Extreme values of the sum of squares of degrees of bipartite graphs

T.C. Edwin Cheng[a], Yonglin Guo[b], Shenggui Zhang[a,b,*]

[a] *Department of Logistics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong SAR, P.R. China*

[b] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China*

Abstract

In this paper we determine the minimum and maximum values of the sum of squares of degrees of bipartite graphs with a given number of vertices and edges.

Keywords: degree squares, bipartite graphs, extreme values

AMS Subject Classification (1991): 05C07 05C35

1 Introduction

All graphs considered in this paper are finite, undirected and simple. For terminology and notation not defined here we follow those in Bondy and Murty [1].

In this paper we study two extremal problems on the degree sequences of bipartite graphs: determine the minimum and maximum values of the sum of squares of degrees of bipartite graphs with a given number of vertices and edges.

Related problems for general graphs have been studied in the literature. It is easy to see that, among all the graphs with a given number of vertices and edges, a graph has the minimum sum of squares of degrees if and only if its maximum degree exceeds its minimum degree by at most one. So this problem is trivial in the minimum case for general graphs. However, the problem is much complicated in the maximum case. Boesch *et al.* [2] studied a more complicated problem for the maximum case: among all the graphs with a given number of vertices and edges, find the ones where the sum of squares of degrees is maximum. They showed that every such graph is a threshold

*Corresponding author at: Department of Applied Mathematics, Northwestern Polytechnical University, Xian, Shaanxi 710072, PR China. Tel.: +86 29 88493415; fax: +86 29 88494314. E-mail address: sgzhang@nwpu.edu.cn (S. Zhang).

graph, and for the given number of vertices and edges, constructed two threshold graphs, and proved that at least one of them has the maximum sum of squares of degrees. Peled *et al.* [3] further studied this problem and showed that, among all the graphs with a given number of vertices and edges, if a graph has the maximum sum of squares of degrees, then it must belong to one of the six particular classes of threshold graphs. In fact, Ahlswede and Kanona [4] had determined the maximum sum of squares of degrees for graphs with a given number of vertices and edges much earlier. For given positive integers n, m and k, they also established a bipartite graph with n vertices, m edges and a bipartite set with k vertices such that the sum of squares of its degrees is maximum.

The rest of this paper is organized as follows. In Section 2 we present some notations and lemmas that will be used later. The minimum and maximum sums of squares of degrees of bipartite graphs with a given number of vertices and edges are presented in Sections 3 and 4, respectively.

2 Notations and lemmas

We use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and maximum degree of a graph G, respectively. By $n_i(G)$ we denote the number of vertices in G with degree i. If S is a set of vertices, we use $\delta(S)$ and $\Delta(S)$ to denote the minimum degree and the maximum degree of the vertices in S, respectively. S^i will represent the set of vertices in S with degree i.

Let n, m and k be three positive integers. We use $B(n,m)$ to denote a bipartite graph with n vertices and m edges, and $B(n,m,k)$ to denote a $B(n,m)$ with a bipartition (X,Y) such that $|X| = k$. By $\mathcal{B}(n,m,k)$ we denote the set of graphs of the form $B(n,m,k)$.

Let $n \geq 2$ be an even integer and t a nonnegative integer. By $B_{n,t}$ we denote the bipartite graph with vertices $x_1, x_2, \ldots, x_{n/2}, y_1, y_2, \ldots, y_{n/2}$ and edges $x_i y_j$ with $i < j \leq i + t$ (where the addition is taken modulo $n/2$) for $i, j = 1, 2, \ldots, n/2$.

For two integers n and m with $n \geq 2$ and $m \geq 0$, let $2m = nt + r$, where $0 \leq r < n$. We define a bipartite graph $B^s(n,m)$ with n vertices and m edges as follows.

Case 1. n is even. Define $B^s(n,m) = B_{n,t} \cup \{x_i y_i | 1 \leq i \leq r/2\}$.

Case 2. n is odd and $nt \leq 2m < nt + t$. Define $B^s(n,m) = B^s(n-1, m-t+1) \cup \{x_i y_0 | (n+r-t+1)/2 + 1 \leq i \leq (n+r+t-1)/2\}$.

