Heavy cycles in k-connected weighted graphs with large weighted degree sums *

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Abstract

A weighted graph is one in which every edge e is assigned a nonnegative number $w(e)$, called the weight of e. The weight of a cycle is defined as the sum of the weights of its edges. The weighted degree of a vertex is the sum of the weights of the edges incident with it. In this paper, we prove that: Let G be a k-connected weighted graph with $k \geq 2$. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$, if G satisfies the following conditions: (1) The weighted degree sum of any $k+1$ pairwise nonadjacent vertices is at least m; (2) In each induced claw and each induced modified claw of G, all edges have the same weight. This generalizes an early result of Enomoto *et al.* on the existence of heavy cycles in k-connected weighted graphs.

Keywords: heavy cycle, weighted degree (sum), induced claw (modified claw)

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1 Terminology and notation

We use Bondy and Murty [5] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. For a subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is

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defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For a vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a subgraph is simply the number of its edges.

An (x, y)*-path* is a path connecting two vertices x and y. Let H be a path or a cycle with a given orientation. By $\overleftarrow{H}$ we mean the same graph as H but with the reverse orientation. If v is a vertex of H, then v_H^{+1} and v_H^{-1} denote the immediate successor and immediate predecessor (if it exists) of v on H, respectively. In the following, we use v_H^+ for v_H^{+1} and v_H^- for v_H^{-1} for simplicity. For an integer $k \geq 2$, v_H^{+k} and v_H^{-k} are defined recursively by $v_H^{+k} = (v_H^{+(k-1)})^+$ and $v_H^{-k} = (v_H^{-(k-1)})^-$. If S is a set of vertices of H, then define $S_H^+ = \{s_H^+ | s \in S\}$. When no confusion occurs, we denote v_H^+, v_H^-, v_H^{+m}, v_H^{-m} and S_H^+ by v^+, v^-, v^{+m}, v^{-m} and S^+, respectively. For two vertices u and v of H, we use $H[u, v]$ to denote the segment of H from u to v. For a path $P[u, v]$, by $P(u, v)$, $P[u, v)$ and $P(u, v]$, we mean the path $P[u, v] - \{u, v\}$, $P[u, v] - \{v\}$ and $P[u, v] - \{u\}$, respectively.

The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. If G is noncomplete, then for a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k pairwise nonadjacent vertices. If G is complete, then both $\sigma_k(G)$ and $\sigma_k^w(G)$ are defined as ∞.

We call the graph $K_{1,3}$ a *claw*, and the graph obtained by joining a pendant edge to some vertex of a triangle a *modified claw*.

2 Results

There have been many results on the existence of long paths and cycles in unweighted graphs. In [3] and [4], Bondy and Fan generalized several classical theorems of Dirac and of Erdös and Gallai on paths and cycles to weighted graphs. A weighted generalization of Ore's theorem was obtained by Bondy *et al.* [2]. In [11], it was shown that if one wants to generalize Fan's theorem on the existence of long cycles to weighted graphs some extra conditions cannot be avoided. By adding two extra conditions, the authors gave a weighted generalization of Fan's theorem.

Among the many results on cycles in unweighted graphs, the following generalization of Ore's theorem is well-known.

Theorem A (Fournier & Fraisse [8]). *Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

A natural question is whether Theorem A also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. *Let G be a k-connected weighted graph where $2 \leq k < \alpha(G)$, such that $\sigma^w_{k+1}(G) \geq m$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$?*

It seems very difficult to settle this problem, even for the case $k = 2$. Motivated by the result in [11], Zhang *et al.* [10] proved that the answer to Problem 1 in the case $k = 2$ is positive with the two same extra conditions as in [11].

Theorem 1 (Zhang *et al.* [10]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $\sigma^w_3(G) \geq m$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

In [7], after giving a characterization of the connected weighted graphs satisfying Conditions (2) and (3) of Theorem 1, Enomoto *et al.* proved that the answer to Problem 1 is positive for any $k \geq 2$ with these two extra conditions.

Theorem 2 (Enomoto *et al.* [7]). *Let G be a k-connected weighted graph where $k \geq 2$. Suppose that G satisfies the following conditions:*
(1) $\sigma^w_{k+1}(G) \geq m$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$.

On the other hand, Fujisawa [9] gave so-called claw conditions for the existence of heavy cycles in weighted graphs, generalizing a result of Bedrossian *et al.* [1] on the existence of long cycles in unweighted graphs.

Theorem 3 (Fujisawa [9]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) *For each induced claw and each induced modified claw of G, all its nonadjacent pair of vertices x and y satisfy $max\{d^w(x), d^w(y)\} \geq s/2$;*
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least s.

A result similar to this theorem was obtained by Chen and Zhang [6]. It also generalizes Theorem 1.

