}_{max}] = [x_{2min}\ \ x_{2max}] = \left[-\frac{\pi}{16}\ \ \frac{\pi}{16}\right]$, $x(t) \in [x_{\min}\ \ x_{\max}] = [x_{3\min}\ \ x_{3\max}] = [-0.12\ \ 0.12]$, $\dot{x}(t) \in [\dot{x}_{\min}\ \ \dot{x}_{\max}] = [x_{4\min}\ \ x_{4\max}] = [-0.2\ \ 0.2]$;

$$f_1(\mathbf{x}(t)) = \frac{(-m_1 l_o + m_1 l_o + m_2 l_c) g \sin\theta(t)/\theta(t)}{J_o + m_1 l_o^2}$$

$$\in [f_{1_{\min}}\ \ f_{1_{\max}}] = [-15.7716\ \ -14.4457],$$

$$f_2(\mathbf{x}(t)) = \frac{m_1(g\cos\theta(t) - \dot{\theta}(t)^2 l_o - 2\dot{\theta}(t)\dot{x}(t))}{J_o + m_1 l_o^2}$$

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$\in \left[f_{2_{\min}} \quad f_{2_{\max}} \right] = [48.4863 \quad 56.4501]$,

$$f_3(\mathbf{x}(t)) = \frac{(J_o g - m_1 l_o^{\ 2} + m_2 l_c l_o) g \sin\theta(t)/\theta(t)}{J_o + m_1 l_o^{\ 2}}$$

$\in \left[f_{3_{\min}} \quad f_{3_{\max}} \right] = [14.2482 \quad 15.0116]$,

$$f_4(\mathbf{x}(t)) = \frac{-m_1^{\ 2} l_o g \cos\theta(t) + J_o \dot{\theta}(t)^2 + 2 m_1 l_o \dot{x}(t) \dot{\theta}(t)}{J_o + m_1 l_o^{\ 2}}$$

$\in \left[f_{4_{\min}} \quad f_{4_{\max}} \right] = [-18.5884 \quad -15.9605]$,

$f_5(t) = \dfrac{-l_o}{J_o + m_1 l_o^{\ 2}} \quad \in \left[f_{5_{\min}} \quad f_{5_{\max}} \right] = [-8.8576 \quad -8.5924]$,

$f_6(t) = \dfrac{J_o / m_1}{J_o + m_1 l_o^{\ 2}} \quad \in \left[f_{6_{\min}} \quad f_{6_{\max}} \right] = [7.5303 \quad 7.6178]$; l_o = 0.330 m is the length of pendulum rod from the pivot to the sliding rod T section, $m_1 = m_{1o} + m_{w1}$ is mass of the complete sliding rod including all attached elements, m_{1o} =0.103 kg is mass of the sliding rod with belt, belt clamps, and rubber end guards(but without sliding rod brass "dount" weights), m_{w1} = 0.110 kg is the combined mass of both of the sliding rod brass "dount" weights, $m_2 = m_{2o} + m_{w2}$ is the mass of the complete assembly minus m_1, m_{2o} = 0.785 kg is the mass of the complete moving assembly minus m_1 and m_{w2}, m_{w2} = 1 kg is the mass of brass balance weight, l_{co} = 0.071 m is the position of the c.g. of the complete pendulum assembly with the sliding rod and balance weight removed, $g = 9.8\text{m/s}^2$ is the acceleration due to gravity, $l_c = \dfrac{m_{w2} l_w + m_{2o} l_{co}}{m_2}$, $l_w = -\dfrac{(T + l_t + l_b)}{2}$, T = 0.05 m, $l_t \in [0.074 \quad 0.080]$ m, $l_b = l_t + 0.031$ m, $J_{oe} = J_o^* + m_1 l_o^{\ 2} + m_2 l_c^{\ 2}$, $J_o^* = 0.0246$ kgm^2 and $J_o = J_{oe} + m_1 x(t)^2$. The system of (24) can be approximated by the TSK model with the following rules:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND $f_2(\mathbf{x}(t))$ is M_2^i AND .... AND $f_6(\mathbf{x}(t))$ is M_6^i

THEN $\dot{\mathbf{x}}_i(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t)$ for $i = 1, 2, 3, \ldots, 64$ (25)

