Design of a Fuzzy Controller for Stabilizing a Ball-and-Beam System[1]

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Abstract – This paper presents the stability analysis and design of fuzzy controller for a nonlinear ball-and beam-system. An improved simple stability condition will be derived based on the Lyapunov's stability theory. The derived stability condition involves a smaller number of Lyapunov's conditions. A design of the membership functions of the fuzzy controller will be given. In our approach, the number of rules of the fuzzy controller can be different from that of the fuzzy plant model. In addition, our approach can be applied to those fuzzy controllers with both positive and negative grades of membership.

I. INTRODUCTION

In this paper, an improved simple stability condition for fuzzy controlled ball-and-beam system [4] will be derived based on the Lyapunov's stability theory. Wang *et.al.* derived a stability condition [3] for this class of nonlinear control systems by using the Lyapunov's stability theory. A sufficient condition for the system stability is obtained by finding a common Lyapunov's function for all the fuzzy sub-systems. For a fuzzy plant model with p rules, a fuzzy controller with p rules is required to close the feedback loop, and $p(p+1)/2$ Lyapunov's conditions will come out. In this paper, the numbers of rules of the fuzzy plant model and that of the fuzzy controller need not be the same. The number of Lyapunov's conditions is reduced to $p+1$. This result eases the finding of the common Lyapunov's function. In most of the previous work, a design guideline to determine the membership functions of the fuzzy controller was not given. In this paper, we provide a way to design the membership functions. The task of finding the common Lyapunov's function can readily be formulated into a linear matrix inequality (LMI) problem [2]. By applying some LMI tools, we can find the design solution easily. In our approach, if the common Lyapunov's function is found, the sign of the membership functions of the fuzzy controller will not affect the system stability.

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This paper is organized as follows. Section II presents the fuzzy plant model of a ball-and-beam system. Based on this model, we design a fuzzy controller which will be presented in Section III. In section IV, the system stability of the fuzzy controlled ball-and-beam system will be analyzed. The stability condition and the membership functions of the fuzzy controller will be derived. In section V, simulation results will be given to verify the analysis results. In section 6, a conclusion will be drawn.

II. FUZZY MODEL OF A BALL-AND-BEAM SYSTEM

A ball and beam system is shown in Fig. 1 [4]. Its dynamic equations are given as follows,

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= B\left(x_1(t)x_4(t)^2 + g\sin(x_3(t))\right)\\ \dot{x}_3(t) &= x_4(t)\\ \dot{x}_4(t) &= u(t)\end{aligned} \tag{1}$$

where $x_1(t)$ is the position of the ball measured from the centre of the beam, $x_2(t)$ is the velocity of the ball, $x_3(t)$ is the angle of the beam with respect to the horizontal axis, $x_4(t)$ is the angular velocity of the beam with respect to the horizontal axis; $B = \dfrac{MR^2}{J_b + MR^2} = 0.7143$, $J_b = 2\times10^{-6}\,\text{kgm}^2$ is the moment of inertia of the ball about the centre of the ball, $M = 0.05$kg is the mass of the ball, $R = 0.01$m is the radius of the ball, $g = 9.8$ ms^{-2} is the acceleration due to gravity. The objective of this application example is to drive the ball to the centre of the beam such that $x_1(t) = 0$. The ball-and-beam system of (1) can be represented by a fuzzy plant model having four rules with the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i and $f_2(\mathbf{x}(t))$ is M_2^i
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$ for $i = 1, 2, 3, 4$ (2)

so that the system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right) \tag{3}$$

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where,

$$\sum_{i=1}^{4} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0\ \ 1] \text{ for } i = 1, 2, 3, 4 \tag{4}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))}{\sum_{k=1}^{4}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\right)} \text{ for } i = 1, 2, 3, 4 \tag{5}$$

are known nonlinear functions and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, 3, 4$, are known membership functions. (Thus, we assume that the fuzzy plant model is known.), $\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)\ \ x_4(t)]^{\mathrm{T}}$, $x_1(t) \in [-0.35\ \ 0.35]$, $x_3(t) \in [-1\ \ 1]$, $x_2(t)$ and $x_4(t)$ have no limits; $f_1(\mathbf{x}(t)) = x_4(t)^2$ and $f_2(\mathbf{x}(t)) = -\frac{\sin(x_3(t))}{x_3(t)}$;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bf_{1_{\min}} & 0 & -Bgf_{2_{\min}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bf_{1_{\min}} & 0 & -Bgf_{2_{\max}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bf_{1_{\max}} & 0 & -Bgf_{2_{\min}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bf_{1_{\max}} & 0 & -Bgf_{2_{\max}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $f_{1_{\min}} = -1$ and $f_{1_{\max}} = -f_{1_{\min}}$, $f_{2_{\min}} = 0.6$ and $f_{2_{\max}} = 1$; $\mu_{\mathrm{M}_1^\beta}(f_1(\mathbf{x}(t))) = \frac{-f_1(\mathbf{x}(t)) + f_{1_{\max}}}{f_{1_{\max}} - f_{1_{\min}}}$ for $\beta = 1, 2$ and $\mu_{\mathrm{M}_1^\delta}(f_1(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_1^1}(f_1(\mathbf{x}(t)))$ for $\delta = 3, 4$; $\mu_{\mathrm{M}_2^\varepsilon}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}}$ for $\varepsilon = 1, 3$ and $\mu_{\mathrm{M}_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_2^1}(f_2(\mathbf{x}(t)))$ for $\phi = 2, 4$ are the membership functions.