Case 3. n is odd and $nt + t \leq 2m \leq nt + n - t - 1$ or $nt + n - t + 1 \leq 2m < nt + n$. Define $B^s(n,m) = B^s(n-1, m-t) \cup \{x_i y_0 | (r-t)/2 + 1 \leq i \leq (r+t)/2\}$.

The degrees of the vertices of the graph $B^s(n,m)$ are shown in Table 1.

Table 1: Degrees of the vertices in the graph $B^s(n,m)$

n is even			
$d(x_i)$	i	$d(y_j)$	j
$t+1$	$1 \le i \le \frac{r}{2}$	$t+1$	$1 \le i \le \frac{r}{2}$
t	$\frac{r}{2}+1 \le i \le n/2$	t	$\frac{r}{2}+1 \le i \le n/2$
n is odd and $nt \le 2m < nt+t$			
$d(x_i)$	i	$d(y_j)$	j
$t+1$	$1 \le i \le \frac{r+t}{2}$	t	$1 \le j \le \frac{n+r-t+1}{2}$
t	$\frac{r+t}{2}+1 \le i \le \frac{n-1}{2}$	$t-1$	$j=0$ or $\frac{n+r-t+1}{2}+1 \le j \le \frac{n-1}{2}$
n is odd and $nt+t \le 2m \le nt+n-t-1$			
$d(x_i)$	i	$d(y_j)$	j
$t+1$	$1 \le i \le \frac{r+t}{2}$	$t+1$	$1 \le j \le \frac{r-t}{2}$
t	$\frac{r+t}{2}+1 \le i \le \frac{n-1}{2}$	t	$j=0$ or $\frac{r-t}{2}+1 \le j \le \frac{n-1}{2}$
n is odd and $nt+n-t+1 \le 2m < nt+n$			
$d(x_i)$	i	$d(y_j)$	j
$t+2$	$1 \le i \le \frac{r+t-n+1}{2}$	$t+1$	$1 \le j \le \frac{r-t}{2}$
$t+1$	$\frac{r+t-n+1}{2}+1 \le i \le \frac{n-1}{2}$	t	$j=0$ or $\frac{r-t}{2}+1 \le j \le \frac{n-1}{2}$

Suppose that n, m and k are three integers with $n \ge 2$, $0 \le m \le \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$ and $\lceil n/2\rceil \le k \le n-1$. Let $m = qk + r$, where $0 \le r < k$. Then $B^l(n,m,k)$ is defined as a bipartite graph in $\mathcal{B}(n,m,k)$ such that q vertices in Y are adjacent to all the vertices of X and one more vertex in Y is adjacent to r vertices in X. We use $B^l(n,m)$ to denote a graph $B^l(n,m,k_0)$ with $k_0 = max\{k | m = qk+r, 0 \le r < k, \lceil n/2\rceil \le k \le n-q-sgn(r)\}$.

Let $D = (d_1, d_2, \ldots, d_n)$ be a nonnegative integer sequence. Define $\sigma_2(D) = \sum_{i=1}^{n} d_i^2$. If D is the degree sequence of a graph G, then we define $\sigma_2(G) = \sigma_2(D)$.

The following lemma is obvious.

Lemma 1. *Let m be a nonnegative integer and $D = (d_1, d_2, \ldots, d_n)$ an integer sequence with $0 \le d_i \le n-1$ for $i = 1, 2, \ldots, n$. If $\sum_{i=1}^{n} d_i = 2m$, then $\sigma_2(D)$ attains the minimum value if and only if $|d_i - d_j| \le 1$ for $1 \le i < j \le n$.*

Lemma 2 ([4]). *Let m, n and k be three integers with $n \ge 2$, $0 \le m \le \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$ and $\lceil n/2\rceil \le k \le n-1$. Suppose $m = qk + r$, where $0 \le r < k$. Then $\sigma_2(B^l(n,m,k))$ attains the maximum value among all the graphs in $\mathcal{B}(n,m,k)$.*

3 Minimum value of the sum of squares of degrees

Theorem 1. *Let n and m be two integers with $n \ge 2$ and $0 \le m \le \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$. Then $\sigma_2(B^s(n,m))$ attains the minimum value among all the bipartite graphs with n vertices* *and m edges.*

Proof. Suppose $2m = nt + r$, where $0 \le r < n$. If n is even, or n is odd and $nt + t \le 2m \le nt + n - t - 1$, then from Table 1 we know that $\Delta(B^s(n,m)) - \delta(B^s(n,m)) \le 1$. By Lemma 1, it is clear that $\sigma_2(B^s(n,m))$ attains the minimum value. If n is odd, then $nt + n - t$ is odd too. Therefore, $2m \neq nt + n - t$ when n is odd. So in the following we need only consider the case where n is odd, $m \ge n$, and $nt \le 2m < nt + t$ or $nt + n - t + 1 \le 2m < nt + n$.

