Stability and Robustness Analysis of Uncertain Multivariable Fuzzy Digital Control Systems[1]

F. H. F. Leung, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
H. K. Lam, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
P. K. S. Tam, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

Abstract

This paper presents the stability and robustness analysis of digital fuzzy control systems subject to parameter uncertainties. To carry out the analysis, we first describe exactly a digital nonlinear system with parameter uncertainties with a fuzzy plant model. Second, a digital fuzzy controller is proposed to close the feedback loop. Three design approaches are introduced. Based on the resultant fuzzy control system, a stability condition is derived for each design approach. A numerical example is given to show the merits.

1. Introduction

Fuzzy control has been successfully applied to some many practical systems [1-2]. Stability conditions of the fuzzy systems have been derived through different ways. Tanaka [4] gave the stability condition for systems formed by fuzzy plant models [3] and fuzzy controllers by using the Lyapunov's method. If there exists a common Lyapunov's function for all local subsystems, the fuzzy system is guaranteed stable. In [7], a fuzzy control system was proved to be equivalent to a switching system and a stability condition was derived. However, the robustness of the fuzzy control system was not considered in these papers. Results on robustness can be found in [5], and the problem is still under active research. In this paper, a digital fuzzy control system subject to parameter uncertainties will be analyzed. Stability conditions and robust areas will be derived for the system with and without parameter uncertainties. Here, the results are developed from those obtained for continuous-time systems [6, 8-9].

2. Fuzzy plant model and fuzzy controller

An uncertain multivariable fuzzy control system can be regarded as consisting of a fuzzy plant model and a digital fuzzy controller closing the feedback loop.

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2.1. Fuzzy plant model with uncertainties

Let p be the number of fuzzy rules describing the plant. The i-th rule is of the following format,

Rule i : IF x_1 is M_1^i and … and x_n is M_n^i

THEN $\mathbf{x}(k+1)=(\mathbf{A}^i+\Delta\mathbf{A}^i)\mathbf{x}(k)+(\mathbf{B}^i+\Delta\mathbf{B}^i)\mathbf{u}(k)$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the state x_α, $\alpha = 1, \ldots, n$, $i = 1, \ldots, p$; $\Delta\mathbf{A}^i \in \Re^{n\times n}$ and $\Delta\mathbf{B}^i \in \Re^{n\times m}$ are the uncertainties of $\mathbf{A}^i \in \Re^{n\times n}$ and $\mathbf{B}^i \in \Re^{n\times m}$ respectively; $\mathbf{x} \in \Re^{n\times 1}$ is the system state vector, $\mathbf{u} \in \Re^{m\times 1}$ is the input vector and $k \in \Re^{+}$ is an integer. The inferred system states are given by

$$\mathbf{x}(k+1)=\sum_{i=1}^{p} w^i(\mathbf{x}(k))\Big((\mathbf{A}^i+\Delta\mathbf{A}^i)\mathbf{x}(k)+(\mathbf{B}^i+\Delta\mathbf{B}^i)\mathbf{u}(k)\Big) \quad (2)$$

$$\sum_{i=1}^{p} w^i(\mathbf{x})=1,\ w^i(\mathbf{x})\in[0,\ 1] \ \text{ for all } i \quad (3)$$

$$w^i(\mathbf{x})=\frac{\mu_{M_1^i}(x_1)\circ\mu_{M_2^i}(x_2)\circ\cdots\circ\mu_{M_n^i}(x_n)}{\sum_{i=1}^{p}(\mu_{M_1^i}(x_1)\circ\mu_{M_2^i}(x_2)\circ\cdots\circ\mu_{M_n^i}(x_n))} \quad (4)$$

$\mu_{M_\alpha^i}(x_\alpha)$ is the grade of membership and '$\circ$' denotes the t-norm operator.

2.2. Fuzzy controller

A fuzzy controller with p fuzzy rules is to be designed. The j-th rule of the controller is of the following format:

