The Ramsey Numbers $R(C_m, K_7)$ and $R(C_7, K_8)$

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Abstract: For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_m denote a cycle of length m and K_n a complete graph of order n. In this paper we show that $R(C_m, K_7) = 6m - 5$ for $m \geq 7$ and $R(C_7, K_8) = 43$, with the former result confirms a conjecture due to Erdös, Faudree, Rousseau and Schelp that $R(C_m, K_n) = (m-1)(n-1) + 1$ for $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$ in the case where $n = 7$.

Key words: Ramsey number, Cycle, Complete graph

1. Introduction

All graphs considered in this paper are finite simple graphs without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree* of G is denoted by $\delta(G)$. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The *independence number* of a graph G is denoted by $\alpha(G)$. For $U \subseteq V(G)$, we write $\alpha(U)$ for $\alpha(G[U])$, where $G[U]$ is the subgraph induced by U in G. Define $\sigma_2(G) = \min\{d(u) + d(v) \mid u, v \in V(G) \textit{ and } uv \notin E(G)\}$. A *Wheel* of order $n+1$ is $W_n = K_1 + C_n$ and W_n^- is a graph obtained from W_n by deleting a spoke from W_n. A *Book* $B_n = K_2 + \overline{K_n}$ is a graph of order $n + 2$. A cycle and a path of order n are denoted by C_n and P_n, respectively. We use mK_n to denote the union of m vertex disjoint K_n. Let $u, v \in V(G)$ and $s \leq t$ be integers. If G contains a (u, v)-path of order l for each l with $s \leq l \leq t$, then we say u and v are (s, t)-*connected* in G. Let C be a cycle. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. If $u = v$, then $u\overrightarrow{C}v = \{u\}$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and vertex sets. We use u^{+t} and u^{-t} to denote the tth successor and predecessor of u, respectively. For convenience, we write $u^{+1} = u^+$ and $u^{-1} = u^-$. For notations not defined here, we follow [2].

The cycle-complete graph Ramsey number $R(C_m, K_n)$ was first studied by Erdös et al. in [5]. In their paper, they asked the following question.

Question 1 (Erdös et al. [5]). For a given n, what is the smallest value of m such that $R(C_m, K_n) = (m-1)(n-1)+1$?

Furthermore, they posed the following conjecture.

Conjecture 1 (Erdös et al. [5]). $R(C_m, K_n) = (m-1)(n-1)+1$ for $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.

The conjecture was confirmed for $n = 3$ in early works due to Faudree et al. [6] and Rosta [9]. Yang et al. proved the conjecture for $n = 4$.

Theorem 1 (Yang et al. [11]). $R(C_m, K_4) = 3m - 2$ for $m \geq 4$.

Bollobás et al. [1] showed that the conjecture is true for $n = 5$.

Theorem 2 (Bollobás et al. [1]). $R(C_m, K_5) = 4m - 3$ for $m \geq 5$.

Schiermeyer [10] confirmed the conjecture for $n = 6$.

Theorem 3 (Schiermeyer [10]). $R(C_m, K_6) = 5m - 4$ for $m \geq 6$.

In a recent paper, Cheng et al. [3] showed that the conjecture is true in the case when $m = n = 7$, and obtained the following result.

Theorem 4 (Cheng et al. [3]). $R(C_7, K_7) = 37$.

For the case where $m \leq n-1$, only 13 exact values of $R(C_n, K_m)$, including 6 classical Ramsey numbers, are known, see Table 1. All the details in Table 1 can be found in the survey [8].

In this paper our main results are the following:

Theorem 5. $R(C_m, K_7) = 6m - 5$ for $m \geq 7$.

Theorem 6. $R(C_7, K_8) = 43$.

Obviously, Theorem 5 shows that Conjecture 1 is true for $n = 7$. Let $f(n)$ be the smallest value of m such that $R(C_m, K_n) = (m-1)(n-1)+1$ for a given n. By the known results (see [8]), we have $f(3) = 4$, $f(4) = 4$, $f(5) = 5$, $f(6) = 5$ and $f(7) = 5$.

Theorem 6 shows that $f(8) \leq 7$.

	K_4	K_5	K_6	K_7	K_8	K_9
C_3	9	14	18	23	28	36
C_4		14	18	22	26	
C_5			21	25		
C_6				31		

Table 1. Known Ramsey Numbers $R(C_m, K_n)$ for $m \leq n-1$

2. Proof of Theorem 5

In order to prove Theorem 5, we first establish some lemmas.

Let G be a graph, C a cycle of length $m \geq 7$ in G and $u \in V(H) = V(G) - V(C)$. Set $N_C(u) = \{x_1, x_2, \ldots, x_k\}$, where the indices follow the orientation of C; $A = \{a_1, a_2, \ldots, a_k\}$, where $a_i = x_i^+$; $B = \{b_1, b_2, \ldots, b_k\}$, where $b_i = x_{i+1}^-$; and $I_i = a_i \overrightarrow{C} b_i$, the subscripts are taken module k. These notations will also be used in Section 3.

If G contains no C_{m+1}, then we have the following lemmas (1-8).

Lemma 1. Both $\{u\} \cup A$ and $\{u\} \cup B$ are independent sets, and u has no consecutive neighbors in C.

Proof. If $a_i a_j \in E(G)$ with $i \neq j$, then $ux_j \overleftarrow{C} a_i a_j \overrightarrow{C} x_i u$ is a C_{m+1}, a contradiction. If $v, v^+ \in N_C(u)$, then $uv^+ \overrightarrow{C} vu$ is a C_{m+1}, again a contradiction. ∎

Lemma 2. Let P be a (u_1, u_2)-path of order $s \geq 2$ in H, $v_1, v_2 \in V(C)$ and $s - |v_1^+ \overrightarrow{C} v_2^-| = t \geq 1$. If $u_1v_1, u_2v_2 \in E(G)$, then $t \neq 1$. Furthermore, if $t \geq 2$ and $w, w^{+t} \in v_2 \overrightarrow{C} v_1$, then $ww^{+t} \notin E(G)$.