The dynamics of the plant is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{64} w_i(\mathbf{x}(t)) \left(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t) \right) \tag{26}$$

where $\mathbf{A}_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1_i} & 0 & f_{2_i} & 0 \\ 0 & 0 & 0 & 1 \\ f_{3_i} & 0 & f_{4_i} & 0 \end{bmatrix}$ and $\mathbf{B}_i = \begin{bmatrix} 0 \\ f_{5_i} \\ 0 \\ f_{6_i} \end{bmatrix}$, $f_{1_i} = f_{1_{\min}}$ for $i = 1, 2, \ldots, 32$; $f_{1_i} = f_{1_{\max}}$ for $i = 33, 34, \ldots, 64$; $f_{2_i} = f_{2_{\min}}$ for $i = 1, 2, \ldots, 16, 33, 34, \ldots 48$; $f_{2_i} = f_{2_{\max}}$ for $i = 17, 18, \ldots, 32, 49, 50, \ldots, 64$; $f_{3_i} = f_{3_{\min}}$ for $i = 1, 2, \ldots, 8, 17, 18, \ldots, 24; 33, 34, \ldots, 40, 49, 50, \ldots, 56$; $f_{3_i} = f_{3_{\max}}$ for $i = 9, 10, \ldots, 16, 25, 26, \ldots, 32, 41, 42, \ldots, 48, 57, 58, \ldots, 64$; $f_{4_i} = f_{4_{\min}}$ for $i = 4\alpha + 1, \ldots, 4\alpha + 4$; $\alpha = 0, 2, \ldots, 14$; $f_{4_i} = f_{4_{\max}}$ for $i = 4\alpha + 1, \ldots, 4\alpha + 4$; $\alpha = 1, 3, \ldots, 15$; $f_{5_i} = f_{5_{\min}}$ for $i = 4\alpha + 1, 4\alpha + 2$; $\alpha = 0, 1, 2, \ldots, 15$; $f_{5_i} = f_{5_{\max}}$ for $i = 4\alpha + 3, 4\alpha + 4$; $\alpha = 0, 1, 2, \ldots, 15$; $f_{6_i} = f_{6_{\min}}$ for $i = 1, 3, \ldots, 63$; $f_{6_i} = f_{6_{\max}}$ for $i = 2, 4, \ldots, 64$. The membership functions are chosen as follows,

$$\mu_{\mathrm{M}_1^\alpha}(f_1(\mathbf{x}(t))) = \begin{cases} \dfrac{-f_1(\mathbf{x}(t)) + f_{1_{\max}}}{f_{1_{\max}} - f_{1_{\min}}} & \text{for } \alpha = 1,2,\ldots,32 \\ 1 - \dfrac{-f_1(\mathbf{x}(t)) + f_{1_{\max}}}{f_{1_{\max}} - f_{1_{\min}}} & \text{for } \alpha = 33,34,\ldots,64 \end{cases} \tag{27}$$

$$\mu_{\mathrm{M}_2^\alpha}(f_2(\mathbf{x}(t))) = \begin{cases} \dfrac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}} & \text{for } \alpha = 1,2\ldots,16,33,34\ldots,48 \\ 1 - \dfrac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}} & \text{for } \alpha = 17,18\ldots,32,49,50\ldots,64 \end{cases} \tag{28}$$

$$\mu_{\mathrm{M}_3^\alpha}(f_3(\mathbf{x}(t))) = \begin{cases} \dfrac{-f_3(\mathbf{x}(t)) + f_{3_{\max}}}{f_{3_{\max}} - f_{3_{\min}}} & \text{for } \alpha = 1,2\ldots,8,17,18,19\ldots,24,33,34,\ldots,40,49,50,51\ldots,56 \\ 1 - \dfrac{-f_3(\mathbf{x}(t)) + f_{3_{\max}}}{f_{3_{\max}} - f_{3_{\min}}} & \text{for } \alpha = 9,10\ldots,16,25,26\ldots,32,41,42\ldots,48,57,58,59\ldots,64 \end{cases} \tag{29}$$

$$\mu_{\mathrm{M}_4^\alpha}(f_4(\mathbf{x}(t))) = \begin{cases} \dfrac{-f_4(\mathbf{x}(t)) + f_{4_{\max}}}{f_{4_{\max}} - f_{4_{\min}}} & \text{for } \alpha = 4i+1,\ldots,4i+4; i = 0,2,\ldots,14 \\ 1 - \dfrac{-f_4(\mathbf{x}(t)) + f_{4_{\max}}}{f_{4_{\max}} - f_{4_{\min}}} & \text{for } \alpha = 4i+1,\ldots,4i+4; i = 1,3,\ldots,15 \end{cases} \tag{30}$$