Suppose that G is a bipartite graph such that $\sigma_2(G)$ attains the minimum value among all the bipartite graphs with n vertices and m edges.

Claim 1. $\delta(G) \ge 1$.

Proof. Since $m \ge n$, there must be one vertex u with $d(u) \ge 2$. If $\delta(G) = 0$, let v be a vertex with $d(v) = 0$. Choose one neighbor w of u and set $G' = G - uw + vw$. Then

$$\sigma_2(G') - \sigma_2(G) = 2(1 - d(u)) < 0,$$

a contradiction. □

Let (X, Y) be the bipartition of G. By the symmetry of X and Y, we assume that $|X| < |Y|$.

Claim 2. $\Delta(X) - \delta(X) \le 1$ and $\Delta(Y) - \delta(Y) \le 1$.

Proof. We only prove $\Delta(X) - \delta(X) \le 1$. The other assertion can be proved similarly.

By contradiction. Suppose that there exist two vertices x and x' in X such that $d(x) = \Delta(X)$, $d(x') = \delta(X)$ and $d(x) - d(x') > 1$. Then there must be one vertex $y \in Y$ such that $xy \in E(G)$ but $x'y \notin E(G)$. Set $G' = G - xy + x'y$. So we have

$$\begin{aligned}\sigma_2(G') - \sigma_2(G) &= (d(x) - 1)^2 + (d(x') + 1)^2 - d(x)^2 - d(x')^2 \\ &= 2(d(x') - d(x) + 1) \\ &< 0,\end{aligned}$$

a contradiction. □

Claim 3. $\Delta(X) = \Delta(G)$ and $\delta(Y) = \delta(G)$.

Proof. Clearly we need only consider the case $\Delta(G) - \delta(G) \ge 1$. We distinguish two cases.

Case 1. $\Delta(G) - \delta(G) = 1$.

Suppose $\Delta(X) \neq \Delta(G)$. Then $\Delta(X) = \delta(X) = \delta(G)$. So we have

$$\sum_{x \in X} d(x) = |X|\delta(G) < |Y|\delta(G) < \sum_{y \in Y} d(y),$$

a contradiction. Suppose $\delta(Y) \neq \delta(G)$. Then $\Delta(Y) = \delta(Y) = \Delta(G)$. So we have

$$\sum_{y \in Y} d(y) = |Y|\Delta(G) > |X|\Delta(G) > \sum_{x \in X} d(x),$$

a contradiction.

Case 2. $\Delta(G) - \delta(G) \geq 2$.

Suppose $\Delta(X) \neq \Delta(G)$. Then $\Delta(Y) = \Delta(G)$. By Claim 2, we have

$$\sum_{y \in Y} d(y) > |Y|(\Delta(G) - 1) > |X|(\Delta(G) - 1) > \sum_{x \in X} d(x),$$

a contradiction. The result $\delta(Y) = \delta(G)$ follows from Claim 2 immediately. $\square$

For simplicity, in the following we use Δ and δ instead of $\Delta(G)$ and $\delta(G)$, respectively.

Claim 4. $\Delta - \delta = 2$.

Proof. Clearly G cannot be a regular bipartite graph. So we have $\Delta - \delta \neq 0$.