III. Fuzzy Controller

A four-rule fuzzy controller is designed for the fuzzy plant model of (3). The j-th rules is given by,

$$\text{Rule } j\text{: IF } \mathbf{x}(t) \text{ is } \mathrm{N}^j \text{ THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) \tag{6}$$

where N^j is a fuzzy term of rule j corresponding to the system state vector $\mathbf{x}(t)$, $j = 1, 2, 3, 4$; $\mathbf{G}_j \in \Re^{1\times 4}$ is the rule j feedback gain vector that is to be designed. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{4} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{7}$$

where,

$$\sum_{j=1}^{4} m_j(\mathbf{x}(t)) = 1 \tag{8}$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}^j}(\mathbf{x}(t))}{\sum_{k=1}^{4}\mu_{\mathrm{N}^k}(\mathbf{x}(t))} \text{ for } j = 1, 2, 3, 4 \tag{9}$$

is a non-linear function of vector $\mathbf{x}(t)$, and $\mu_{\mathrm{N}^j}(\mathbf{x}(t))$, $j = 1, 2, 3, 4$, are membership functions to be designed. It should be noted that the fuzzy controller does not require $m_j(\mathbf{x}(t)) \in [0\ \ 1]$ for all j.

IV. Stability Analysis of the Fuzzy Control System

A closed-loop system can be obtained by combining (3) and (7). Writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, the fuzzy control system then becomes,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t) \tag{10}$$

where,

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j \tag{11}$$

To investigate the stability of the fuzzy control system of (10), we consider the following Lyapunov's function in quadratic form,

$$V(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{12}$$

where $\mathbf{P} \in \Re^{4\times 4}$ is a symmetric positive definite matrix. Then,

$$\dot{V}(\mathbf{x}(t)) = \frac{1}{2}\left(\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\right) \tag{13}$$

From (10), (13) and the property of $\sum_{i=1}^{4} w_i = \sum_{j=1}^{4} m_j = \sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j = 1$, we have,

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \frac{1}{2}\left(\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right)^{\mathrm{T}} \mathbf{P}\mathbf{x}(t) \\
&\quad + \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t) \\
&= \frac{1}{2}\left[\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \left(\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m\right)\mathbf{x}(t)\right]^{\mathrm{T}} \mathbf{P}\mathbf{x}(t) \\
&\quad + \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \left(\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m\right)\mathbf{x}(t) \\
&= \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{H}_m^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right)\mathbf{x}(t) \\
&\quad + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{H}_{ij} - \mathbf{H}_m\right)\right]\mathbf{x}(t) \\
&= -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)
\end{aligned} \tag{14}$$

where $\mathbf{H}_m \in \Re^{4\times 4}$ is a stable matrix to be designed. $\mathbf{Q}_m \in \Re^{4\times 4}$ is a symmetric positive definite matrix and $\mathbf{Q}_{ij} \in \Re^{4\times 4}$ is a symmetric matrix. They are defined as follows,

$$\mathbf{Q}_m = -\left(\mathbf{H}_m^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right) \tag{15}$$

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right) \text{ for } i = 1,2,3,4;\ j = 1, 2, 3, 4 \tag{16}$$

From (14),

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) \\
&\quad - \frac{1}{2}\sum_{j=1}^{4} m_j \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)
\end{aligned} \tag{17}$$

and we let,

$$m_j = \frac{\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)}{\sum_{k=1}^{4}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]} \text{ for } j = 1, 2, 3, 4 \tag{18}$$

Comparing (18) with (9), (18) gives the design of the membership functions of the fuzzy controller (3) such that $\mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)$.