Proof. If $t = 1$, then $u_1 \overrightarrow{P} u_2 v_2 \overrightarrow{C} v_1 u_1$ is a C_{m+1}, a contradiction. If $t \geq 2$ and $w, w^{+t} \in v_2 \overrightarrow{C} v_1$, then $u_1 \overrightarrow{P} u_2 v_2 \overrightarrow{C} w w^{+t} \overrightarrow{C} v_1 u_1$ is a C_{m+1}, again a contradiction. ∎

Lemma 3. Let $v, w \in V(H) - \{u\}$. If $v \in N(a_i^+)$, $w \in N(a_j^+)$ and $i \neq j$, then $vw \notin E(G)$. Similarly, if $v \in N(b_i^-)$, $w \in N(b_j^-)$ and $i \neq j$, then $vw \notin E(G)$.

Proof. Otherwise, $ux_j \overleftarrow{C} a_i^+ vwa_j^+ \overrightarrow{C} x_i u$ is a C_{m+1}, a contradiction. As for the latter part, the proof is similar. ∎

Lemma 4. If $v \in N_H(a_i^+)$ and $u \neq v$, then $\{u, v\} \cup A$ is an independent set. Similarly, if $v \in N_H(b_i^-)$ and $u \neq v$, then $\{u, v\} \cup B$ is an independent set.

Proof. By Lemma 2, $uv \notin E(G)$. By Lemma 1, $a_i v \notin E(G)$. Let $j \neq i$ and $a_j v \in E(G)$. If $|I_i| \geq 2$ or $|I_i| = 1$ and $j \neq i+1$, then $ux_i \overleftarrow{C} a_j v a_i^+ \overrightarrow{C} x_j u$ is a C_{m+1}, a contradiction. If $|I_i| = 1$ and $j = i+1$, then $a_j = a_i^{+2}$, which contradicts Lemma 1. Thus, noting that $\{u\} \cup A$ is an independent set by Lemma 1, we see that $\{u, v\} \cup A$ is an independent set. As for the latter part, the proof is similar. ∎

Lemma 5. Let $|I_i| \geq 2$, $|I_{i-1}| = 1$ and $k \geq 3$. Suppose $y \in V(H)$ and $a_j^+ \in N_C(y)$ for all j with $|I_j| \geq 2$. If uvw is a P_3 in $H - \{y\}$, then $\{w\} \cup A$ is an independent set.

Proof. By Lemma 2, $wa_i \notin E(G)$. If $wa_{i-1} \in E(G)$, then by Lemma 2, we have $|I_{i-2}| \geq 2$. Thus, $uvwa_{i-1} \overleftarrow{C} a_{i-2}^+ y a_i^+ \overrightarrow{C} x_{i-2} u$ is a C_{m+1}, a contradiction. Let $j \neq i, i-1$. Assume $wa_j \in E(G)$. If $|I_j| \geq 2$, then $uvwa_j \overleftarrow{C} a_i^+ y a_j^+ \overrightarrow{C} x_{i-1} u$ is a C_{m+1}, a contradiction. If $|I_j| = 1$, then $|I_{j+1}| \geq 2$ by Lemma 2. Thus, $uvwa_j \overleftarrow{C} a_i^+ y a_{j+1}^+ \overrightarrow{C} x_i u$ is a C_{m+1}, again a contradiction. ∎

Lemma 6. Let $v', v \in V(H)$ and $d_C(v') = l \geq 1$. If v' and v are $(3, m-l+1)$-connected in H, then $N_C(v) = \emptyset$.

Proof. Let $N_C(v') = \{y_1, y_2, \ldots, y_l\}$, where the indices follow the orientation of C. Suppose $w \in N_C(v)$. Choose y_i such that $p = \min\{|w \overrightarrow{C} y_i|, |y_i \overrightarrow{C} w|\}$ is as large as possible. Obviously, $p \leq m/2 + 1$. Since G contains no C_{m+1}, we have $l \leq m/2$. Thus we have $m - l + 1 \geq p$. If $p \geq 4$, then by Lemma 2, H contains no (v', v)-path of order $p - 1$, which contradicts that v' and v are $(3, m-l+1)$-connected in H. Thus we may assume $p \leq 3$. In this case, we must have $p \geq l$ by the choice of y_i. Thus, since v' and v are $(3, m-l+1)$-connected in H, H contains a (v', v)-path of order $m - p + 1$, which implies that G contains a C_{m+1}, a contradiction. ∎

Let $k \geq 1$ and $Z_i = \{v \mid v \in V(H) \textit{ and } d_H(u, v) = i\}$ for $i = 1, 2$. Suppose $\delta(G) \geq m$ and $d_C(h) \leq 2$ for each $h \in V(H)$. We have the following two lemmas(7-8).

Lemma 7. If $G[Z_1]$ contains a hamiltonian path, then there are three vertices $z_1, z_2, z_3 \in Z_2$ such that $N_C(z_i) = \emptyset$ and $\{z_1, z_2, z_3\}$ is an independent set.

Proof. Let $P = y_1 \cdots y_p$ be a hamiltonian path in $G[Z_1]$ and $Y_i = N_{Z_2}(y_i)$ for $1 \leq i \leq p$. Since G contains no C_{m+1}, $\delta(G) \geq m \geq 7$ and $d_C(u) \leq 2$, we have $5 \leq m - 2 \leq p \leq m - 1$. Obviously, u and y_i are $(2, m-k+1)$-connected in H for $i = 1, p$. By Lemma 6, $N_C(y_i) = \emptyset$ for $i = 1, p$, which implies that $Y_i \neq \emptyset$ for $i = 1, p$.

If $p = m - 1$, then y_2 and u are $(2, m-1)$-connected in H. If $d_C(y_2) \geq 2$, then by Lemma 6 we have $k = 0$, which contradicts $k \geq 1$, and hence we have $d_C(y_2) \leq 1$. If $Y_2 = \emptyset$, then since $d_H(y_2) \geq m - 1$, we have $y_2 y_p \in E(G)$, which implies that u and y_3 are $(2, m)$-connected in H. By Lemma 6, $N_C(y_3) = \emptyset$, which implies that $Y_3 \neq \emptyset$.

Take $z_1 \in Y_1$, $z_2 \in Y_2$ if $Y_2 \neq \emptyset$ and $z_2 \in Y_3$ if $Y_2 = \emptyset$, and $z_3 \in Y_p$. If $z_i = z_j$ or $z_iz_j \in E(G)$ for some $i, j \in \{1, 2, 3\}$ and $i \neq j$, then G contains a C_{m+1}, a contradiction. Obviously, u and z_i are $(3, m)$-connected in H for $1 \leq i \leq 3$. By Lemma 6, $N_C(z_i) = \emptyset$ for $1 \leq i \leq 3$. Thus, z_1, z_2 and z_3 are the vertices as required.