Rule j: IF x_1 is N_1^j and … and x_n is N_n^j

THEN $\mathbf{u}(k)=\mathbf{G}^j\mathbf{x}(k)+\mathbf{r}(k)$ (5)

where N_β^j is a fuzzy term of rule j corresponding to the state x_β, $\beta = 1, \ldots, n$, $j = 1, \ldots, p$; $\mathbf{G}^j \in \Re^{m\times n}$ is the feedback gain of rule j, $\mathbf{r} \in \Re^{m\times 1}$ is the input vector. The inferred output of the fuzzy controller is given by

$$\mathbf{u}(k)=\sum_{j=1}^{c} m^j(\mathbf{x}(k))(\mathbf{G}^j\mathbf{x}(k)+\mathbf{r}(k)) \quad (6)$$

$$\sum_{j=1}^{c} m^j(\mathbf{x})=1,\ m^j(\mathbf{x})\in[0,\ 1] \ \text{ for all } j \quad (7)$$

$$m^j(\mathbf{x}) = \frac{\mu_{\mathrm{N}_1^j}(x_1) \circ \mu_{\mathrm{N}_2^j}(x_2) \circ \cdots \circ \mu_{\mathrm{N}_n^j}(x_n)}{\sum_{j=1}^{c} (\mu_{\mathrm{N}_1^j}(x_1) \circ \mu_{\mathrm{N}_2^j}(x_2) \circ \cdots \circ \mu_{\mathrm{N}_n^j}(x_n))} \quad (8)$$

$\mu_{\mathrm{N}_\beta^j}(x_\beta)$ is the grade of membership.

3. Stability and robustness analysis

The stability and robustness of an uncertain fuzzy control system are to be analyzed in this section. Three cases of design approaches are to be investigated.

3.1. General Design Approach (GDA)

General design approach allows difference in the rule antecedents between the fuzzy plant model and the fuzzy controller. This approach gives designers the largest freedom on designing the fuzzy controller. From (1) to (8), the closed-loop fuzzy system is given by,

$$\mathbf{x}(k+1) = \sum_{i=1}^{p} \sum_{j=1}^{c} w^i m^j ((\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x}(k) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}) \quad (9)$$

$$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j, \Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j \quad (10)$$

3.2. Parallel Design Approach (PDA)

Parallel design approach uses the same rule antecedents of the plant model in the fuzzy controller. Hence, some of the terms in (9) can be grouped together. This makes the stability criterion to be satisfied more easily. The closed-loop fuzzy system is given by,

$$\begin{aligned}\mathbf{x}(k+1) = &\sum_{i=1}^{p} w^i w^i (\mathbf{H}^{ii} + \Delta\mathbf{H}^{ii})\mathbf{x}(k) \\ &+ 2\sum_{i<j}^{p} w^i w^j (\mathbf{J}^{ij} + \Delta\mathbf{J}^{ij})\mathbf{x} + \sum_{i=1}^{p} w^i (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\end{aligned} \quad (11)$$

$$\mathbf{J}^{ij} = \frac{\mathbf{H}^{ij} + \mathbf{H}^{ji}}{2},\ \Delta\mathbf{J}^{ij} = \frac{\Delta\mathbf{H}^{ij} + \Delta\mathbf{H}^{ji}}{2} \quad (12)$$

$$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j,\ \Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j \quad (13)$$

3.3. Simplified Design Approach (SDA)

Simplified design approach requires the sub-system in each rule of the fuzzy plant model possesses a common input matrix **B**, and the fuzzy controller has the same number of rules with the same antecedents as the fuzzy plant model. In this simplified case, the closed-loop fuzzy system is given by,

$$\mathbf{x}(k+1) = \sum_{j=1}^{p} w^j ((\mathbf{H}^j + \Delta\mathbf{H}^j)\mathbf{x}(k) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}) \quad (14)$$

$$\mathbf{H}^j = \mathbf{A}^j + \mathbf{B}\mathbf{G}^j,\ \Delta\mathbf{H}^j = \Delta\mathbf{A}^j + \Delta\mathbf{B}\mathbf{G}^j \quad (15)$$

Because the input matrices are common, we have,

$$\mathbf{B} = \mathbf{B}^i,\ \Delta\mathbf{B} = \Delta\mathbf{B}^i \text{ or,} \quad (16)$$

$$\sum_{i=1}^{p} w^i\mathbf{B}^i = \mathbf{B},\ \sum_{i=1}^{p} w^i\Delta\mathbf{B}^i = \Delta\mathbf{B} \quad (17)$$

In (16), **B** and $\Delta\mathbf{B}$ are constants. In (17), **B** and $\Delta\mathbf{B}$ vary during the operation as w^i in each rule varies. Still in both cases, when $\mathbf{G}^j$ is designed such that $\mathbf{H}^j = \mathbf{A}^j + \mathbf{B}\mathbf{G}^j = \mathbf{H}$ for all *j*, and the system has no parameter uncertainty (i.e. $\Delta\mathbf{A}^j = \Delta\mathbf{B}^j = \Delta\mathbf{H}^j = 0$ for all *j*), a linear closed-loop control system can be obtained. If (16) holds, the linear system is obtained by feedback compensation (i.e. pole placement technique); otherwise, it is obtained by feedback linearization with respect to linear sub-systems satisfying (17). Nevertheless, the structure of the fuzzy controller for the latter is more complicated than that of the former.