Considering the denominator at the right hand side of (18), we have,

$$\begin{aligned}
&\sum_{k=1}^{4}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right] \\
&= \sum_{i=1}^{4} w_i\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{k=1}^{4}\mathbf{Q}_{ik} - 4\mathbf{Q}_m\right)\mathbf{x}(t)\right]
\end{aligned} \tag{19}$$

We design $\mathbf{Q}_{ik}$ and $\mathbf{Q}_m$ such that,

$$\sum_{k=1}^{4}\mathbf{Q}_{ik} - 4\mathbf{Q}_m > \mathbf{0} \text{ for } i = 1, 2, 3, 4 \tag{20}$$

As (20) is a positive definite matrix and $w_i(\mathbf{x}(t)) \in [0 \quad 1]$ for all i, and at least one of the $w_i \neq 0$ (this is a property of a fuzzy plant model), (19) will be always greater than or equal to zero. It is equal to zero only when $\mathbf{x}(t) = \mathbf{0}$. Under this condition, the output of the fuzzy controller of (7) is zero and we shall choose $m_j = \frac{1}{4}$ for satisfying the condition of (8). From (17) and (18),

$$\dot{V}(\mathbf{x}(t)) = -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{\sum_{j=1}^{4}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]^2}{2\sum_{k=1}^{4}\left[\mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]} \tag{21}$$

As the second term at the right side of (21) is semi-positive definite, we have,

$$\dot{V}(\mathbf{x}(t)) \leq -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) \leq 0 \tag{22}$$

Hence, we can conclude that the fuzzy control system is stable. The sufficient stability condition of the fuzzy control system is summarized by the following lemma.

Lemma 1: *A fuzzy control system of* (10) *is guaranteed to be stable if we choose the membership* *functions of the fuzzy controller as,*

$$\begin{cases} \mu_{N^j}(\mathbf{x}(t)) = \mathbf{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{4} w_i \mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t) \text{ when } \mathbf{x}(t) \neq \mathbf{0} \\ \mu_{N^j}(\mathbf{x}(t)) = \frac{1}{4} \text{ when } \mathbf{x}(t) = \mathbf{0} \end{cases}$$

for $j = 1, 2, 3, 4$

and there is a common solution of **P** *for the following linear matrix inequalities,*

$$\begin{cases} \mathbf{Q}_m > \mathbf{0} \\ \sum_{k=1}^{4} \mathbf{Q}_{ik} - 4\mathbf{Q}_m > \mathbf{0} \textit{ for all } i = 1, 2, 3, 4 \end{cases}$$

where,

$$\begin{cases} \mathbf{Q}_m = -\left(\mathbf{H}_m{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right) \\ \mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right) \quad \text{for } i = 1,2,3,4; j = 1,2,3,4 \end{cases}$$

Lemma 1 governs the way of choosing the membership functions of the fuzzy controller. The numbers of rules of the fuzzy plant model and its fuzzy controller are not necessarily the same. The number of linear matrix inequalities is 5, instead of 10 as stated in [3].

V. Simulation Result

In the following, the simulation result of the fuzzy controlled ball-and-beam system will be given. The feedback gains of the fuzzy controller are designed as $\mathbf{G}_1 = [-0.1166 \quad 10.2041 \quad -34.2857 \quad -10.0000]$, $\mathbf{G}_2 = [-0.0700 \quad 6.1224 \quad -34.2857 \quad -10.0000]$, $\mathbf{G}_3 = [11.7881 \quad 13.6054 \quad -35.714 \quad -10.0000]$, $\mathbf{G}_4 = [7.0729 \quad 8.1633 \quad -35.7143 \quad -10.0000]$, so that the eigenvalues of $\mathbf{H}_{11} = \mathbf{H}_{22} = \mathbf{H}_{33} = \mathbf{H}_{44}$ are –1, –2, –3 and –4. The membership functions are designed as listed in Lemma 1. To check the stability, we choose

$$\mathbf{H}_m = \frac{\sum_{i=1}^{4}\sum_{j=1}^{4}\mathbf{H}_{ij}}{100} = \begin{bmatrix} 0 & 0.1600 & 0 & 0 \\ 0 & 0 & -0.8960 & 0 \\ 0 & 0 & 0 & 0.1600 \\ 0.7470 & 1.5238 & -5.6000 & -1.6000 \end{bmatrix}$$

which is a stable matrix. We have 5 linear matrix inequalities. One common solution of them is

$$\mathbf{P} = \begin{bmatrix} 0.2034 & 0.0651 & -0.0913 & -0.0019 \\ 0.0651 & 0.0775 & -0.0628 & -0.0027 \\ -0.0913 & -0.0628 & 0.1734 & 0.0041 \\ -0.0019 & -0.0027 & 0.0041 & 0.0009 \end{bmatrix}.$$

By Lemma 1, we can conclude that the closed-loop system is stable. Fig. 2 to 5 show the responses of the system states under the initial condition of $\mathbf{x}(0) = [0.35 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. The control signal of the ball-and-beam system is shown in Fig. 6.