If $p = m - 2$, then since $\delta(G) \geq m$ and $d_C(u) \leq 2$, we have $k = 2$ and $|Y_i| \geq 2$ for $i = 1, p$. Since G contains no C_{m+1}, we have $E(Y_i, Y_j) = \emptyset$ for $i \in \{1, p\}$ and $j \neq i$. If $|Y_1 \cap Y_p| = 1$ or $|Y_1 \cap Y_p| = 2$ and $|Y_1 \cup Y_p| \geq 3$ or $|Y_1 \cap Y_p| \geq 3$, then we have $\alpha(Y_1 \cup Y_p) \geq 3$. Let $\{z_1, z_2, z_3\} \subseteq Y_1 \cup Y_p$ be an independent set. Since u and z_i are $(3, m)$-connected in H, by Lemma 6, $N_C(z_i) = \emptyset$ for $1 \leq i \leq 3$. Thus, z_1, z_2 and z_3 are the vertices as required. If $|Y_1 \cap Y_p| = |Y_1 \cup Y_p| = 2$, we assume that $Y_1 = Y_p = \{z_1, z_2\}$. In this case, noting that $y_2y_1z_1y_p\overleftarrow{P}y_4$ and $y_2y_1z_1y_p\overleftarrow{P}y_3$ are (y_2, u)-paths of order $m - 1$ and m, respectively, we see that u and y_2 are $(2, m)$-connected in H. By the symmetry of y_2 and y_{p-1}, u and y_{p-1} are also $(2, m)$-connected in H. Thus, by Lemma 6, we have $N_C(y_i) = \emptyset$ for $i = 2, p-1$, which implies that $|Y_i| \geq 2$ for $i = 2, p-1$. If $Y_2 \cup Y_{p-1} \subseteq \{z_1, z_2\}$, then $uy_1z_1y_2\overrightarrow{P}y_{p-1}z_2y_pu$ is a C_{m+1}, a contradiction. If $Y_2 \cup Y_{p-1} \nsubseteq \{z_1, z_2\}$, say $z_3 \in Y_2 \cup Y_{p-1} - \{z_1, z_2\}$, then z_1, z_2 and z_3 are the vertices as required. Thus we may assume that $Y_1 \cap Y_p = \emptyset$. If $\alpha(Y_1) \geq 2$, say $z_1, z_2 \in Y_1$ and $z_1z_2 \notin E(G)$, then for any $z_3 \in Y_p$, z_1, z_2 and z_3 are the vertices as required. Thus by the symmetry of Y_1 and Y_p, we may assume Y_i is a clique of order at least 2 for $i = 1, p$. Since $p = m - 2$, y_2 and u are $(2, m - 2)$-connected in H. If $d_C(y_2) \geq 3$, then by Lemma 6 we have $k = 0$, which contradicts $k = 2$, and hence $d_C(y_2) \leq 2$. Noting that $\delta(G) \geq m$, we have $d_H(y_2) \geq m - 2$. Thus, if $Y_2 = \emptyset$, then we have $y_2y_p \in E(G)$, which implies that u and y_3 are $(2, m - 1)$-connected in H. By Lemma 6, $N_C(y_3) = \emptyset$, which implies that $Y_3 \neq \emptyset$. Let $z_1 \in Y_1$, $z_2 \in Y_2$ if $Y_2 \neq \emptyset$ and $z_2 \in Y_3$ if $Y_2 = \emptyset$, and $z_3 \in Y_p$, then u and z_i are $(3, m - 1)$-connected in H. By Lemma 6, $N_C(z_i) = \emptyset$ for $1 \leq i \leq 3$. Since Y_i is a clique of order at least 2 for $i = 1, p$ and $E(Y_i, Y_j) = \emptyset$ for $i \in \{1, p\}$ and $j \neq i$, we see that $z_2 \notin \{z_1, z_3\}$, and hence z_1, z_2 and z_3 are the vertices as required. ∎

Lemma 8. If $G[Z_1] = K_p \cup K_q$, then $\alpha(Z_2) \geq 4$.

Proof. Let $Z_1 = Z_{11} \cup Z_{12}$ and $G[Z_{1i}]$ a clique for $i = 1, 2$. Set $Z_{11} = \{y_1, \ldots, y_p\}$, $Z_{12} = \{y_{p+1}, \ldots, y_{p+q}\}$, $Y_i = N_{Z_2}(y_i)$ for $1 \leq i \leq p + q$ and $Z_{2i} = N(Z_{1i}) \cap Z_2$ for $i = 1, 2$. Since $\delta(G) \geq m$ and $d_C(u) \leq 2$, we have $p + q \geq m - 2$.

If $\max\{p, q\} \geq m - 2$, then since G contains no C_{m+1}, we have $p \leq m - 1$. If $p = m - 1$, then u and y_i are $(2, m)$-connected in H for $1 \leq i \leq p$. By Lemma 6, $N_C(y_i) = \emptyset$, which implies that $Y_i \neq \emptyset$ for $1 \leq i \leq p$. If $Y_i \cap Y_j \neq \emptyset$ or $E(Y_i, Y_j) \neq \emptyset$ for some $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, then G contains a C_{m+1}, which implies that $\alpha(\cup_{i=1}^4 Y_i) \geq 4$, and hence $\alpha(Z_2) \geq 4$. If $p = m - 2$, then u and y_i are $(2, m - 1)$-connected in H for $1 \leq i \leq p$. By Lemma 6, $d_C(y_i) \leq 1$, which implies that $Y_i \neq \emptyset$ for $1 \leq i \leq p$. If $Y_1 \cap Y_2 \neq \emptyset$, then u and y_i are $(2, m)$-connected in H for $1 \leq i \leq p$. By Lemma 6, $N_C(y_i) = \emptyset$, which implies that $|Y_i| \geq 2$ for $1 \leq i \leq p$. Let $z_1 \in Y_1 \cap Y_2$ and

$z_i \in Y_i - \{z_1\}$ for $3 \le i \le 5$. If $z_i = z_j$ for some $i, j \in \{3, 4, 5\}$ with $i \ne j$ or $z_i z_j \in E(G)$ for some $i, j \in \{1, 3, 4, 5\}$ with $i \ne j$, then G contains a C_{m+1}. Thus, $\{z_1, z_3, z_4, z_5\}$ is an independent set of size 4, and hence $\alpha(Z_2) \ge 4$. By symmetry, we may assume that $Y_i \cap Y_j = \emptyset$ for all $1 \le i < j \le p$. Since G contains no C_{m+1}, we have $E(Y_i, Y_j) = \emptyset$ for $i \ne j$, which implies that $\alpha(\cup_{i=1}^{4} Y_i) \ge 4$, and hence $\alpha(Z_2) \ge 4$. Thus we may assume that $\max\{p, q\} \le m - 3$.