$$\mu_{\mathrm{M}_5^\alpha}(f_5(t)) = \begin{cases} \dfrac{-f_5(t) + f_{5_{\max}}}{f_{5_{\max}} - f_{5_{\min}}} & \text{for } \alpha = 4i+1, 4i+2; i = 0,1,2,\ldots,15 \\ 1 - \dfrac{-f_5(t) + f_{5_{\max}}}{f_{5_{\max}} - f_{5_{\min}}} & \text{for } \alpha = 4i+3, 4i+4; i = 0,1,2,\ldots,15 \end{cases} \tag{31}$$

$$\mu_{\mathrm{M}_6^\alpha}(f_6(t)) = \begin{cases} \dfrac{-f_6(t) + f_{6_{\max}}}{f_{6_{\max}} - f_{6_{\min}}} & \text{for } \alpha = 1,3,\ldots,63 \\ 1 - \dfrac{-f_6(t) + f_{6_{\max}}}{f_{6_{\max}} - f_{6_{\min}}} & \text{for } \alpha = 2,4,\ldots,64 \end{cases} \tag{32}$$

2) A 16-rule fuzzy controller is designed to balance the inverted pendulum based on the TSK fuzzy plant model of (25). The rules of the fuzzy controller are defined as follows.

Rule j: IF $x_1(t)$ is N_1^j AND $x_2(t)$ is N_2^j AND $x_3(t)$ is N_3^j AND $x_4(t)$ is N_4^j

THEN $u_j(t) = \mathbf{G}_j \mathbf{x}(t)$, $j = 1, 2, \ldots, 16$ (33)

The output of the fuzzy controller is defined as,

$$u(t) = \sum_{j=1}^{16} m_j(\mathbf{x}(t)) \mathbf{G}_j \mathbf{x}(t) \tag{34}$$

The membership functions for the 4 fuzzy sets are given by,

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$$\mu_{N_1^i}(\theta(t)) = \begin{cases} e^{\frac{-(|\theta(t)|-\overline{m}_1)^2}{2\overline{\sigma}_1^2}} & \text{for } i = 1,2,\ldots,8 \\ 1-e^{\frac{-(|\theta(t)|-\overline{m}_1)^2}{2\overline{\sigma}_1^2}} & \text{for } i = 9,10,\ldots,16 \end{cases} \quad (35)$$

$$\mu_{N_2^i}(\dot{\theta}(t)) = \begin{cases} e^{\frac{-(|\dot{\theta}(t)|-\overline{m}_2)^2}{2\overline{\sigma}_2^2}} & \text{for } i = 1,2,3,4,9,10,11,12 \\ 1-e^{\frac{-(|\dot{\theta}(t)|-\overline{m}_2)^2}{2\overline{\sigma}_2^2}} & \text{for } i = 5,6,7,8,13,14,15,16 \end{cases} \quad (36)$$

$$\mu_{N_3^i}(x(t)) = \begin{cases} e^{\frac{-(|x(t)|-\overline{m}_3)^2}{2\overline{\sigma}_3^2}} & \text{for } i = 1,2,5,6,9,10,13,14 \\ 1-e^{\frac{-(|x(t)|-\overline{m}_3)^2}{2\overline{\sigma}_3^2}} & \text{for } i = 3,4,7,8,11,12,15,16 \end{cases} \quad (37)$$

$$\mu_{N_4^i}(\dot{x}(t)) = \begin{cases} e^{\frac{-(|\dot{x}(t)|-\overline{m}_4)^2}{2\overline{\sigma}_4^2}} & \text{for } i = 1,3,5,\ldots,15 \\ 1-e^{\frac{-(|\dot{x}(t)|-\overline{m}_4)^2}{2\overline{\sigma}_4^2}} & \text{for } i = 2,4,6,\ldots,16 \end{cases} \quad (38)$$

$\overline{\sigma}_k = \frac{x_{k_{max}} - x_{k_{min}}}{6}$ and $\overline{m}_k = \frac{x_{k_{max}} - x_{k_{min}}}{2}$, $k = 1,2,3,4$ are the parameters of the membership functions of (35) to (38).