VI. Conclusion

The stability analysis and design of fuzzy controlled ball-and-beam system has been discussed. An improved stability criterion has been derived. This criterion involves p+1 linear matrix inequalities where p is the number of rules of the fuzzy plant model. A design on the membership functions of the fuzzy controller has been given. The stability criterion can also be applied to fuzzy controllers having membership functions of both positive and negative grades of membership.

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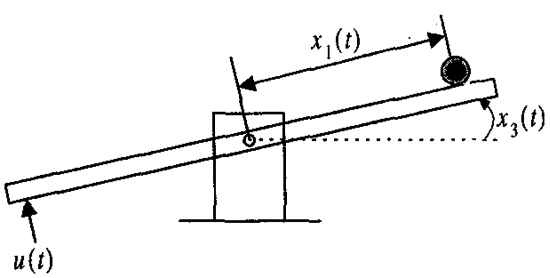

Fig. 1. A ball-and-beam system.

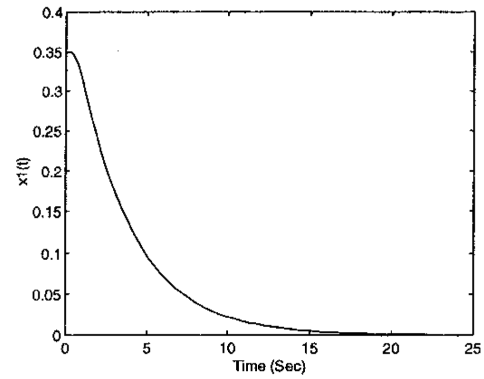

Fig. 2. Response of $x_1(t)$ of the ball-and-beam system.

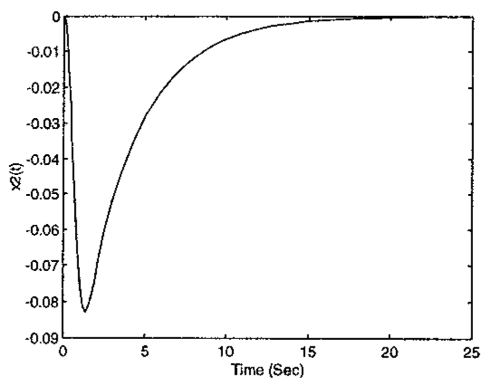

Fig. 3. Response of $x_2(t)$ of the ball-and-beam system.

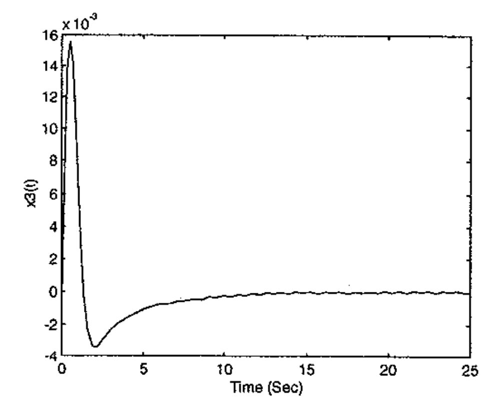

Fig. 4. Response of $x_3(t)$ of the ball-and-beam system.

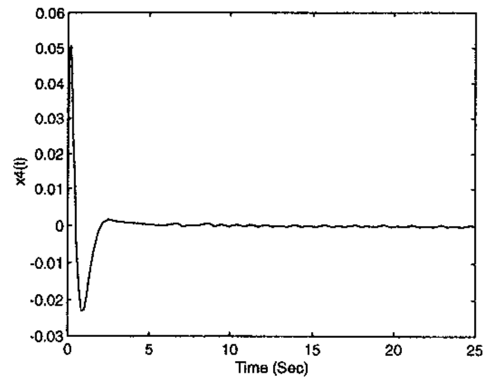

Fig. 5. Response of $x_4(t)$ of the ball-and-beam system.

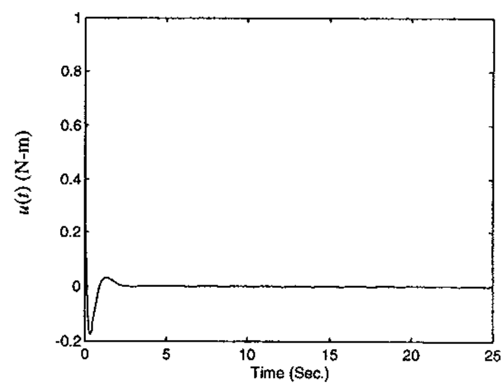

Fig. 6. Control signal of the ball-and-beam system.