If $Z_{21} \cap Z_{22} \ne \emptyset$, we assume that $z_4 \in N_{Z_2}(y_p) \cap N_{Z_2}(y_{p+q})$. In this case, we have $p + q = m - 2$ for otherwise G contains a C_{m+1}. Assume without loss of generality that $p \ge q$. It is easy to see that u and y_i are $(2, m)$-connected in H for $1 \le i \le p - 1$. By Lemma 6, $N_C(y_i) = \emptyset$ for $1 \le i \le p - 1$. Thus, noting that $p \le m - 3$ and $\delta(G) \ge m$, we have $|Y_i| \ge 3$ for $1 \le i \le p - 1$. Since $p + q = m - 2$, $m \ge 7$ and $p \ge q$, we have $p \ge 3$. Let $z_i \in Y_i - \{z_4\}$ for $i = 1, 2$. If $p \ge 4$, we let $z_3 \in Y_3 - \{z_4\}$. If $p = 3$, then $2 \le q \le 3$, which implies that $|Y_{p+1}| \ge 2$. In this case, we let $z_3 \in Y_{p+1} - \{z_4\}$. If $z_i = z_j$ for some $i, j \in \{1, 2, 3\}$ with $i \ne j$ or $z_i z_j \in E(G)$ for some $i, j \in \{1, 2, 3, 4\}$ with $i \ne j$, then G contains a C_{m+1}. Thus, $\{z_1, z_2, z_3, z_4\}$ is an independent set of size 4, which implies that $\alpha(Z_2) \ge 4$. Hence, we may assume that $Z_{21} \cap Z_{22} = \emptyset$.

If $E(Z_{21}, Z_{22}) \ne \emptyset$, then G contains a C_{m+1}, and hence we have $E(Z_{21}, Z_{22}) = \emptyset$.

Assume that $\min\{p, q\} \ge 2$. If $\alpha(Z_2) \le 3$, then since $Z_{21} \cap Z_{22} = \emptyset$ and $E(Z_{21}, Z_{22}) = \emptyset$, we may assume $\alpha(Z_{21}) = 1$. Since $p \le m - 3$, $\delta(G) \ge m$ and $d_C(y_i) \le 2$, we have $Y_i \ne \emptyset$ for $1 \le i \le p$. Let $|Z_{21}| = r$. Since $\delta(G) \ge m$ and $d_C(y_i) \le 2$ for $1 \le i \le p$, we have $p + r \ge m - 2$. If $p = 2$ and $Y_1 \cap Y_2 = \emptyset$, then since $d_C(y_i) \le 2$ and $\delta(G) \ge m \ge 7$, we have $|Y_1 \cup Y_2| \ge 2(m - 4) \ge m - 1$. Noting that both $G[Z_{11}]$ and $G[Z_{21}]$ are cliques, we see that G contains a C_{m+1}. If $p = 2$ and $Y_1 \cap Y_2 \ne \emptyset$ or $p \ge 3$, then since both $G[Z_{11}]$ and $G[Z_{21}]$ are cliques and $Y_i \ne \emptyset$ for $1 \le i \le p$, we see u and y_i are $(2, p+r+1)$-connected in H for $1 \le i \le p$. If $p + r = m - 2$, then by Lemma 6, we have $d_C(y_1) \le 1$, which implies that $p + r \ge m - 1$, a contradiction. If $p + r = m - 1$, then by Lemma 6, we have $N_C(y_i) = \emptyset$, which implies that $p + r \ge m$, again a contradiction. Thus we have $p + r \ge m$. In this case, we see that G contains a C_{m+1}, and hence $\min\{p, q\} = 1$.

Since $\min\{p, q\} = 1$, $\max\{p, q\} \le m - 3$ and $p + q \ge m - 2$, we may assume that $p = m-3$ and $q = 1$. Obviously, u and y_i are $(2, m-2)$-connected in H for $1 \le i \le p$. By Lemma 6, we have $d_C(y_i) \le 2$, which implies that $Y_i \ne \emptyset$ for $1 \le i \le p$. Since G contains no C_{m+1}, we have $d_C(y_{p+1}) \le m/2$, which implies that $Y_{p+1} \ne \emptyset$ since $\delta(G) \ge m \ge 7$. Thus, noting that $Z_{21} \cap Z_{22} = \emptyset$ and $E(Z_{21}, Z_{22}) = \emptyset$, we have $\alpha(Z_2) \ge \alpha(Z_{21}) + 1$, and hence we need only to show $\alpha(Z_{21}) \ge 3$ in the following proof. If $Y_i \cap Y_j = \emptyset$ for $1 \le i < j \le p$, we let $z_i \in Y_i$ for $1 \le i \le 4$. If $E(G[\{z_1, z_2, z_3, z_4\}]) \ge 2$, then G contains a C_{m+1}, a contradiction. If $E(G[\{z_1, z_2, z_3, z_4\}]) \le 1$, then $\alpha(\{z_1, z_2, z_3, z_4\}) \ge 3$, which implies that $\alpha(Z_{21}) \ge 3$. Thus we may assume that $z_1 \in Y_1 \cap Y_2$. In this case, u and y_i are $(2, m - 1)$-connected in H for $1 \le i \le p$. By Lemma 6, $d_C(y_i) \le 1$, which implies that $|Y_i| \ge 2$ for $1 \le i \le p$. Let $z_i \in Y_i - \{z_1\}$ for $i = 2, 3, 4$. If $Y_i \cap Y_j = \emptyset$ for $\{i, j\} \ne \{1, 2\}$, then $\{z_2, z_3, z_4\}$ is an independent set of size 3, which implies that $\alpha(Z_{21}) \ge 3$. If there is some $Y_i \cap Y_j \ne \emptyset$ for $\{i, j\} \ne \{1, 2\}$, we may assume that