Theorem 4 (Chen and Zhang [6]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $\sigma_3^w(G) \geq m$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

Clearly, Condition (2) of Theorem 4 is weaker than Conditions (2) and (3) of Theorem 2. Thus, we have the following problem: Can Conditions (2) and (3) in Theorem 2 be weakened by Condition (2) of Theorem 4? In this paper, we give a positive answer to this problem. Our result is a generalization of Theorem 2.

Theorem 5. *Let G be a k-connected weighted graph where $k \geq 2$. Suppose that G satisfies the following conditions:*
(1) $\sigma_{k+1}^w(G) \geq m$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$.

We postpone the proof of Theorem 5 to the next section.

3 Proof of Theorem 5

To prove Theorem 5, we need the following lemmas. Lemma 1 can be proved by a minor modification of the proof of Lemma 5 in Bondy and Fan [4], while the proof of Lemma 2 is almost immediate.

Lemma 1. *Let G be a 2-connected weighted graph which is non-hamiltonian and P an (s,t)-path in G. Then there is a cycle $\tilde{C}$ in G with $w(\tilde{C}) \geq d^w(s) + d^w(t)$, if the following conditions are satisfied :*
(i) $N(s) \cup N(t) \subseteq V(P)$;

(ii) $N_P(s) \cap N_P(t)^+ = \emptyset$;
(iii) $w(x^-x) \geq w(sx)$ *if* $x \in N_P(s)$ *and* $w(xx^+) \geq w(xt)$ *if* $x \in N_P(t)$.

Lemma 2. *Let* G *be a* k*-connected weighted graph where* $2 \leq k < \alpha(G)$ *and* $\{u_1, u_2, \cdots, u_{k+1}\}$ *an independent set of* G. *Then there exist* u_i *and* u_j *with* $1 \leq i < j \leq k+1$ *such that* $d^w(u_i) + d^w(u_j) \geq \frac{2}{k+1}\sigma^w_{k+1}(G)$.

Lemma 3 (Fujisawa [9]). *Let* G *be a weighted graph satisfying Condition (2) of Theorem 5. If* x_1yx_2 *is an induced path with* $w(x_1y) \neq w(x_2y)$ *in* G, *then each vertex* $x \in N(y)\backslash\{x_1, x_2\}$ *is adjacent to both* x_1 *and* x_2.

Lemma 4 (Fujisawa [9]). *Let* G *be a weighted graph satisfying Condition (2) of Theorem 5. Suppose* x_1yx_2 *is an induced path such that* $w_1 = w(x_1y)$ *and* $w_2 = w(x_2y)$ *with* $w_1 \neq w_2$, *and* yz_1z_2 *is a path such that* $\{z_1, z_2\} \cap \{x_1, x_2\} = \emptyset$ *and* $x_2z_2 \notin E(G)$. *Then*
(i) $\{z_1x_1, z_1x_2, z_2x_1\} \subseteq E(G)$, *and* $yz_2 \notin E(G)$. *Moreover, all edges in the subgraph induced by* $\{x_1, y, x_2, z_1, z_2\}$, *other than* x_1y, *have the same weight* w_2.
(ii) Let Y *be the component of* $G - \{x_2, z_1, z_2\}$ *with* $y \in V(Y)$. *For each vertex* $v \in V(Y)\backslash\{x_1, y\}$, v *is adjacent to all of* x_1, x_2, y *and* z_2. *Furthermore,* $w(vx_1) = w(vx_2) = w(vy) = w(vz_2) = w_2$.

Proof of Theorem 5. Let G be a k-connected weighted graph satisfying the conditions of Theorem 5. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/(k+1)$.

Choose a cycle C in G such that
(1) C is as long as possible;
(2) $w(C)$ is as large as possible, subject to (1).
Then from the assumption that G does not contain a Hamilton cycle, we can immediately see that $R = V(G)\backslash V(C) \neq \emptyset$. Choose $u_0 \in R$ such that $d^w(u_0) = min\{d^w(u)|u \in R\}$ and denote by A_0 the component containing u_0 in $G - V(C)$. Since G is k-connected, there exist k paths $P_i = u_0 \cdots w_iv_i$ $(i = 1, 2, \cdots, k)$, such that $V(P_i) \cap V(P_j) = \{u_0\}$, $V(P_i) \cap V(C) = \{v_i\}$, and $v_i \neq v_j$ for $i \neq j$.

Assign an orientation to C. We can assume that the vertices $v_1, v_2, \cdots, v_k$ appear on C along this orientation. Now let $u_i = v_i^+$. It is easy to verify that $\{u_0, u_1, u_2, \cdots, u_k\}$ is an independent set of G by the choice of C.