3.4. Stability and Robustness Analysis

In this section, the stability and the robustness of the fuzzy control systems are analyzed. Theorem 1 to 3 summarize the stability and robustness analysis results for the three design approaches. Theorem 1 and 2 are applicable only to those systems without parameter uncertainties, whereas Theorem 3 is for systems subject to parameter uncertainties. Although Theorem 2 is applicable to systems without parameter uncertainties, Theorem 1 is less conservative. The use of Theorem 2 is to serve as a basis to determine the robust areas of the uncertain systems as stated in Theorem 3.

Theorem 1: *Under GDA, the fuzzy control system as given by (9) without uncertainty, i.e.* $\Delta\mathbf{H}^{ij} = 0$, *is stable if the following inequality hold.*

$$\left\|\mathbf{T}\mathbf{H}_\rho\mathbf{T}^{-1}\right\| < 1$$

where $\mathbf{H}_\rho = \underbrace{\mathbf{H}^{ij} \times \cdots \times \mathbf{H}^{ij}}_{\rho \text{ times}}$, *the values of i, j in each term of the product can be different from one another, i = 1, ..., p, j = 1, ..., c, ρ is a nonzero positive integer arbitrarily chosen by the designer,* $\mathbf{T} \in \Re^{n \times n}$ *is a transformation matrix of rank n,* $\|\cdot\|$ *denotes the* l_2 *norm for a vector or the induced norm for a matrix.*

Under PDA, the fuzzy control system as given by (12) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ii} = 0$ *and* $\Delta\mathbf{J}^{ij} = 0$, *is stable if the following inequalities hold.*

$$\begin{cases}\left\|\mathbf{T}\mathbf{H}_\rho^{\mathbf{H}}\mathbf{T}^{-1}\right\| < 1 \\ \left\|\mathbf{T}\mathbf{H}_\upsilon^{\mathbf{J}}\mathbf{T}^{-1}\right\| < 1 \\ \left\|\mathbf{T}\mathbf{J}_\upsilon^{\mathbf{H}}\mathbf{T}^{-1}\right\| < 1 \\ \left\|\mathbf{T}\mathbf{J}_\rho^{\mathbf{J}}\mathbf{T}^{-1}\right\| < 1\end{cases}$$

where $\mathbf{H}_\rho^{\mathbf{H}} = \underbrace{\mathbf{H}^{ii}\times\cdots\times\mathbf{H}^{ii}}_{\rho\text{ times}}$, $\mathbf{H}_\upsilon^{\mathbf{J}} = \underbrace{\mathbf{H}^{ii}\times\mathbf{J}^{ij}\times\mathbf{H}^{ii}\times\mathbf{J}^{ij}\times\cdots}_{\upsilon\text{ times}}$, $\mathbf{J}_\upsilon^{\mathbf{H}} = \underbrace{\mathbf{J}^{ij}\times\mathbf{H}^{ii}\times\mathbf{J}^{ij}\times\mathbf{H}^{ii}\times\cdots}_{\upsilon\text{ times}}$, $\mathbf{J}_\rho^{\mathbf{J}} = \underbrace{\mathbf{J}^{ij}\times\cdots\times\mathbf{J}^{ij}}_{\rho\text{ times}}$, *the values of i, j in each term of the product can be different from one another, i = 1, ..., p, j = 1, ..., c, it should be noted that i < j for* $\mathbf{J}^{ij}$, ρ *and* υ *are nonzero positive integers.*

Under SDA, the fuzzy control system of (15) *without uncertainty, i.e.* $\Delta\mathbf{H}^j = 0$, *is stable if the following inequalities hold.*

$$\left\|\mathbf{T}\mathbf{H}_\rho\mathbf{T}^{-1}\right\| < 1$$

where $\mathbf{H}_\rho = \underbrace{\mathbf{H}^i\times\cdots\times\mathbf{H}^i}_{\rho\text{ times}}$, *the values of i in each term of the product can be different from one another, i = 1, ..., p,* ρ *is a nonzero positive integer.*

Theorem 2 (more conservative stability condition):