$Y_2 \cap Y_3 \neq \emptyset$ or $Y_3 \cap Y_4 \neq \emptyset$. In both cases, u and y_i are $(2, m)$-connected in H for $1 \leq i \leq p$. By Lemma 6, $N_C(y_i) = \emptyset$, which implies that $|Y_i| \geq 3$ for $1 \leq i \leq p$. If $Y_2 \cap Y_3 \neq \emptyset$, then $Y_3 \cap Y_4 = \emptyset$ for otherwise G contains a C_{m+1} and vice versa. If $Y_2 \cap Y_3 \neq \emptyset$, we let $z_2 \in Y_2 \cap Y_3$ and $z_i \in Y_i - \{z_1, z_2\}$ for $i = 3, 4$. If $Y_3 \cap Y_4 \neq \emptyset$, we let $z_4 \in Y_3 \cap Y_4$ and $z_i \in Y_i - \{z_1, z_4\}$ for $i = 2, 3$. Thus, $\{z_2, z_3, z_4\}$ is an independent set of size 3, which implies that $\alpha(Z_{21}) \geq 3$. ∎

Lemma 9 (Chvátal and Erdös [4]). If $\alpha(G) \leq \kappa(G) + 1$, then G has a hamiltonian path.

Lemma 10 (Cheng et al. [3]). Let G be a graph of order $6m - 5$ ($m \geq 6$) with $\alpha(G) \leq 6$. If G contain no C_m, then G contains no W_{m-2}.

Proof of Theorem 5. We use induction on m. If $m = 7$, then Theorem 5 holds by Theorem 4. Assume that Theorem 5 holds for some given $m \geq 7$, we now show that Theorem 5 holds for $m + 1$.

Let G be a graph of order $6(m + 1) - 5 = 6m + 1$. Suppose to the contrary that neither G contains a C_{m+1} nor $\overline{G}$ contains a K_7. If there is some vertex $v \in V(G)$ such that $d(v) \leq m - 1$, then $G' = G - N[v]$ has an order of at least $5m + 1$. Obviously, G' contains no C_{m+1}. Thus by Theorem 3, G' contains an independent set of size at least 6. Clearly, any independent set of size 6 in G' and v form an independent set of size 7 in G, a contradiction. Hence we have

$$\delta(G) \geq m. \tag{1}$$

For any $v \in V(G)$, since G contains no C_{m+1}, by (1) we see that

$G[N(v)]$ contains no hamiltonian path. (2)

By the induction hypothesis, G contains a cycle of length m. Let C be a cycle of length m, $H = G - C$ and $d_C(u_0) = k = \max\{d_C(h) \mid h \in H\}$. Define $N_C(u_0), A, B$ and I_i as in Section 2. Among all the cycles of length m in G, we choose C such that k is as large as possible and subject to this, $\min\{|I_i| \mid 1 \leq i \leq k\}$ is as small as possible. Let $U_i = \{u \mid u \in V(H) \textit{ and } d_H(u_0, u) = i\}$ for $i = 1, 2$.

If $\max\{|I_i| \mid 1 \leq i \leq k\} = 1$, then since $m \geq 7$, we have $k \geq 4$. We now show that this case cannot occur. Let $v \in U_1$. If $N(v) \cap A \neq \emptyset$, say $va_1 \in E(G)$, then by Lemma 2, $x_2x_3, x_3x_4 \notin E(G)$. Thus we have $d_H(x_3) \geq 3$ and $d_H(x_4) \geq 2$ by (1). Assume that $y \in N_H(x_4) - \{u_0\}$ and $z \in N_H(x_3) - \{u_0, y\}$. By Lemmas 2 and 3, $\{u_0, y, z\} \cup A$ is an independent set of size at least 7, a contradiction. By Lemma 2, we have $N_C(v) \cap N_C(u_0) = \emptyset$. Thus we have

$N_C(v) = \emptyset$ for any $v \in U_1$. (3)

For any $v \in U_1$ and $w \in N_H(v)$, by Lemmas 1 and 2, we see that

$$\{w\} \cup A \text{ is an independent set.} \tag{4}$$

Now, let $v \in U_1$ be given. By (1) and (3), we have $d(v) = d_H(v) \geq m$. If $\alpha(G[N_H(v)]) \geq 3$, then by (4) we have $\alpha(G) \geq 7$, a contradiction. Hence we have $\alpha(G[N_H(v)]) \leq 2$. By Lemma 9 and (2), we may assume that $G[N_H(v)] = K_p \cup K_q$, where $p+q = d(v) \geq m$. If $p \geq m-1$ or $q \geq m-1$, then G contains a K_m, which contradicts Lemma 10. Thus we have $p \leq m-2$ and $q \leq m-2$, which implies that $q \geq 2$ and $p \geq 2$. Let $u_0, u_1 \in K_p$. If $N_H(u_1) \subseteq U_1 \cup \{u_0\}$, then $G[N_H(u_1)]$ is connected. By (1) and (3), $d(u_1) = d_H(u_1) \geq m$. By Lemma 9 and (2), we may assume that $\alpha(G[N_H(u_1)]) \geq 3$. In this case, we have $\alpha(G) \geq 7$ by (4), a contradiction. Thus, there is some $u_2 \in N_H(u_1)$ such that $u_0u_2 \notin E(G)$. If $N_H(u_2) \cap K_q = \emptyset$, then for any $u_3 \in K_q$, $\{u_0, u_2, u_3\} \cup A$ is an independent set of size at least 7, a contradiction. If $N_H(u_2) \cap K_q \neq \emptyset$, then it is easy to see that G contains a C_{m+1}, again a contradiction. Thus we have

$$\max\{|I_i| \mid 1 \leq i \leq k\} \geq 2. \tag{5}$$

Since $|H| = 5m+1$, by Theorem 3, H contains an independent set I of size 6. Obviously, I is also a maximum independent set of G. Since $|I| = 6$ and $m \geq 7$, by the choice of u_0, we have $k \geq 2$. By Lemma 1, $k \leq 5$. Thus we have $2 \leq k \leq 5$.

If $k = 5$, then by (5), there is some i such that $|I_i| \geq 2$. Since $\delta(G) \geq m$, we have $N_H(a_i^+) \neq \emptyset$. Assume that $v \in N_H(a_i^+)$. By Lemma 4, $\{u_0, v\} \cup A$ is an independent set of size 7, a contradiction. Thus we have $2 \leq k \leq 4$.