If $N_R(u_i) \neq \emptyset$, choose a path $Q_i = u_iy_i \cdots z_i$ in $G[R \cup \{u_i\}]$ such that
(1) Q_i is as long as possible;
(2) $w(Q_i)$ is as large as possible, subject to (1).
Then from the choice of Q_i, we know that $N_R(z_i) \subseteq Q_i$. Let A_i be the component of $G - V(C)$ such that $y_i \in V(A_i)$. Without loss of generality, we can assume $N_R(u_i) = \emptyset$ for $i = 1, 2, \cdots, q$ and $N_R(u_i) \neq \emptyset$ for $i = q+1, q+2, \cdots, k$.

Claim 1. *Let P be an (s,t)-path, such that $|V(P)| > |V(C)|$. Then $N_P(s) \cap N_P(t)^+ = \emptyset$.*

Proof. Suppose $N_P(s) \cap N_P(t)^+ \neq \emptyset$. Let x be a vertex in $N_P(s) \cap N_P(t)^+$. Then we get a cycle $C' = sP[s,x^-]x^-t\overleftarrow{P}[t,x]xs$ which is longer than C, a contradiction. □

Claim 2. *$A_0, A_{q+1}, \cdots, A_k$ are different components of $G - V(C)$.*

Proof. If $A_0 = A_i$ for some $i \in \{q+1, q+2, \cdots, k\}$, then there exists a (w_i, y_i)-path P_i^* in this component. So we can get a cycle $C' = w_i P_i^* y_i u_i C[u_i, v_i] v_i w_i$ which is longer than C, a contradiction.

If $A_i = A_j$ for some $q+1 \leq i < j \leq k$, then there exists a (y_i, y_j)-path P_{ij}^* in this component. So we can get a cycle $C' = y_i P_{ij}^* y_j u_j C[u_j, v_i] v_i \overleftarrow{P}_i u_0 P_j v_j \overleftarrow{C}[v_j, u_i] u_i y_i$ which is longer than C, a contradiction. □

Claim 3. *$\{u_0, u_1, u_2, \cdots, u_q, z_{q+1}, z_{q+2}, \cdots, z_k\}$ is an independent set.*

Proof. Since $\{u_0, u_1, u_2, \cdots, u_k\}$ is an independent set, we need only prove that $u_0 z_j \notin E(G)$ for $j = q+1, \cdots, k$; $u_i z_j \notin E(G)$ for $i = 1, \cdots, q$ and $j = q+1, \cdots, k$; and $z_i z_j \notin E(G)$ for $q+1 \leq i < j \leq k$. By Claim 2, it is obvious that $u_0 z_j \notin E(G)$ and $z_i z_j \notin E(G)$. By the assumption that $N_R(u_i) = \emptyset$ when $i = 1, 2, \cdots, q$, and $z_j \in R$, we have $u_i z_j \notin E(G)$ for any $i = 1, 2, \cdots, q$ and $j = q+1, \cdots, k$. □

Apply Lemma 2 to the independent set $\{u_0, u_1, u_2, \cdots, u_q, z_{q+1}, z_{q+2}, \cdots, z_k\}$, there must be two vertices s and t in this set such that $d^w(s) + d^w(t) \geq 2m/(k+1)$.

We distinguish two cases.

Case 1. *$u_0 \notin \{s, t\}$.*

Case 1.1. *$s = u_i$ and $t = u_j$ for some i and j with $1 \leq i < j \leq q$.*

Consider the path $P = sC[s, v_j] v_j \overleftarrow{P}_j u_0 P_i v_i \overleftarrow{C}[v_i, t]t$. It is obvious that $V(C) \subset V(P)$. Then, from $N(s) \subset V(C)$ and $N(t) \subset V(C)$, we have $N(s) \cup N(t) \subset V(P)$; from $|V(P)| > |V(C)|$ and Claim 1 we have $N_P(s) \cap N_P(t)^+ = \emptyset$. In the following, let's prove that $w(x_P^- x) \geq w(sx)$ if $x \in N_P(s)$; and $w(x x_P^+) \geq w(xt)$ if $x \in N_P(t)$. We prove $w(x_P^- x) \geq w(sx)$ if $x \in N_P(s)$ in details and leave the proof of $w(x x_P^+) \geq w(xt)$ if $x \in N_P(t)$ to the readers.

Let x be a vertex in $N_P(s)$. Since $N(s) \subset V(C)$, we have $x \in V(C)$. If $w(x_P^- x) = w(sx)$, then there is nothing to prove. So we make the following assumption.

Assumption 1. *$w(x_P^- x) \neq w(sx)$.*

Let's prove that $w(x_P^- x) > w(sx)$.

Claim 4. *$x \neq u_j, v_i, v_j$.*

Proof. By Claims 3, it is obvious that $x \neq u_j$.