Under GDA, the fuzzy control system as given by (9) without uncertainty, i.e. $\Delta\mathbf{H}^{ij} = 0$, *is stable if the following inequality holds.*

$\left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| < 1$ *for all i = 1, ..., p and j = 1, ..., c*

Under PDA, the fuzzy control system as given by (12) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ii} = 0$ *and* $\Delta\mathbf{J}^{ij} = 0$, *is stable if the following inequalities hold:*

$$\begin{cases}\left\|\mathbf{T}\mathbf{H}^{ii}\mathbf{T}^{-1}\right\| < 1 \textit{ for all } i\\ \left\|\mathbf{T}\mathbf{J}^{ij}\mathbf{T}^{-1}\right\| < 1 \textit{ for all } i < j\end{cases}$$

Under SDA, the fuzzy control system of (15) *without uncertainty, i.e.* $\Delta\mathbf{H}^j = 0$, *is stable if the following inequality holds:*

$\left\|\mathbf{T}\mathbf{H}^i\mathbf{T}^{-1}\right\| < 1$ *for all i = 1, ..., p*

Definition 1: *The robust area of a fuzzy control system is defined as the area in the parameter space inside which uncertainties are allowed to exist without affecting the system stability.*

Theorem 3: *Under GDA, with the uncertain fuzzy control system given by* (9), *the robust area is governed by,*

$\left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} < 1 - \left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|$ *for all i = 1, ..., p and j = 1, ..., c.*

The uncertain fuzzy control system is stable if the uncertainty $\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|$, *with* $\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$$\left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| \le \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\max} \le \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} \text{ for all } i \text{ and } j$$

Under PDA, with the uncertain fuzzy control system given by (12), *the robust area is governed by,*

$$\begin{cases}\left\|\mathbf{T}\mathbf{H}^{ii}\mathbf{T}^{-1}\right\|_{\text{Robust area}} < 1 - \left\|\mathbf{T}\mathbf{H}^{ii}\mathbf{T}^{-1}\right\| \textit{ for all } i\\ \left\|\mathbf{T}\mathbf{J}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} < 1 - \left\|\mathbf{T}\mathbf{J}^{ij}\mathbf{T}^{-1}\right\| \textit{ for all } i < j\end{cases}$$

The uncertain fuzzy control system is stable if the uncertainties $\left\|\mathbf{T}\Delta\mathbf{H}^{ii}\mathbf{T}^{-1}\right\|$ and $\left\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\right\|$, *with* $\left\|\mathbf{T}\Delta\mathbf{H}^{ii}\mathbf{T}^{-1}\right\|_{\max}$ and $\left\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\right\|_{\max}$ *as their maximum values respectively, satisfy the following conditions:*

$$\begin{cases}\left\|\mathbf{T}\Delta\mathbf{H}^{ii}\mathbf{T}^{-1}\right\| \le \left\|\mathbf{T}\Delta\mathbf{H}^{ii}\mathbf{T}^{-1}\right\|_{\max}\\ \le \left\|\mathbf{T}\Delta\mathbf{H}^{ii}\mathbf{T}^{-1}\right\|_{\text{Robust area}} \text{ for all } i\\ \left\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\right\| \le \left\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\right\|_{\max}\\ \le \left\|\mathbf{T}\Delta\mathbf{J}^{ij}\mathbf{T}^{-1}\right\|_{\text{Robust area}} \text{ for all } i < j\end{cases}$$

Under SDA, with the uncertain fuzzy control system given by (15), *the robust area is governed by*

$\left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|_{\text{Robust area}} \le 1 - \left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|$ *for all j.*

The uncertain fuzzy control system is stable if the uncertainty $\left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|$, *with* $\left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$\left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\| \le \left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|_{\max} \le \left\|\mathbf{T}\Delta\mathbf{H}^j\mathbf{T}^{-1}\right\|_{\text{Robust area}}$, *for all j.*

Proof: As the proofs for PDA and SDA are similar to that for PDA, they are omitted in this paper. On multiplying **T** to the fuzzy system of (9), we obtain

$$\mathbf{T}\mathbf{x}(k+1) = \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{T}((\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x}(k) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}) \tag{18}$$