Claim 1. Let $|I_i| \geq 2$ and $v \in N_H(a_i^+)$. If $k = 4$, then $a_j^+ \in N(v)$ for all I_j with $|I_j| \geq 2$.

Proof. If there exists some $j \neq i$ such that $|I_j| \geq 2$ and $a_j^+ v \notin E(G)$, then by (1), there exists some $w \in N_H(a_j^+)$. By Lemma 3, $wv \notin E(G)$. Thus, $\{u_0, v, w\} \cup A$ is an independent set of size 7 by Lemma 4, a contradiction. ∎

We now distinguish the following two cases separately.

Case 1. $\min\{|I_i| \mid 1 \leq i \leq k\} = 1$.

Since $\min\{|I_i| \mid 1 \leq i \leq k\} = 1$, by (5), there exists some i such that $|I_i| = 1$ and $|I_{i+1}| \geq 2$. Assume without loss of generality that $|I_1| = 1$, $|I_2| \geq 2$ and $v_0 \in N_H(a_2^+)$. Set $V_i = \{v \mid v \in V(H) \textit{ and } d_H(v_0, v) = i\}$ for $i = 1, 2$.

Claim 2. $U_1 \cap V_1 = \emptyset$.

Proof. Assume that $U_1 \cap V_1 \neq \emptyset$ and $w_0 \in U_1 \cap V_1$. Let $W = N_H(w_0) - \{u_0, v_0\}$. By Lemma 2, we have $W \cap (U_1 \cup V_1) = \emptyset$ and $x_1x_2, a_2^+a_2^{+3} \notin E(G)$. Thus we have $d_H(x_2) \geq 2$ and $d_H(a_2^+) \geq 2$. Let $v_0' \in N_H(a_2^+) - \{v_0\}$. By Lemma 2, $v_0v_0' \notin E(G)$.

If $\alpha(W) \geq 2$, then for any $w_1, w_2 \in W$ with $w_1 w_2 \notin E(G)$, by Lemmas 1 and 2, $\{u_0, v_0, v_0', w_1, w_2, a_1, a_2\}$ is an independent set of size 7, and hence W is a clique. If $k = 4$, then by Lemma 4, $\{u_0, v_0, v_0'\} \cup A$ is an independent set of size 7, and hence we may assume that $k \leq 3$. Assume that $k = 3$. If $|I_3| = 1$, then by Lemmas 2 and 4, we see that for any $w \in W$, $\{u_0, v_0, v_0', w\} \cup A$ is an independent set of size 7, a contradiction. If $|I_3| \geq 2$, we let $y \in N_H(a_3^+)$. If $y \notin \{v_0, v_0'\}$, then by Lemmas 2, 3 and 4, $\{u_0, v_0, v_0', y\} \cup A$ is an independent set of size 7. If $y \in \{v_0, v_0'\}$, then by Lemmas 2, 4 and 5, $\{u_0, v_0, v_0', w\} \cup A$ is an independent set of size 7 for any $w \in W$, a contradiction. Thus, we may assume that $k = 2$. By (1) and the choice of u_0, we have $|W| \geq m - 4$. Since $d_H(x_2) \geq 2$, we may let $z \in N_H(x_2) - \{u_0\}$. If $z \in W$ or $N_W(z) \neq \emptyset$, then since W is a clique of order at least $m-4$, it is easy to see that G contains a C_{m+1}, and hence $z \notin W$ and $N_W(z) = \emptyset$. If $z \notin N_H(a_2^+)$, then by Lemmas 1 and 2, $\{u_0, v_0, v_0', z, w\} \cup A$ is an independent set of size 7 for any $w \in W$, and hence we have $z \in N_H(a_2^+)$. In this case, $C' = u_0 x_2 z a_2^+ \overrightarrow{C} x_1 u_0$ is a C_m. By the choice of C and u_0, we have $a_1 a_2^+ \notin E(G)$, which implies that $d_H(a_2^+) \geq 3$. Let $v_0'' \in N_H(a_2^+) - \{v_0, v_0'\}$. If $v_0' v_0'' \notin E(G)$, then by Lemmas 1 and 2, $\{u_0, v_0, v_0', v_0'', w\} \cup A$ is an independent set of size 7 for any $w \in W$, and hence $v_0' v_0'' \in E(G)$. By Lemma 2, $z \notin \{v_0', v_0''\}$. If $z \neq v_0$, then $\{u_0, v_0, v_0', z, w\} \cup A$ is an independent set of size 7 for any $w \in W$, and hence $z = v_0$. Thus, by Lemma 2, we have $x_2 a_2^+ \notin E(G)$, which implies that $d_H(x_2) \geq 3$. Let $z' \in N_H(x_2) - \{u_0, z\}$. Since $v_0' v_0'' \in E(G)$, by Lemma 2, $z' \neq v_0'$. Thus, for any $w \in W$, $\{u_0, v_0, v_0', z', w\} \cup A$ is an independent set of size 7, a contradiction. ∎

Claim 3. Let $u_0 u_1 u_2$ be a P_3 in $H - \{v_0\}$. If $k = 3$, $|I_3| \geq 2$ and $N_H(a_2^+) \cap N_H(a_3^+) = \emptyset$, then $\{u_2\} \cup A$ is an independent set.

Proof. By Lemma 2, we have $u_2 a_2 \notin E(G)$. If $u_2 a_1 \in E(G)$, then by Lemma 2, $a_1 a_2^+, x_2 a_2^{+2}, a_3^+ b_2 \notin E(G)$. If $u_2 a_3 \in E(G)$, then by Lemma 2, $a_1 a_2^+, x_2 b_3, a_3^+ a_3^{+4} \notin E(G)$. Thus we have $d_H(a_2^+) \geq 2$, $d_H(x_2) \geq 2$ and $d_H(a_3^+) \geq 2$ in both cases. Let $v_0' \in N_H(a_2^+) - \{v_0\}$, $y \in N_H(x_2) - \{u_0\}$ and $w_0, w_0' \in N_H(a_3^+)$. If $v_0 v_0' \notin E(G)$, then by Lemmas 3 and 4, $\{u_0, w_0, v_0, v_0'\} \cup A$ is an independent set of size 7, and hence $v_0 v_0' \in E(G)$. Similarly, $w_0 w_0' \in E(G)$. Thus, we have $y \notin \{v_0, v_0'\}$ and $y v_0 \notin E(G)$ by Lemma 2, and $y \notin \{w_0, w_0'\}$ and $y w_0 \notin E(G)$ by Lemma 3. By Lemmas 1 and 4, we see that $\{u_0, v_0, w_0, y\} \cup A$ is an independent set of size 7, a contradiction. Thus we have $u_2 a_1, u_2 a_3 \notin E(G)$, and hence $\{u_2\} \cup A$ is an independent set. ∎

Claim 4. If $k \geq 3$ and $U_2 \neq \emptyset$, then U_2 is a clique.