If $x = v_i$, then $x_P^- = w_i$ and $\{x, s, x_P^-, x_P^+\}$ induces a claw or a modified claw. So $w(x_P^- x) = w(sx)$, contradicting Assumption 1.

If $x = v_j$, then $\{x, w_j, s, x_P^-\}$ induces a claw or a modified claw. So $w(x_P^- x) = w(sx)$, contradicting Assumption 1. □

Claim 5. *$w_i x \notin E(G)$ and $w_j x \notin E(G)$.*

Proof. If $w_i x \in E(G)$, then by the choice of C, we have $w_i x_P^- \notin E(G)$. Now $\{x, x_P^-, s, w_i\}$ induces a claw or a modified claw, so $w(x_P^- x) = w(sx)$, contradicting Assumption 1.

Similarly, we can prove that $w_j x \notin E(G)$. □

Claim 6. *$t x_P^- \notin E(G)$.*

Proof. This follows from Claim 1 and the choice of P immediately. □

Claim 7. *If $x \in V(P(s, v_j))$, then for every vertex $v \in V(P(x, v_j])$, if $x_P^- v \in E(G)$, then $w_i v_P^- \notin E(G)$; If $x \in V(P(v_i, t))$, then for every vertex $v \in V(P[x, t))$, if $x_P^- v \in E(G)$, then $w_i v_P^+ \notin E(G)$.*

Proof. Suppose $x \in V(P(s, v_j))$ and there exists a vertex $v \in V(P(x, v_j])$ such that $x_P^- v \in E(G)$ and $w_i v_P^- \in E(G)$. Then we have another cycle $C' = w_i v_i \overleftarrow{C}[v_i, v] v x_P^- \overleftarrow{C}[x_P^-, s] s x C[x, v_P^-]\ v_P^- w_i$ which is longer than C, a contradiction.

Suppose $x \in V(P(v_i, t))$ and there exists a vertex $v \in V(P[x, t))$ such that $x \in V(P(v_i, t))$ and $w_i v_P^+ \in E(G)$. Then we have another cycle $C' = w_i v_i \overleftarrow{C}[v_i, x_P^-] x_P^- v\ C[v, x] x s C[s, v_P^+] v_P^+ w_i$ which is longer than C, a contradiction. □

Case 1.1.1 $s x_P^- \in E(G)$.

Claim 8. *$w_i x_P^- \notin E(G)$, $w_j x_P^- \notin E(G)$ and $w_i s_P^+ \notin E(G)$.*

Proof. If $w_i x_P^- \in E(G)$, then from Claim 5 and $w_i s \notin E(G)$, we know that $\{x_P^-, x, s, w_i\}$ induces a modified claw. So $w(x_P^- x) = w(sx)$, contradicting Assumption 1. Similarly, we can prove that $w_j x_P^- \notin E(G)$.

Suppose $x \in V(P(s, v_j))$. If $w_i s_P^+ \in E(G)$, then we have another cycle $C' = w_i v_i \overleftarrow{C}[v_i, x]\ x s x_P^-\ \overleftarrow{C}[x_P^-, s_P^+] s_P^+ w_i$, which is longer than C, a contradiction.

Suppose $x \in V(P(v_i, t))$. If $w_i s_P^+ \in E(G)$, then we have another cycle $C' = w_i v_i \overleftarrow{C}[v_i, x_P^-]\ x_P^- s x\ \overleftarrow{C}[x, s_P^+] s_P^+ w_i$, which is longer than C, a contradiction. □

Claim 9. *Exactly on one of $v_i x$ and $v_i x_P^-$ is an edge of G.*

Proof. If $v_i x \notin E(G)$ and $v_i x_P^- \notin E(G)$, then $\{s, x, x_P^-, v_i\}$ induces a modified claw. If $v_i x \in E(G)$ and $v_i x_P^- \in E(G)$, then by Claims 5 and 8, both $\{v_i, x, x_P^-, w_i\}$ and $\{v_i, s, x, w_i\}$ induce modified claws. We can always get $w(x_P^- x) = w(sx)$, contradicting Assumption 1. □

Case 1.1.1.1 $v_i x \in E(G)$ and $v_i x_P^- \notin E(G)$.

Claim 10. $w(x_P^- x) \neq w(v_i x)$.