To prove Theorem 1, let $\Delta\mathbf{H}^{ij} = \mathbf{0}$. The fuzzy system becomes,

$$\mathbf{T}\mathbf{x}(k+1) = \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{T}(\mathbf{H}^{ij}\mathbf{x}(k) + \mathbf{B}^i\mathbf{r})$$

$$= \sum_{i_{k_o},\cdots,i_k=1}^{p}\ \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}}\cdots\mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}(\mathbf{T}\mathbf{x}(k_o))$$

$$+ \sum_{l=1}^{k-k_o}\ \sum_{i_{k_o},\cdots,i_k=1}^{p}\ \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \times$$

$$\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}}\cdots\mathbf{H}^{i_{k_o+l-1} j_{k_o+l-1}}\mathbf{T}^{-1}\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r} + \sum_{i_k=1}^{p}\sum_{j_k=1}^{c} w^{i_k} m^{j_k}\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}$$

where $k_o \in \Re^+ \le k$ is an integer,

$$\begin{aligned}
\left\|\mathbf{T}\mathbf{x}(k+1)\right\| \le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o+l-1} j_{k_o+l-1}}\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\| \\
& + \sum_{i_k=1}^{p} \sum_{j_k=1}^{c} w^{i_k} m^{j_k} \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\| \\
\le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o+l-1} j_{k_o+l-1}}\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\|_{\max} \\
& + \sum_{i_k=1}^{p} \sum_{j_k=1}^{c} w^{i_k} m^{j_k} \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\|_{\max} \\
\le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o+l-1} j_{k_o+l-1}}\mathbf{T}^{-1}\right\| \max_{i_{k_o+l-1}} \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\|_{\max} \\
& + \max_{i_k} \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\|_{\max}
\end{aligned} \tag{19}$$

where,

$$\left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\| \le \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\|_{\max} \le \max_{i_{k_o+l-1}} \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\|_{\max} \quad \text{and}$$

$$\left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\| \le \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\|_{\max} \le \max_{i_k} \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\|_{\max}$$

For (19), two cases can be identified: $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \ne \mathbf{0}$. For the former case, the necessary and sufficient condition for $\left\|\mathbf{T}\mathbf{x}(k+1)\right\| \to 0$ (i. e. $\left\|\mathbf{x}(k+1)\right\| \to 0$) as $k \to \infty$ is

$$\left\|\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}\right\| \to 0 \text{ as } k \to \infty \tag{20}$$

For the latter case, if the condition of (20) holds, $\left\|\mathbf{T}\mathbf{x}(k+1)\right\|$ can be proved to be bounded as $k \to \infty$ because the second and the third summation terms of (19) are bounded. Consequently, $\left\|\mathbf{x}(k+1)\right\|$ is also bounded as $k \to \infty$. However, the condition of (20) is difficult to be found. Thus, we divide $\mathbf{T}\mathbf{H}^{i_k j_k}\mathbf{H}^{i_{k-1} j_{k-1}} \cdots \mathbf{H}^{i_{k_o} j_{k_o}}\mathbf{T}^{-1}$ into segments with length smaller than or equal to ρ. As a result, the norm of these segments can be calculated easily, but, the result may be more conservative.

$$\begin{aligned}
\left\|\mathbf{T}\mathbf{x}(k+1)\right\| \le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \\
& \times \prod_{s=1}^{\left\lceil \frac{k-k_o+1}{\rho} \right\rceil - 1} \left\| \prod_{q=\rho(s-1)+1}^{s\rho} \mathbf{T}\mathbf{H}^{i_{k-k_o-q+2} j_{k-k_o-q+2}}\mathbf{T}^{-1} \right\| \left\| \prod_{v=\rho\left(\left\lceil \frac{k-k_o+1}{\rho} \right\rceil - \right)+1}^{k-k_o+1} \mathbf{T}\mathbf{H}^{i_{k-k_o-v+2} j_{k-k_o-v+2}}\mathbf{T}^{-1} \right\| \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \prod_{s=1}^{\left\lceil \frac{k-k_o-l+1}{\rho} \right\rceil - 1} \left\| \prod_{q=\rho(s-1)+1}^{s\rho} \mathbf{T}\mathbf{H}^{i_{k-k_o-q+2} j_{k-k_o-q+2}}\mathbf{T}^{-1} \right\| \left\| \prod_{v=\rho\left(\left\lceil \frac{k-k_o-l+1}{\rho} \right\rceil - \right)+1}^{k-k_o-l+1} \mathbf{T}\mathbf{H}^{i_{k-k_o-v+2} j_{k-k_o-v+2}}\mathbf{T}^{-1} \right\| \max_{i_{k_o+l-1}} \left\|\mathbf{T}\mathbf{B}^{i_{k_o+l-1}}\mathbf{r}\right\|_{\max} \\
& + \max_{i_k} \left\|\mathbf{T}\mathbf{B}^{i_k}\mathbf{r}\right\|_{\max}
\end{aligned}$$