Suppose $\Delta - \delta \geq 3$. If $|X^{\Delta-1}| = 0$ or $|Y^{\delta+1}| = 0$, then by Claim 1, there exist two vertices $x^* \in X^\Delta$ and $y^* \in Y^\delta$ such that $x^*y^* \in E(G)$. If $|X^{\Delta-1}| \neq 0$ and $|Y^{\delta+1}| \neq 0$, and there exist no edges connecting vertices in X^Δ and vertices in Y^δ, then by Claim 1, we can choose three vertices $x^* \in X^\Delta$, $x' \in X^\delta$ and $y^* \in Y^\delta$ such that $x'y^* \in E(G)$ but $x^*y^* \notin E(G)$. Since Y^Δ contains all the neighbors of x^* and $d(x^*) = d(x') + 1$, there must exist one vertex $y' \in Y^\Delta$ such that $x^*y' \in E(G)$ but $x'y' \notin E(G)$. Set $G' = G - x^*y' - x'y^* + x^*y^* + x'y'$. Then G' has the same degree sequence as G and there is one edge connecting a vertex in X^Δ and a vertex in Y^δ. So we can always assume that there is at least one vertex $x^* \in X^\Delta$ and one vertex $y^* \in Y^\delta$ such that $x^*y^* \in E(G)$.

Let $x_1 = x^*, x_2, \ldots, x_{\delta(Y)}$ be the neighbors of y^*. Choose $\delta(Y)$ vertices $y_1, y_2, \ldots, y_{\delta(Y)}$ in $Y \setminus \{y^*\}$. Then we have $d(x_1) \geq d(y_1) + 2$ and $d(x_i) \geq d(y_i) + 1$ for $i = 2, 3, \ldots, \delta(Y)$ by Claim 2. Let G^* be the graph obtained from G by deleting the edges x_iy^* and adding the edges y^*y_i for $i = 1, 2, \ldots, \delta(Y)$. Then we have

$$\begin{aligned}
\sigma_2(G^*) - \sigma_2(G) = &\sum_{i=1}^{\delta(Y)} [(d(x_i) - 1)^2 + (d(y_i) + 1)^2 - d(x_i)^2 - d(y_i)^2] \\
= &2(d(y_1) - d(x_1) + 1) + 2\sum_{i=2}^{\delta(Y)} (d(y_i) - d(x_i) + 1) \\
< &0,
\end{aligned}$$

a contradiction.

Suppose $\Delta - \delta = 1$. Then it is easy to see that $\Delta = \Delta(X) = t+1$, $\delta = \delta(Y) = t$. It follows from $\sum_{x\in X} d(x) = \sum_{y\in Y} d(y)$ that

$$|X^\Delta|(t+1) + |X^{\Delta-1}|t = |Y^{\delta+1}|(t+1) + |Y^\delta|t.$$

Then

$$|X^\Delta| = (|Y^{\delta+1}| + |Y^\delta| - |X^\Delta| - |X^{\Delta-1}|)t + |Y^{\delta+1}| \geq t + |Y^{\delta+1}|. \tag{1}$$

So we have

$$|X^\Delta| + |Y^{\delta+1}| \geq t + 2|Y^{\delta+1}| \geq t. \tag{2}$$

Since $|X| = |X^\Delta| + |X^{\Delta-1}| < |Y| = |Y^{\delta+1}| + |Y^\delta|$, by (1) we have

$$|Y^\delta| > |X^\Delta| + |X^{\Delta-1}| - |Y^{\delta+1}| \geq t + |X^{\Delta-1}|.$$

This implies that

$$|X^\Delta| + |Y^{\delta+1}| = n - |X^{\Delta-1}| - |Y^\delta| \leq n - |Y^\delta| < n - (t + |X^{\Delta-1}|) \leq n - t. \tag{3}$$

By $|X^\Delta| + |Y^{\delta+1}| = r = 2m - nt$, and (2) and (3), we have

$$nt + t \leq 2m < nt + n - t,$$

a contradiction.

From the above discussions, we know that $\Delta - \delta = 2$. $\square$

Claim 5. $|X^\Delta| = m - \frac{(n-1)(\delta+1)}{2}$, $|X^{\Delta-1}| = \frac{(n-1)(\delta+2)}{2} - m$, $|Y^{\delta+1}| = m - \frac{(n+1)\delta}{2}$ and $|Y^\delta| = \frac{(n+1)(\delta+1)}{2} - m$.

Proof. We distinguish two cases.

Case 1. $|X^\Delta| + |X^{\Delta-1}| = |Y^{\delta+1}| + |Y^\delta| - 1$.