3) The transformation matrix $\mathbf{T}$ and the feedback gains $\mathbf{G}_j$ will be solved based on the fitness function defined in (23) using the GA with arithmetic crossover and non-uniform mutation [5]. In order to reduce the searching space of the GA process, T_{ij} is set to be T_{ji} if $i \neq j$. We set $n_{ij} = 1$ for $i = 1, 2, \ldots, 64$, $j = 1, 2, \ldots, 16$. The shape parameters, the probabilities of crossover and mutation of the GA are chosen to be 1, 0.8, 0.01 respectively. The number of iteration is 50000. After the GA process, $\mathbf{T} = 10^{-8} \times \begin{bmatrix} 0.199436499 & 0.011850747 & 0.183240064 & 0.012038775 \\ 0.011850747 & 0.049340392 & 0.036607074 & 0.050838952 \\ 0.183240064 & 0.036607074 & 0.545770386 & 0.047468967 \\ 0.012038775 & 0.050838952 & 0.047468967 & 0.076149585 \end{bmatrix}$ and the feedback gains obtained are listed in Table I. It can be verified that by applying the results to $\mu[\mathbf{T}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{T}^{-1}]$, the matrix measures for all i and j are negative. The maximum value of $\mu[\mathbf{T}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{T}^{-1}]$ among all i and j is -0.000582145, from Lemma I, ε is arbitrary chosen to be 0.0001. By Lemma 1, the system is guaranteed to be exponentially stable.

We have realized the designed fuzzy controller of (34) practically using a real-time digital signal processor (DSP) based controller unit, servo/actuator interfaces, servo amplifiers, and auxiliary power supplies. To test the stability and robustness, we adjust the height of the balance mass l_t from 0.07m to 0.1 m. The responses under the initial condition of $\mathbf{x}(0) = [0.17 \quad 0 \quad -0.55 \quad 0]^T$ are obtained from simulations and experiments respectively. Fig. 3 to Fig. 5 show the simulation and experimental results with l_t = 0.07 m, 0.08 m, 0.09 m and 0.1 m respectively. It can be seen from the system responses that the proposed fuzzy controller can stabilize the inverted pendulum even when the value of l_t is a bit outside the operating range of the TSK fuzzy plant model.

VII. Conclusion

Fuzzy control of nonlinear systems has been investigated. The nonlinear systems are represented by the TSK fuzzy plant models. Based on the TSK model, a fuzzy controller has been proposed, and the stability conditions have been derived. GA with arithmetic crossover and non-uniform mutation has been used to help find the solution to the stability conditions and the feedback gains of the fuzzy controller. An application example on stabilizing an inverted pendulum system has been presented to illustrate the merits of the proposed fuzzy controller.

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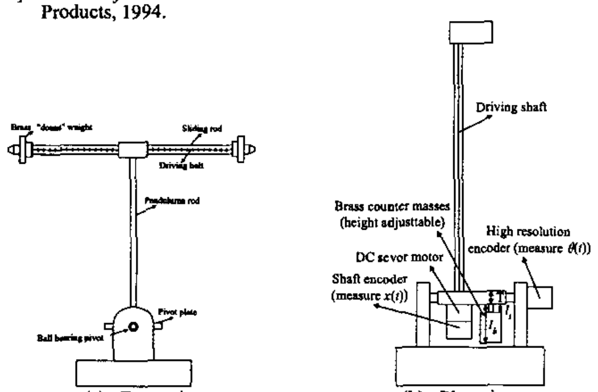
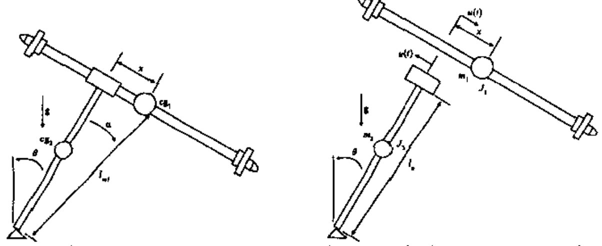


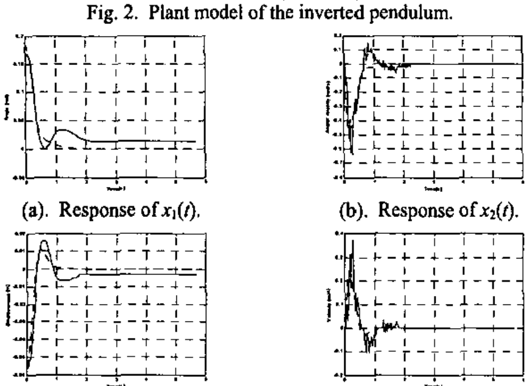
(a). Front view. (b). Plan view.

Fig. 1. Inverted pendulum system.



(a). Whole plant. (b). Two principal structure members.

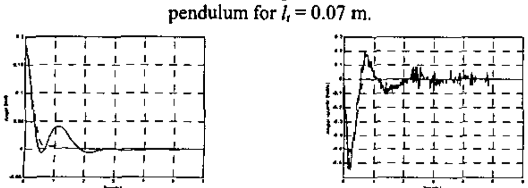
Fig. 2. Plant model of the inverted pendulum.