It should be noted that if $\left\lceil \frac{k-k_o+1}{\rho} \right\rceil < 1$ in the last equation, we take $\prod_{s=1}^{\left\lceil \frac{k-k_o+1}{\rho} \right\rceil - 1} \left\| \prod_{q=\rho(s-1)+1}^{s\rho} \mathbf{T}\mathbf{H}^{i_{k-k_o-q+2} j_{k-k_o-q+2}}\mathbf{T}^{-1} \right\| = 1$. The stability of the system is guaranteed for the two cases if the following condition is satisfied.

$$\left\| \prod_{q=\rho(s-1)+1}^{s\rho} \mathbf{T}\mathbf{H}^{i_{k-k_o-q+2} j_{k-k_o-q+2}}\mathbf{T}^{-1} \right\| < 1 \quad \text{for all } i_{k-k_o-q+2} \text{ and } j_{k-k_o-q+2} \tag{21}$$

This condition is equivalent to the condition of GDA in Theorem 1. This ends the proof of Theorem 1. Now, we consider the situation of $\Delta\mathbf{H}^{ij} \ne \mathbf{0}$. From (18),

$$\begin{aligned}
\left\|\mathbf{T}\mathbf{x}(k+1)\right\| \le & \sum_{i=1}^{p} \sum_{j=1}^{c} w^i m^j \left(\left\|\mathbf{T}(\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{T}^{-1}\right\| \left\|\mathbf{T}\mathbf{x}(k)\right\| + \left\|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\| \right) \\
\le & \sum_{i=1}^{p} \sum_{j=1}^{c} w^i m^j \left(\left(\left\|\mathbf{T}\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{ij}\mathbf{T}^{-1}\right\| \right) \left\|\mathbf{T}\mathbf{x}(k)\right\| + \left\|\mathbf{T}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\| \right) \\
\le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \prod_{q=k}^{k_o} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| \right) \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \prod_{q=k}^{k_o+l-1} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| \right) \left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\| \\
& + \sum_{i_k=1}^{p} \sum_{j_k}^{c} w^{i_k} m^{j_k} \left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\| \\
\le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \prod_{q=k}^{k_o} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\|_{\max} \right) \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \prod_{q=k}^{k_o+l-1} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\|_{\max} \right) \left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\|_{\max} \\
& + \sum_{i_k=1}^{p} \sum_{j_k}^{c} w^{i_k} m^{j_k} \left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\|_{\max} \\
\le & \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o}} m^{j_{k_o}} \prod_{q=k}^{k_o} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\|_{\max} \right) \left\|\mathbf{T}\mathbf{x}(k_o)\right\| \\
& + \sum_{l=1}^{k-k_o} \sum_{i_{k_o},\cdots,i_k=1}^{p} \sum_{j_{k_o},\cdots,j_k=1}^{c} w^{i_k} m^{j_k} w^{i_{k-1}} m^{j_{k-1}} \cdots w^{i_{k_o+l-1}} m^{j_{k_o+l-1}} \\
& \times \prod_{q=k}^{k_o+l-1} \left(\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\|_{\max} \right) \max_{i_{k_o+l-1}} \left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\|_{\max} \\
& + \sum_{i_k=1}^{p} \sum_{j_k}^{c} w^{i_k} m^{j_k} \max_{i_{k_o}} \left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\|_{\max}
\end{aligned} \tag{22}$$

where,

$$\left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\| \le \left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\|_{\max} \le \max_{i_{k_o+l-1}} \left\|\mathbf{T}(\mathbf{B}^{i_{k_o+l-1}} + \Delta\mathbf{B}^{i_{k_o+l-1}})\mathbf{r}\right\|_{\max}$$

and

$$\left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\| \le \left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\|_{\max} \le \max_{i_{k_o}} \left\|\mathbf{T}(\mathbf{B}^{i_k} + \Delta\mathbf{B}^{i_k})\mathbf{r}\right\|_{\max}$$

The stability of the system of (22) is guaranteed if the following condition holds.

$$\left\|\mathbf{T}\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\| + \left\|\mathbf{T}\Delta\mathbf{H}^{i_q j_q}\mathbf{T}^{-1}\right\|_{\max} < 1 \text{ for all } i_q \text{ and } j_q \tag{23}$$

For GDA, Theorem 2 and 3 can be obtained from (23). This ends the proofs of Theorem 2 and 3. It should be noted that Theorem 2 ($\rho = 1$) is a special case of Theorem 1 ($\rho \ge 1$), hence, we can conclude that Theorem 1 is less conservative than Theorem 2. **QED**

By using a transformation matrix **T**, we can transform matrices with norms greater than 1 into matrices with norms smaller than 1, so that the stability conditions derived can be applied.