Clearly we have

$$|X^\Delta| + |X^{\Delta-1}| = \frac{n-1}{2} \tag{4}$$

and

$$|Y^{\delta+1}| + |Y^\delta| = \frac{n+1}{2}. \tag{5}$$

On the other hand, it follows from $|X^\Delta|(\delta+2) + |X^{\Delta-1}|(\delta+1) = |Y^{\delta+1}|(\delta+1) + |Y^\delta|\delta = m$, $|X^\Delta| + |X^{\Delta-1}| + |Y^{\delta+1}| + |Y^\delta| = n$ and $|X^\Delta| + |X^{\Delta-1}| = |Y^{\delta+1}| + |Y^\delta| - 1$ that

$$2|X^\Delta| + |X^{\Delta-1}| - |Y^{\delta+1}| = \delta \tag{6}$$

and

$$2|X^\Delta| + |X^{\Delta-1}| + |Y^{\delta+1}| = 2m - n\delta. \tag{7}$$

Solving the equations (4) to (7), we get $|X^{\Delta}| = m - \frac{(n-1)(\delta+1)}{2}$, $|X^{\Delta-1}| = \frac{(n-1)(\delta+2)}{2} - m$, $|Y^{\delta+1}| = m - \frac{(n+1)\delta}{2}$ and $|Y^{\delta}| = \frac{(n+1)(\delta+1)}{2} - m$.

Now let us determine δ. If $\delta \le t-2$, then

$$\begin{aligned} 2m &= |X^{\Delta}|(\delta+2) + |X^{\Delta-1}|(\delta+1) + |Y^{\delta+1}|(\delta+1) + |Y^{\delta}|\delta \\ &\le (|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| + |Y^{\delta}|)t - |X^{\Delta-1}| - |Y^{\delta+1}| - 2|Y^{\delta}| \\ &\le nt - 2, \end{aligned}$$

a contradiction. If $\delta \ge t+1$, then

$$\begin{aligned} 2m &= |X^{\Delta}|(\delta+2) + |X^{\Delta-1}|(\delta+1) + |Y^{\delta+1}|(\delta+1) + |Y^{\delta}|\delta \\ &\ge (|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| + |Y^{\delta}|)(t+1) + 2|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| \\ &\ge n(t+1) + 2, \end{aligned}$$

a contradiction. So we have $\delta = t$ or $t-1$.

Suppose $nt \le 2m < nt + t$. If $\delta = t$, then

$$2|Y^{\delta}| = (n+1)(t+1) - 2m > n+1.$$

Since $2m = |X^{\Delta}|(\delta+2) + |X^{\Delta-1}|(\delta+1) + |Y^{\delta+1}|(\delta+1) + |Y^{\delta}|\delta > n\delta = nt$, we have $t < \frac{2m}{n} \le \frac{2\times\frac{n^2-1}{4}}{n} < \frac{n}{2} < \frac{n+1}{2} < |Y^{\delta}|$. On the other hand, it follows from $|X^{\Delta}| + |Y^{\delta}| = \delta + 1$ that $|Y^{\delta}| \le \delta = t$, a contradiction. Therefore, we have $\delta = t-1$ when $nt \le 2m < nt + t$.

Suppose $nt + n - t + 1 \le 2m < nt + n$. If $\delta = t-1$, then

$$2|Y^{\delta}| = (n+1)t - 2m \le (n+1)t - (nt + n - t + 1) = 2t - (n+1) < 0,$$

a contradiction. Therefore, we have $\delta = t$ when $nt + n - t + 1 \le 2m < nt + n$.

Case 2. $|X^{\Delta}| + |X^{\Delta-1}| \le |Y^{\delta+1}| + |Y^{\delta}| - 3$.

If $|Y^{\delta}| \ge \delta + 1$, then as in the proof of Claim 2, we can assume that there exists an edge $x^*y^* \in E(G)$ with $x^* \in X^{\Delta}$ and $y^* \in Y^{\delta}$. Denote the neighbors of y^* by $x_1 = x^*, x_2, x_3, \ldots, x_{\delta}$ and choose δ vertices $y_1, y_2, \ldots, y_{\delta}$ in $Y^{\delta} \setminus \{y^*\}$. Let G' be the graph obtained from G by deleting the edges x_iy^* and adding the edges y^*y_i for $i = 1, 2, \ldots, \delta$. It is easy to see that $\sigma_2(G') - \sigma_2(G) < 0$, a contradiction.