Proof. Since $\{v_i, s, x, w_i\}$ induces a modified claw, $w(sx) = w(v_i x)$. From Assumption 1, we have $w(x_P^- x) \neq w(v_i x)$. $\square$

If $x \in V(P(s, v_j))$, then by Claim 8, there exists some vertex $x_P^{+p} \in V(P(x, w_j])$ such that $x_P^{+p} \notin N(v_i) \cap N(x_P^-)$. If $x \in V(P(v_i, t))$, then by Claim 6, there exists some vertex $x_P^{+p} \in V(P(x, t])$ such that $x_P^{+p} \notin N(v_i) \cap N(x_P^-)$. In both cases, choose the vertex x_P^{+p} such that p is as small as possible. From Claim 10 and Lemma 3, we know that $x_P^+ \in N(v_i) \cap N(x_P^-)$. So we have $p \geq 2$. Clearly, if $p = 2$, then $x_P^{+(p-2)} = x$.

Claim 11. $w(sv_i) = w(v_i x_P^{+(p-2)})$ *and* $w(x_P^- x_P^{+(p-1)}) = w(x_P^{+(p-2)} x_P^{+(p-1)})$.

Proof. By the choice of x_P^{+p}, we have $x_P^- x_P^{+(p-2)} \in E(G)$ and $v_i x_P^{+(p-2)} \in E(G)$. It follows from Claims 5 and 7 that $w_i x_P^{+(p-2)} \notin E(G)$. So $\{v_i, s, x_P^{+(p-2)}, w_i\}$ induces a claw or a modified claw, which implies that $w(sv_i) = w(v_i x_P^{+(p-2)})$.

If $w_i x_P^{+(p-1)} \in E(G)$, then it follows from Claims 7 and 8 that $\{x_P^{+(p-1)}, x_P^{+(p-2)}, x_P^-, w_i\}$ induces a modified claw. So $w(x_P^- x_P^{+(p-1)}) = w(x_P^{+(p-2)} x_P^{+(p-1)})$. If $w_i x_P^{+(p-1)} \notin E(G)$, then from the choice of x_P^{+p} and Claim 7, $\{v_i, x_P^{+(p-2)}, x_P^{+(p-1)}, w_i\}$ induces a modified claw. On the other hand, $\{x_P^{+(p-1)}, x_P^{+p}, x_P^-, v_i\}$ induces a claw or a modified claw. So $w(x_P^{+(p-2)} x_P^{+(p-1)}) = w(v_i x_P^{+(p-1)})$ and $w(x_P^- x_P^{+(p-1)}) = w(v_i x_P^{+(p-1)})$. So $w(x_P^- x_P^{+(p-1)}) = w(x_P^{+(p-2)} x_P^{+(p-1)})$. $\square$

If $x \in V(P(s, v_j))$, let $C' = v_i \overleftarrow{C}[v_i, x_P^{+(p-1)}] x_P^{+(p-1)} x_P^- \overleftarrow{C}[x_P^-, s] sxC[x, x_P^{+(p-2)}]\, x_P^{+(p-2)} v_i$. If $x \in V(P(v_i, t))$, let $C = v_i \overleftarrow{C}[v_i, x_P^-] x_P^- x_P^{+(p-1)} \overleftarrow{C}[x_P^{+(p-1)}, s]\ sxC[x, x_P^{+(p-2)}]\ x_P^{+(p-2)} v_i$. In both cases, C' is a longest cycle different from C. By the choice of C, we have $w(C') \geq w(C)$. This implies that

$$w(sv_i) + w(x_P^{+(p-2)} x_P^{+(p-1)}) + w(x_P^- x) \geq w(sx) + w(v_i x_P^{+(p-2)}) + w(x_P^- x_P^{+(p-1)}).$$

From Claim 11 and Assumption 1, we get $w(x_P^- x) > w(sx)$.

Case 1.1.1.2 $v_i x \notin E(G)$ and $v_i x_P^- \in E(G)$.

Claim 12. $w(sx_P^-) = w(sv_i) = w(v_i x_P^-)$.

Proof. From Claim 8, we can easily see that $\{v_i, s, x_P^-, w_i\}$ induces a modified claw. So $w(sx_P^-) = w(sv_i) = w(v_i x_P^-)$. $\square$

Suppose $x \in V(P(s, v_j))$. Consider the longest cycle $C' = sC[s, x_P^-] x_P^- v_i \overleftarrow{C}[v_i, x] xs$. By the choice of C, we have $w(sv_i) + w(x_P^- x) \geq w(sx) + w(v_i x_P^-)$. From Claim 12 and Assumption 1, $w(x_P^- x) > w(sx)$. So from now on, we only consider the case $x \in V(P(v_i, t))$.

Claim 13. $v_i s_P^+ \in E(G)$.

Proof. Suppose $v_i s_P^+ \notin E(G)$. Then $\{s, x, v_i, s_P^+\}$ induced a claw or a modified claw. So $w(sv_i) = w(sx)$. From Claim 12 and Assumption 1, we have $w(v_i x_P^-) \neq w(x_P^- x)$. It is clear that t is a vertex of the component of $G - \{v_i, s, s_P^+\}$ which contains x_P^-. So, applying Lemma 4 (ii) to $\{x, x_P^-, v_i, s, s_P^+\}$, we can know that $tx_P^- \in E(G)$, contradicting Claim 6. □

Claim 14. $w(v_i s_P^+) = w(ss_P^+)$.