Proof. If $k = 4$, then by Lemma 5 and Claim 1, we have $E(U_2, A) = \emptyset$. If $k = 3$, then by Lemma 5 and Claim 3, we have $E(U_2, A) = \emptyset$. By Lemma 2, $N_{U_2}(v_0) = \emptyset$. By Claim 2, $v_0 \notin U_2$. Thus, if $\alpha(U_2) \geq 2$, then by Lemma 4, we have $\alpha(U_2 \cup A \cup \{u_0, v_0\}) \geq 7$, a contradiction. ∎

Claim 5. Let $P = y_1 \cdots y_p$ be a longest path in $G[U_1]$. If $k \geq 3$, then $p \leq m - k - 1$.

Proof. If $p \geq m-k$, then u_0 and y_i are $(2, m-k+1)$-connected in H for $i = 1, p$. By Lemma 6, $N_C(y_i) = \emptyset$ for $i = 1, p$. Since G contains no C_{m+1}, we have $p \leq m-1$. By (1) and the maximality of P, we have $d_{U_2}(y_i) \geq m-p \geq 1$ for $i = 1, p$. By Claim 4, U_2 is a clique. Let $P' = y_1u_0y_2\overrightarrow{P}y_p$, then $|P'| = p+1$. If $|(N(y_1) \cup N(y_p)) \cap U_2| = 1$, then $p = m-1$. Let $z \in (N(y_1) \cup N(y_p)) \cap U_2$, then $y_1\overrightarrow{P'}y_pzy_1$ is a C_{m+1}, a contradiction. If $|(N(y_1) \cup N(y_p)) \cap U_2| \geq 2$, then there are two vertices $z_1, z_p \in U_2$ such that $y_iz_i \in E(G)$ for $i = 1, p$. Since $|U_2| \geq m-p$ and U_2 is a clique, $G[U_2]$ contains a (z_1, z_p)-path P'' of order $m-p$. Thus, the paths P',P'', together with the edges y_1z_1, y_pz_p form a C_{m+1}, again a contradiction. ∎

Claim 6. If $k \geq 3$, then for any $u \in U_1$, $N_{U_2}(u) \neq \emptyset$.

Proof. Let $U_0 = U_1 \cup \{u_0\}$. If $N_{U_2}(u) = \emptyset$, then $N_H[u] \subseteq U_0$. Let $N(u) \cap U_1 = U_1'$. By Lemma 2, $N(v_0) \cap U_1' = \emptyset$. By Lemma 5 and Claims 1, 3, we have $E(U_1', A) = \emptyset$. Thus, if $\alpha(U_1') \geq 3$, then by Lemma 4, we have $\alpha(U_1' \cup A \cup \{v_0\}) \geq 7$, and hence $\alpha(U_1') \leq 2$. By Lemma 9, $G[U_1' \cup \{u\}]$ contains a hamiltonian path, which implies that $G[U_1]$ contains a path of order at least $m-k$. By Claim 5, this is a contradiction. ∎

By Claim 2, $U_1 \cap V_1 = \emptyset$. By Lemma 2, $N(v_0) \cap U_2 = \emptyset$. If $k = 4$, then $U_2 \neq \emptyset$ by Claim 6. By Lemma 5 and Claim 1, $E(U_2, A) = \emptyset$. Thus, by Lemma 4, we have $\alpha(U_2 \cup A \cup \{u_0, v_0\}) \geq 7$, a contradiction. If $k = 3$, then $U_2 \neq \emptyset$ By Claim 6. By Lemma 5 and Claim 3, $E(U_2, A) = \emptyset$. By Claim 4, U_2 is a clique. If $G[U_1]$ contains an isolated vertex, say $u' \in U_1$ and $d_{U_1}(u') = 0$, then by (1) and the choice of u_0, we have $d_{U_2}(u') \geq m-4$, which implies that U_2 is a clique of order at least $m-4$. By Claim 6, $N(u) \cap U_2 \neq \emptyset$ for any $u \in U_1$. Thus, noting that $d_{U_2}(u') \geq m-4 \geq 3$, it is easy to see that H contains a (u_0, u)-path of order $m-1$ for any $u \in U_1$. This implies that $E(U_1, A) = \emptyset$ for otherwise G contains a C_{m+1}. If $\alpha(U_1) \geq 3$, then by Lemmas 2 and 4, we have $\alpha(U_1 \cup A \cup \{v_0\}) \geq 7$, a contradiction. Hence, $\alpha(U_1) \leq 2$. If $G[U_1]$ contains no isolated vertices, then by Lemmas 2, 5 and Claim 3, $\{u, v_0\} \cup A$ is an independent set for any $u \in U_1$, which implies that $\alpha(U_1) \leq 2$. Thus we have $\alpha(U_1) \leq 2$ in both cases. If $G[U_1]$ is connected, then by Lemma 9, $G[U_1]$ contains a hamiltonian path, which contradicts Claim 5. Thus, we may assume that $G[U_1] = K_p \cup K_q$, where $p+q \geq m-3$. If $p+q+|U_2| \geq m$, then by Claims 4 and 6, it is easy to see that G contains a C_{m+1}. Hence we have $p+q+|U_2| \leq m-1$, which implies that $p+q \leq m-2$ and $|U_2| \leq 2$. If $|U_2| = 1$, then for any $u \in K_p$, by (1) we have $m \leq d(u) \leq d_C(u) + p + |U_2| \leq 3 + \lfloor (m-2)/2 \rfloor + 1$, which implies that $m \leq 6$, a contradiction. Therefore, we have $|U_2| = 2$ and $p+q = m-3$. For any $u \in K_p$, by (1) we have $m \leq d(u) \leq 3 + 2 + \lfloor (m-3)/2 \rfloor$, which implies that $m = 7$, $p = 2$ and $U_2 \subseteq N(u)$. In this case, u and u_0 are $(2, 5)$-connected in H. By Lemma 6, $N_C(u) = \emptyset$, which implies that $d(u) \leq 4$, a contradiction. Therefore, we may assume that $k = 2$. If $\alpha(G[U_1]) \geq 3$ and $\alpha(G[V_1]) \geq 3$, then by Lemma 2, we see that $\alpha(U_1 \cup V_1 \cup \{x_2\}) \geq 7$, a contradiction. Thus we have $\alpha(G[U_1]) \leq 2$ or $\alpha(G[V_1]) \leq 2$. If $\alpha(G[U_1]) \leq 2$, then by Lemma 9, either $G[U_1]$ has a hamiltonian path or $G[U_1]$ is the disjoint union of two complete graphs. Thus, we have $\alpha(U_2 \cup A \cup \{u_0, v_0\}) \geq 7$ by Lemmas 4 and 7 in the former case and $\alpha(U_2 \cup \{u_0, v_0, a_2\}) \geq 7$ by Lemmas 2, 4 and 8 in the latter case. If $\alpha(G[V_1]) \leq 2$, then by Lemma 9, either $G[V_1]$ has a hamiltonian path or $G[V_1]$ is the disjoint union of two complete graphs. Thus, we have $\alpha(V_2 \cup A \cup \{u_0, v_0\}) \geq 7$ by Lemmas 4 and 7 in the former case and $\alpha(V_2 \cup \{u_0, v_0, a_1\}) \geq 7$ by Lemmas 2, 4 and 8 in the latter case.