If $|Y^{\delta}| \le \delta$, then it follows from $|X^{\Delta}|(\delta+2) + |X^{\Delta-1}|(\delta+1) = |Y^{\delta+1}|(\delta+1) + |Y^{\delta}|\delta$ that

$$\begin{aligned} |X^{\Delta}| &= (|Y^{\delta+1}| + |Y^{\delta}| - |X^{\Delta}| - |X^{\Delta-1}|)\delta + |Y^{\delta+1}| - |X^{\Delta}| - |X^{\Delta-1}| \\ &\ge 3\delta + |Y^{\delta+1}| - |X^{\Delta}| - |X^{\Delta-1}| \\ &\ge 2\delta + |Y^{\delta+1}| + |Y^{\delta}| - |X^{\Delta}| - |X^{\Delta-1}| \\ &\ge 2\delta + 3. \end{aligned}$$

Now let B denote the graph $B^s(n,m)$. Then we have

$$\begin{aligned}\sigma_2(B) &= n_{\Delta(B)}(\delta+2)^2 + n_{\delta(B)}\delta^2 + (n - n_{\Delta(B)} - n_{\delta(B)})(\delta+1)^2 \\ &< n_{\Delta(B)}(\delta+2)^2 + (n - n_{\Delta(B)})(\delta+1)^2\end{aligned}$$

and

$$\begin{aligned}\sigma_2(G) &= |X^\Delta|(\delta+2)^2 + |Y^\delta|\delta^2 + (n - |X^\Delta| - |Y^\delta|)(\delta+1)^2 \\ &= (|X^\Delta| - |Y^\delta|)(\delta+2)^2 + (n - |X^\Delta| + |Y^\delta|)(\delta+1)^2 + |Y^\delta|((\delta+2)^2 + \delta^2 - 2(\delta+1)^2) \\ &> (|X^\Delta| - |Y^\delta|)(\delta+2)^2 + (n - |X^\Delta| + |Y^\delta|)(\delta+1)^2.\end{aligned}$$

At the same time, from Table 1, we can see that $n_{\Delta(B)} \leq \delta$. Therefore,

$$\begin{aligned}\sigma_2(B) - \sigma_2(G) &< (n_{\Delta(B)} + |Y^\delta| - |X^\Delta|)((\delta+2)^2 - (\delta+1)^2) \\ &\leq -3((\delta+2)^2 - (\delta+1)^2) \\ &< 0,\end{aligned}$$

a contradiction. □

From Claims 4 and 5, we know that $\Delta(G) - \delta(G) = 2$, $n_{\delta(G)} = \frac{(n+1)(\delta+1)}{2} - m$, $n_{\delta(G)+1} = n - \delta - 1$ and $n_{\delta(G)+2} = m - \frac{(n-1)(\delta+1)}{2}$, where

$$\delta = \begin{cases} t-1, & \text{if } nt \leq 2m < nt+t; \\ t, & \text{if } nt+n-t+1 \leq 2m < nt+n. \end{cases}$$

At the same time, it is easy to see from Table 1 that the graph $B^s(n,m)$ has the same degree sequence as G. Therefore, $\sigma_2(B^s(n,m))$ attains the minimum value among all the bipartite graph with n vertices and m edges. This completes the proof of the Theorem. □

Corollary 1. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, and G a bipartite graph with n vertices and m edges. Then the minimum possible value of $\sigma_2(G)$ is*

$$\begin{cases} (4m-n-nt)t+2m, & \text{if } n \text{ is even,} \\ & \text{or } n \text{ is odd and } nt+t \leq 2m \leq nt+n-t-1; \\ (4m+1-nt)t, & \text{if } n \text{ is odd and } nt \leq 2m < nt+t; \\ (4m-n+1-nt)(t+1), & \text{if } n \text{ is odd and } nt+n-t+1 \leq 2m < nt+n, \end{cases}$$

where $t = \lfloor \frac{2m}{n} \rfloor$.