Proof. From Claims 8 and 13, $\{v_i, s, s_P^+, w_i\}$ induces a modified claw. So we have $w(v_i s_P^+) = w(ss_P^+)$. □

Let $C' = v_i \overleftarrow{C}[v_i, x_P^-] x_P^- sx\ \overleftarrow{C}[x, s_P^+] s_P^+ v_i$. Then C' is another longest cycle. By the choice of C, we have $w(sv_i) + w(ss_P^+) + w(x_P^- x) \geq w(sx_P^-) + w(v_i s_P^+) + w(sx)$. From Claims 12 , 14 and Assumption 1, we get $w(x_P^- x) > w(sx)$.

Case 1.1.2 $sx_P^- \notin E(G)$.

Claim 15. $v_i x \notin E(G)$.

Proof. Suppose $v_i x \in E(G)$. From Assumption 1, $w(sx) \neq w(x_P^- x)$. Since t is a vertex of the component of $G - \{s, v_i, w_i\}$ containing x_P^-, by applying Lemma 4 (ii) to $\{x_P^-, x, s, v_i, w_i\}$, we have $tx_P^- \in E(G)$, contradicting Claim 6. □

If $x \in V(P(s, v_j))$, then, since $w_j s \notin E(G)$, there exists a vertex $x_P^{+q} \in V(P(x, w_j])$ such that $x_P^{+q} \notin N(s) \cap N(x_P^-)$. If $x \in V(P(v_i, t))$, then since $tx_P^- \notin E(G)$, there exists a vertex $x_P^{+q} \in V(P(x, t])$ such that $x_P^{+q} \notin N(s) \cap N(x_P^-)$. In both cases, choose x_P^{+q} such that q is as small as possible. Apply Lemma 3 to the induced path sxx_P^- and x^+, we have $x_P^+ \in N(s) \cap N(x_P^-)$. So we have $q \geq 2$.

Claim 16. $xx_P^{+q} \notin E(G)$.

Proof. If $xx_P^{+q} \in E(G)$, then from the choice of x_P^{+q}, $\{x, x_P^-, x_P^{+q}, s\}$ induces a claw or a modified claw. So $w(x_P^- x) = w(sx)$, contradicting Assumption 1. □

Case 1.1.2.1 $xx_P^{+(q-1)} \in E(G)$.

Claim 17. $x_P^- x_P^{+q} \in E(G)$, $sx_P^{+q} \notin E(G)$.

Proof. From Assumption 1, $w(sx) \neq w(x_P^- x)$. If $x_P^- x_P^{+q} \notin E(G)$, then since t is a vertex of the component of $G - \{x_P^-, x_P^{+(q-1)}, x_P^{+q}\}$ containing x, by applying Lemma 4 (ii) to $\{x_P^-, x, s, x_P^{+(q-1)}, x_P^{+q}\}$, we have $tx_P^- \in E(G)$, contradicting Claim 6. So $x_P^- x_P^{+q} \in E(G)$. By the choice of x_P^{+q}, we have $sx_P^{+q} \notin E(G)$. □

Claim 18. $x \notin V(P(v_i, t))$.

Proof. Suppose $x \in V(P(v_i, t))$. Then from Claim 17, $sx_P^{+q} \notin E(G)$. Since t is a vertex of the component of $G - \{s, x_P^{+(q-1)}, x_P^{+q}\}$ containing x, by applying Lemma 4 (ii) to $\{x_P^-, x, s, x_P^{+(q-1)}, x_P^{+q}\}$, we have $tx_P^- \in E(G)$, contradicting Claim 6. So $x \notin V(P(v_i, t))$. □

From Claims 17, 18 and Assumption 1, since x_P^{-2} is a vertex of the component of $G - \{s, x_P^{+(q-1)}, x_P^{+q}\}$ containing x, by applying Lemma 4 to $\{x_P^-, x, s, x_P^{+(q-1)}, x_P^{+q}\}$, we have $x_P^{-2}x \in E(G)$, and

$$w(x_P^{-2}x_P^-) = w(x_P^{-2}x) = w(x_P^-x_P^{+(q-1)}) = w(x_P^-x_P^{+q}) = w(x_P^{+(q-1)}x_P^{+q}) = w(sx). \qquad (1)$$

Let $C' = x_P^{-2}\overleftarrow{C}[x_P^{-2}, x_P^{+q}]x_P^{+q}x_P^- \, x_P^{+(q-1)} \overleftarrow{C}[x_P^{+(q-1)}, x]xx_P^{-2}$. Then C' is another longest cycle. By the choice of C, we have $w(x_P^{-2}x_P^-) + w(x_P^-x) + w(x_P^{+(q-1)}x_P^{+q}) \geq w(x_P^{-2}x) + w(x_P^-x_P^{+q}) + w(x_P^-x_P^{+(q-1)})$. By (1), we get $w(x_P^-x) \geq w(x_P^-x_P^{+q})$. This implies that $w(x_P^-x) \geq w(sx)$. From Assumption 1, $w(x_P^-x) > w(sx)$.