4. Application Example

A numerical example is given in this section to illustrate the design procedures and the application of the theorems derived. Let an uncertain nonlinear plant be described by the following fuzzy rules.

Rule i: IF x_1 is $\mathbf{M}_1^i$ and x_2 is $\mathbf{M}_2^i$

THEN $\mathbf{x}(k+1)=(\mathbf{A}^i+\Delta\mathbf{A}^i)\mathbf{x}(k)+(\mathbf{B}^i+\Delta\mathbf{B}^i)u(k)$ (24)

with the t-norm operation being the multiplication, $i = 1, 2, 3, 4$. The operating range of the states is assumed to be within -1.5 and 1.5. The fuzzy rules of the fuzzy controller are defined as follows,

Rule 1: IF x_1 is $\mathbf{M}_1^j$ and x_2 is $\mathbf{M}_2^j$

THEN $u(k)=\mathbf{G}^j\mathbf{x}(k)$, $j = 1, 2, 3, 4$ (25)

where the membership functions of $\mathbf{M}_\alpha^i$, $i = 1, 2, 3, 4$, $\alpha = 1, 2$, are $\mu_{\mathbf{M}_1^1}(x_1)=\mu_{\mathbf{M}_1^2}(x_1)=1-\dfrac{x_1^{\,2}}{2.25}$, $\mu_{\mathbf{M}_1^3}(x)=\mu_{\mathbf{M}_1^4}(x)=\dfrac{x_1^{\,2}}{2.25}$, $\mu_{\mathbf{M}_2^1}(x_2)=\mu_{\mathbf{M}_2^3}(x_2)=1-\dfrac{x_2^{\,2}}{2.25}$, $\mu_{\mathbf{M}_2^2}(x_2)=\mu_{\mathbf{M}_2^4}(x_2)=\dfrac{x_2^{\,2}}{2.75}$; $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$;

$\mathbf{A}^1=\begin{bmatrix}-0.2 & 0\\-0.801 & 0.1\end{bmatrix}$, $\mathbf{A}^2=\begin{bmatrix}-0.2 & 0\\-0.801 & 0.1\end{bmatrix}$, $\mathbf{A}^3=\begin{bmatrix}-0.2 & 0\\-0.8535 & 0.1\end{bmatrix}$, $\mathbf{A}^4=\begin{bmatrix}-0.2 & 0\\-0.8535 & 0.1\end{bmatrix}$;

$\Delta\mathbf{A}^1=\Delta\mathbf{A}^2=\Delta\mathbf{A}^3=\Delta\mathbf{A}^4=\begin{bmatrix}0 & 0.1\\c_1(k) & c_2(k)\end{bmatrix}$,

$\mathbf{B}^1=\begin{bmatrix}0\\0.14387\end{bmatrix}$, $\mathbf{B}^2=\begin{bmatrix}0\\0.05613\end{bmatrix}$, $\mathbf{B}^3=\begin{bmatrix}0\\0.14387\end{bmatrix}$, $\mathbf{B}^4=\begin{bmatrix}0\\0.05613\end{bmatrix}$, $\Delta\mathbf{B}^1=\Delta\mathbf{B}^2=\Delta\mathbf{B}^3=\Delta\mathbf{B}^4=\begin{bmatrix}0\\c_3(k)\end{bmatrix}$;

$c_1(k)=\dfrac{c_1^U+c_1^L}{2}+(c_1^L-\dfrac{c_1^U+c_1^L}{2})\sin(10k)$ so that $c_1(k)\in[c_1^L,c_1^U]$,

$c_2(k)=\dfrac{c_2^U+c_2^L}{2}+(c_2^L-\dfrac{c_2^U+c_2^L}{2})\cos(5k)$ so that $c_2(k)\in[c_2^L,c_2^U]$,

$c_3(k)=\dfrac{c_3^U+c_3^L}{2}+(c_3^L-\dfrac{c_3^U+c_3^L}{2})\cos(5k)$ so that $c_3(k)\in[c_3^L,c_3^U]$, $c_1^L=-0.11$, $c_2^L=-0.01$, $c_3^L=-0.01$, $c_1^U=0.11$, $c_2^U=0.01$, $c_3^U=0.01$. Although the parameter uncertainties c_1, c_2 and c_3 are modeled as functions of k in order to illustrate the performance of the designed controller, the exact parameter values are unknown practically. The equations used to describe the uncertain parameters are just for the purpose of illustrating the robustness property of the controller.