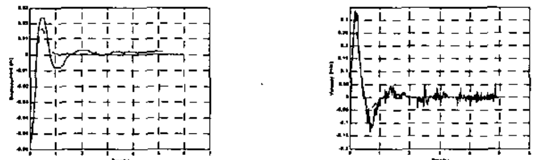
(a). Response of $x_1(t)$. (b). Response of $x_2(t)$.

(c). Response of $x_3(t)$. (d). Response of $x_4(t)$.

Fig. 3. Simulation (dotted lines) and experimental (solid lines) results of the pendulum for $l_t = 0.07$ m.

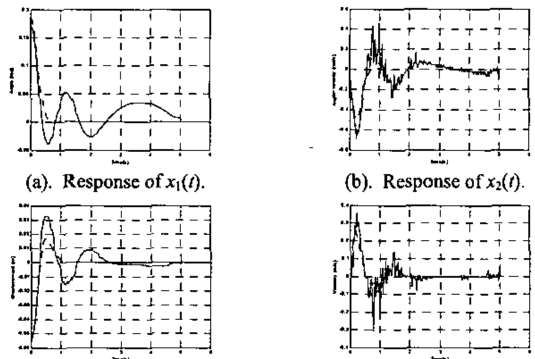


(a). Response of $x_1(t)$. (b). Response of $x_2(t)$.



(c). Response of $x_3(t)$. (d). Response of $x_4(t)$.

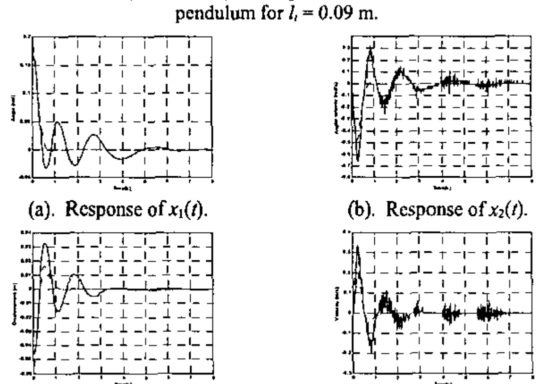
Fig. 4. Simulation (dotted lines) and experimental (solid lines) results of the pendulum for $l_t = 0.08$ m.



(a). Response of $x_1(t)$. (b). Response of $x_2(t)$.

(c). Response of $x_3(t)$. (d). Response of $x_4(t)$.

Fig. 5. Simulation (dotted lines) and experimental (solid lines) results of the pendulum for $l_t = 0.09$ m.



(a). Response of $x_1(t)$. (b). Response of $x_2(t)$.

(c). Response of $x_3(t)$. (d). Response of $x_4(t)$.

Fig. 6. Simulation (dotted lines) and experimental (solid lines) results of the pendulum for $l_t = 0.1$ m.

$\mathbf{G}_1$	[−17.6400 −7.6010 −57.4994 −12.6397]
$\mathbf{G}_2$	[−16.9238 −7.3720 −56.0060 −12.2283]
$\mathbf{G}_3$	[−16.3253 −7.2394 −55.3780 −11.9594]
$\mathbf{G}_4$	[−15.6844 −7.0323 −54.0189 −11.5885]
$\mathbf{G}_5$	[−21.9391 −9.0288 −62.2073 −14.3193]
$\mathbf{G}_6$	[−20.9478 −8.7117 −60.3399 −13.7860]
$\mathbf{G}_7$	[−20.0617 −8.4880 −59.2924 −13.3842]
$\mathbf{G}_8$	[−19.1952 −8.2083 −57.6338 −12.9149]
$\mathbf{G}_9$	[−19.6967 −8.0138 −59.9186 −13.1253]
$\mathbf{G}_{10}$	[−18.8748 −7.7600 −58.2745 −12.6795]
$\mathbf{G}_{11}$	[−18.1757 −7.6020 −57.4894 −12.3732]
$\mathbf{G}_{12}$	[−17.4452 −7.3744 −56.0049 −11.9743]
$\mathbf{G}_{13}$	[−15.3442 −6.4928 −54.4500 −11.3363]
$\mathbf{G}_{14}$	[−14.7450 −6.3050 −53.1277 −10.9876]
$\mathbf{G}_{15}$	[−14.2592 −6.2031 −52.6715 −10.7770]
$\mathbf{G}_{16}$	[−13.7193 −6.0320 −51.4574 −10.4601]

Table I. Feedback gains $\mathbf{G}_j$.

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