Case 1.1.2.2 $xx_P^{+(q-1)} \notin E(G)$.

Claim 19. $w(sx) = w(x_P^-x_P^{+(q-1)})$.

Proof. By the choice of x_P^{+q}, $sx_P^{+(q-1)} \in E(G)$. From Claim 15, $\{s, v_i, x_P^{+(q-1)}, x\}$ induces a claw or a modified claw. So $w(sx) = w(sx_P^{+(q-1)})$. At the same time, by the choice of x_P^{+q}, $\{x_P^{+(q-1)}, x_P^{+q}, s, x_P^-\}$ induces a claw or a modified claw. So $w(sx_P^{+(q-1)}) = w(x_P^-x_P^{+(q-1)})$. This implies that $w(sx) = w(x_P^-x_P^{+(q-1)})$. □

Claim 20. $x_P^{-2}x, x_P^{-2}x_P^{+(q-1)}, sx_P^{-2} \in E(G)$, $x_P^-x_P^{+q}, v_ix_P^{-2}, x_P^{-2}x_P^{+q}, x_P^{+(q-2)}x_P^{+q} \notin E(G)$.

Proof. From Claim 19 and Assumption 1, we get $w(x_P^-x) \neq w(x_P^-x_P^{+(q-1)})$. So, applying Lemma 3 to the induced path $xx_P^-x_P^{+(q-1)}$ and the vertex x_P^{-2}, we get $x_P^{-2}x \in E(G)$ and $x_P^{-2}x_P^{+(q-1)} \in E(G)$. Now applying Lemma 3 to the induced path sxx_P^- and the vertex x_P^{-2}, we get $sx_P^{-2} \in E(G)$.

If $x_P^-x_P^{+q} \in E(G)$, then from Claim 16 and $xx_P^{+(q-1)} \notin E(G)$, $\{x_P^-, x_P^{+q}, x_P^{+(q-1)}, x\}$ induces a modified claw. So $w(x_P^-x_P^{+(q-1)}) = w(x_P^-x)$. From Claim 19, we get $w(sx) = w(x_P^-x)$, contradicting Assumption 1.

If $v_ix_P^{-2} \in E(G)$, then since t is a vertex of the component of $G - \{x, x_P^{-2}, v_i\}$ containing x_P^-, by applying Lemma 4 (ii) to $\{x_P^{+(q-1)}, x_P^-, x, x_P^{-2}, v_i\}$, we have $tx_P^- \in E(G)$, contradicting Claim 6. Similarly, we can prove $x_P^{-2}x_P^{+q} \notin E(G)$ and $x_P^{+(q-2)}x_P^{+q} \notin E(G)$. □

Claim 21. $w(x_P^{-2}x_P^-) = w(x_P^{-2}x) = w(x_P^-x_P^{+(q-2)}) = w(x_P^{+(q-2)}x_P^{+(q-1)})$.

Proof. By the choice of x_P^{+q}, $x_P^-x_P^{+(q-1)} \in E(G)$ and $x_P^-x_P^{+(q-2)} \in E(G)$. From Claim 20, both $\{x_P^{+(q-1)}, x_P^{-2}, x_P^-, x_P^{+q}\}$ and $\{x_P^{+(q-1)}, x_P^{+(q-2)}, x_P^-, x_P^{+q}\}$ induce modified claws, so we have $w(x_P^-x_P^{+(q-1)}) = w(x_P^{-2}x_P^-)$ and $w(x_P^-x_P^{+(q-1)}) = w(x_P^-x_P^{+(q-2)}) = w(x_P^{+(q-2)}x_P^{+(q-1)})$.