PDA is applied to design the fuzzy controller, the feedback gains are designed as $\mathbf{G}^{11}$ = [1.3971 –2.0852], $\mathbf{G}^{12}$ = [3.5810 –5.3447], $\mathbf{G}^{13}$ = [1.5535 –2.0852] and $\mathbf{G}^{14}$ = [3.9818 –5.3447] so that $\mathbf{H}^{11}=\mathbf{H}^{22}=\mathbf{H}^{33}=\mathbf{H}^{44}=\begin{bmatrix}-0.2 & 0\\0 & -0.2\end{bmatrix}$; The transformation matrix is chosen to be $\mathbf{T}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Figure 1 and 2 show the responses of $x_1(k)$ and $x_2(k)$ with (solid line) and without (dotted line) parameter uncertainties with initial states of $\mathbf{x}(0)=[-1,\ -1]$. The stability and robustness analysis results are tabulated in Table I. By Theorem 2, the system without uncertainties is guaranteed stable as the values of column 2 are all smaller than one. As stated in Theorem 3, the robust areas are listed in column 3. Column 4 lists the maximum norms of the parameter uncertainties specified before. As we find that the maximum norms of parameter uncertainties are smaller than the corresponding robust areas (the differences are listed in column 5), we can conclude that the system with parameter uncertainties is stable.

5. Conclusions

The stability and robustness of digital multivariable fuzzy control systems subject to parameter uncertainties have been analyzed. Stability conditions and robust areas for three design approaches have been derived. An example has been given to illustrate the design procedures and the stabilizability and robustness property of the digital fuzzy controller designed based on the developed theorems.

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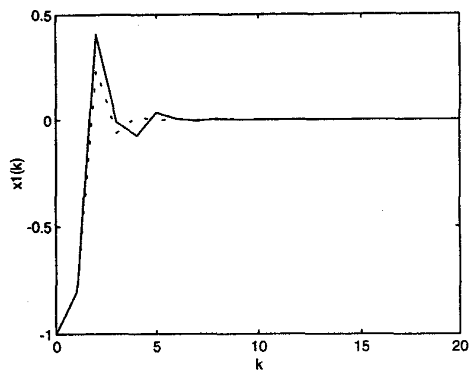

Figure 1. The responses of $x_1(k)$ of the system with (solid line) and without (dotted line) parameter uncertainties.

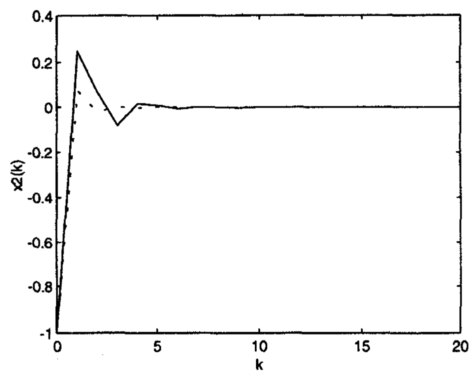

Figure 2. The responses of $x_2(k)$ of the system with (solid line) and without (dotted line) parameter uncertainties.

i,j	$\|\mathbf{H}^{ii}\|$ or $\|\mathbf{J}^{ij}\|$	$\|\Delta\mathbf{H}^{ii}\|_{\text{Robust area}}$ or $\|\Delta\mathbf{J}^{ij}\|_{\text{Robust area}}$	$\|\Delta\mathbf{H}^{ii}\|_{\max}$ or $\|\Delta\mathbf{J}^{ij}\|_{\max}$	Column (3) minus Column (4)
1, 1	0.6606	0.3394	0.2569	0.0825
1, 2	0.6325	0.3675	0.3173	0.0502
1, 3	0.6658	0.3342	0.2585	0.0757
1, 4	0.6228	0.3772	0.3212	0.0560
2, 2	0.6606	0.3394	0.2867	0.0527
2, 3	0.6429	0.3571	0.2875	0.0696
2, 4	0.6658	0.3342	0.2895	0.0447
3, 3	0.6711	0.3289	0.2585	0.0704
3, 4	0.6332	0.3668	0.2895	0.0774
4, 4	0.6711	0.3289	0.3212	0.0077

Table I. The stability and robustness analysis result.