4 Maximum value of the sum of squares of degrees

Theorem 2. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Then $\sigma_2(B^l(n,m))$ attains the maximum value among all the bipartite graphs with n vertices and m edges.*

Proof. Suppose that G is a bipartite graph such that $\sigma_2(G)$ attains the maximum value among all the bipartite graphs with n vertices and m edges. Let (X, Y) be the bipartition of G. Without loss of generality, we assume $|X| = k \geq \lceil \frac{n}{2} \rceil$. Let $m = qk + r$, where $0 \leq r < k$.

First, let us prove that $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor \leq 1$. Suppose $m = \lfloor \frac{m}{k+1} \rfloor (k+1) + r'$, where $0 \leq r' < k+1$. If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor > 1$, then

$$\begin{aligned} r' &= \lfloor \tfrac{m}{k} \rfloor k + r - \lfloor \tfrac{m}{k+1} \rfloor (k+1) \\ &\geq \lfloor \tfrac{m}{k} \rfloor k + r - (\lfloor \tfrac{m}{k} \rfloor - 2)(k+1) \\ &= r + 2(k+1) - \lfloor \tfrac{m}{k} \rfloor \\ &\geq r + 2(k+1) - \lfloor \tfrac{n}{2} \rfloor \\ &\geq r + 2(k+1) - k \\ &> k + 1, \end{aligned}$$

a contradiction.

By Lemma 2, we can assume that q vertices in Y are all adjacent to all the vertices in X and one more vertex in Y is adjacent to r vertices in X. So we have

$$\begin{aligned} \sigma_2(G) &= r(q+1)^2 + (k-r)q^2 + qk^2 + r^2 \\ &= (m-qk)(q+1)^2 + (k+qk-m)q^2 + qk^2 + (m-qk)^2 \\ &= q(k-1)(k+qk-2m) + m^2 + m \\ &= \lfloor \tfrac{m}{k} \rfloor (k-1)(k + \lfloor \tfrac{m}{k} \rfloor k - 2m) + m^2 + m. \end{aligned}$$

Set $f(k) = \sigma_2(G)$. Then

$$f(k+1) - f(k) = \lfloor \tfrac{m}{k+1} \rfloor k(k+1+\lfloor \tfrac{m}{k+1} \rfloor (k+1) - 2m) - \lfloor \tfrac{m}{k} \rfloor (k-1)(k + \lfloor \tfrac{m}{k} \rfloor k - 2m).$$

If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor = 0$, then

$$f(k+1) - f(k) = 2\lfloor \tfrac{m}{k} \rfloor (k(\lfloor \tfrac{m}{k} \rfloor + 1) - m) \geq 0.$$

If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor = 1$, then

$$f(k+1) - f(k) = 2(\lfloor \tfrac{m}{k} \rfloor - k)(\lfloor \tfrac{m}{k} \rfloor k - m) \geq 0.$$

Thus, $f(k)$ is a nondecreasing function. So we can assume that $k = k_0 = max\{k | m = qk + r, 0 \leq r < k, \lceil n/2 \rceil \leq k \leq n - q - sgn(r)\}$. The proof follows from the construction of $B^l(n, m)$ immediately. □

Corollary 2. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, and G a bipartite graph with n vertices and m edges. Then the maximum possible value of $\sigma_2(G)$ is*

$$\lfloor \frac{m}{k_0} \rfloor (k_0 - 1)(k_0 + \lfloor \frac{m}{k_0} \rfloor k_0 - 2m) + m^2 + m,$$

where $k_0 = \{k | m = qk + r, 0 \leq r < k, \lceil \frac{n}{2} \rceil \leq k \leq n - q - sgn(r)\}$.

Acknowledgements

This work was supported by NSFC (No. 60642002) and SRF for ROCS of SEM. The second and the third author were also supported in part by The Hong Kong Polytechnic University under grant number G-YX42.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London; Elsevier, New York, 1976.

[2] F. Boesch, R. Brigham, S. Burr, R. Dutton, R. Tindell, Maximizing the. sum of the squares of the degrees of a graph, Tech. Rep., Stevens Inst. Tech., Hoboken, NJ, c. 1990.

[3] U. Peled, R. Petreschi, A. Sterbini, (n, e)-graphs with maximum sum of squares of degrees, J. Graph Theory **31** (1999) 283-295.

[4] R. Ahlswede, G.O.H. Katona, Graphs with maximal number of adjacent pairs of edges, Acta Math. Acad. Sci. Hungar. **32** (1978) 97-120.