From Claims 15 and 20, $\{s, x_P^{-2}, x, v_i\}$ induces a modified claw, so $w(x_P^{-2}x) = w(sx)$. The result follows from Claim 19 immediately. $\square$

If $x \in V(P(s, v_j))$, let $C' = x_P^{-2}xC[x, x_P^{+(q-2)}]x_P^{+(q-2)}x_P^{-}x_P^{+(q-1)}\ C[x_P^{+(q-1)}, x_P^{-2}]x_P^{-2}$. If $x \in V(P(v_i, t))$, let $C' = x_P^{-2}x\overleftarrow{C}[x, x_P^{+(q-2)}]x_P^{+(q-2)}x_P^{-}x_P^{+(q-1)}\ \overleftarrow{C}[x_P^{+(q-1)}, x_P^{-2}]x_P^{-2}$. In both cases, C' is a longest cycle different from C. By the choice of C, we have $w(C') \geq w(C)$. This implies that

$$w(x_P^{-2}x_P^{-}) + w(x_P^{+(q-2)}x_P^{+(q-1)}) + w(x_P^{-}x) \geq w(x_P^{-2}x) + w(x_P^{-}x_P^{+(q-2)}) + w(x_P^{-}x_P^{+(q-1)}).$$

It follows from Claim 21 that $w(x_P^{-}x) \geq w(x_P^{-}x_P^{+(q-1)})$. From Claim 19 and Assumption 1, we get $w(x_P^{-}x) > w(sx)$.

From the above discussion, we see that $w(x_P^{-}x) \geq w(sx)$ if $x \in N_P(s)$. So, by Lemma 1, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geq d^w(s) + d^w(t) \geq 2m/(k+1)$.

Case 1.2. *$s = u_i$ for some i with $1 \leq i \leq q$ and $t = z_j$ for some j with $q+1 \leq j \leq k$.*

Consider the path $P = u_iC[u_i, v_j]v_j\overleftarrow{P}_ju_0P_iv_i\overleftarrow{C}[v_i, u_j]u_jQ_jz_j$. As in the proof of Case 1.1, we can show that $N(s) \cup N(t) \subseteq V(P)$, $N_P(s) \cap N_P(t)^+ = \emptyset$, and $w(x_P^{-}x) \geq w(sx)$ if $x \in N_P(s)$. Now let's prove that $w(xx_P^{+}) \geq w(xt)$ for $x \in N_P(t)$.

Suppose $x \in V(Q_j)$. Then $w(xx_P^{+}) \geq w(xt)$ by the choice of Q_j.

Suppose $x \in V(C) \setminus \{u_j\}$. Then, by the choice of C, $x_P^{-}t \notin E(G)$ and $x_P^{+}t \notin E(G)$. So $\{t, x_P^{+}, x, x_P^{-}\}$ induces a claw or a modified claw. Thus $w(xx_P^{+}) = w(xt)$.

Then by Lemma 1, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geq d^w(s)+d^w(t) \geq 2m/(k+1)$.

Case 1.3. *$s = z_i$ and $t = z_j$ for some i and j with $q+1 \leq i < j \leq k$.*

Consider the path $P = z_i\overleftarrow{Q}_iu_iC[u_i, v_j]v_j\overleftarrow{P}_ju_0P_iv_i\overleftarrow{C}[v_i, u_j]u_jQ_jz_j$. Similar to Case 1.2, we can prove that the path P satisfies the three conditions of Lemma 1. So G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geq d^w(s) + d^w(t) \geq 2m/(k+1)$.

This completes the proof of Case 1.

Case 2. *$u_0 \in \{s, t\}$.*

Without loss of generality, we assume that $s = u_0$.

Case 2.1. *$t = u_i$ for some i with $1 \leq i \leq q$.*

Choose a path $Q' = v_iy_0' \cdots z_0'$ in $G[V(A_0) \cup \{v_i\}]$ such that

(1) Q' is as long as possible;

(2) $w(Q')$ is as large as possible, subject to (1) .

Then by the choice of u_0, we have $d^w(z_0') \geq d^w(u_0)$. Now consider the path $P = z_0'\overleftarrow{Q}'v_i\overleftarrow{C}[v_i, u_i]u_i$. As in Case 1.2, we can prove that this path satisfies the three conditions of Lemma 1. Therefore, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \geq d^w(z_0') + d^w(u_i) \geq d^w(s) + d^w(t) \geq 2m/(k+1)$.

Case 2.2. *$t = z_j$ for some j with $q+1 \le j \le k$.*

Choose a path $Q'' = v_j y_0'' \cdots z_0''$ in $G[V(A_0) \cup \{v_j\}]$ such that

(1) Q'' is as long as possible;

(2) $w(Q'')$ is as large as possible, subject to (1) .

Then by the choice of u_0, we have $d^w(z_0'') \ge d^w(u_0)$. Consider the path $P = z_0'' \overleftarrow{Q}'' v_j \overleftarrow{C}[v_j, u_j] u_j Q_j z_j$. As before, we can prove that this path satisfies the three conditions of Lemma 1. Therefore, G contains a cycle $\tilde{C}$ of weight $w(\tilde{C}) \ge d^w(z_0'') + d^w(z_j) \ge d^w(s) + d^w(t) \ge 2m/(k+1)$.

The proof of the theorem is complete.

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