Case 2. $\min\{|I_i| \mid 1 \leq i \leq k\} \geq 2$.

In this case, we still let $v_0 \in N_H(a_2^+)$ and $V_i = \{v \mid v \in V(H) \text{ and } d_H(v_0, v) = 2\}$ for $i = 1, 2$.

If $k = 4$, then by Claim 1, we have $a_i^+ \in N_C(v_0)$ for $1 \leq i \leq 4$. Obviously, $C' = u_0 x_2 \overleftarrow{C} a_1^+ v_0 a_2^+ \overrightarrow{C} x_1 u_0$ is a C_m. By the choice of C and u_0, we have $\{x_2, x_3, x_4\} \not\subseteq N(a_1)$, which implies that there is some x_i with $2 \leq i \leq 4$ such that $d_H(x_i) \geq 2$. Let $w_0 \in N_H(x_i) - \{u_0\}$. By the choice of u_0, we have $w_0 \neq v_0$. Thus, by Lemmas 3 and 4, we see that $\{u_0, v_0, w_0\} \cup A$ is an independent set of size 7, a contradiction.

Let $k = 3$. If $N_H(a_i^+) \cap N_H(a_j^+) = \emptyset$ for $1 \leq i < j \leq 3$, then by Lemmas 3 and 4, we have $\alpha(G) \geq 7$, a contradiction. Hence we may assume without loss of generality that $v_0 \in N_H(a_2^+) \cap N_H(a_3^+)$. Obviously, $C' = u_0 x_3 \overleftarrow{C} a_2^+ v_0 a_3^+ \overrightarrow{C} x_2 u_0$ is a C_m. By the choice of C and u_0, we have $x_2, a_2^+ \notin N(a_3)$ and $x_3, a_3^+ \notin N(a_2)$. Thus we have $d_H(a_i^+) \geq 2$ and $d_H(x_i) \geq 2$ for $i = 2, 3$. Let $v_0' \in N_H(a_3^+) - \{v_0\}$, then $v_0 v_0' \notin E(G)$ by Lemma 3. Let $y \in N_H(a_2^+) - \{v_0\}$. If $y \neq v_0'$, then by Lemmas 3 and 4, $\{u_0, y, v_0, v_0'\} \cup A$ is an independent set of size 7, and hence $y = v_0'$. Let $z \in N_H(a_1^+)$. If $z \notin \{v_0, v_0'\}$, then by Lemmas 3 and 4, $\{u_0, z, v_0, v_0'\} \cup A$ is an independent set of size 7, and hence we may assume that $z = v_0$. In this case, $C'' = u_0 x_2 \overleftarrow{C} a_1^+ v_0 a_2^+ \overrightarrow{C} x_1 u_0$ is a C_m. By the choice of C and u_0, we have $a_1 a_2^+ \notin E(G)$, which implies that $d_H(a_2^+) \geq 3$. Let $v_0'' \in N_H(a_2^+) - \{v_0, v_0'\}$, then by Lemmas 3 and 4, $\{u_0, v_0, v_0', v_0''\} \cup A$ is an independent set of size 7, a contradiction.

Let $k = 2$. If $N_H(a_1^+) \cap N_H(a_2^+) \neq \emptyset$, we assume that $v_0 \in N_H(a_1^+) \cap N_H(a_2^+)$ and $|I_1| \leq |I_2|$. In this case, $C' = u_0 x_2 \overleftarrow{C} a_1^+ v_0 a_2^+ \overrightarrow{C} x_1 u_0$ is a C_m. By the choice of C and u_0, we have

$$N_C(a_i) = N_{C'}(a_i) = \{x_i, a_i^+\} \text{ for } i = 1, 2. \tag{6}$$

Since $m \geq 7$, we have $|I_2| \geq 3$, which implies that $a_2^{+2} \neq x_1$. If $b_1 a_2^{+2} \in E(G)$, then $u_0 x_2 \overrightarrow{C} a_2^+ v_0 a_1^+ \overrightarrow{C} b_1 a_2^{+2} \overrightarrow{C} x_1 u_0$ is a C_{m+1}, a contradiction. Hence we have $b_1 a_2^{+2} \notin E(G)$. By (1) and (6), we have $d_H(a_2^+) \geq 2$. Assume $v_0' \in N_H(a_2^+) - \{v_0\}$. By Lemma 3, $v_0 v_0' \notin E(G)$. By Lemma 1, $v_0, v_0' \notin N(a_2^{+2})$. If $|I_1| \geq 3$, then by the choice of C and u_0, we have $v_0, v_0' \notin N(b_1)$. Thus, by Lemmas 1, 4 and (6), $\{u_0, v_0, v_0', b_1, a_2^{+2}\} \cup A$ is an independent set of size 7, a contradiction. Hence we may assume that $|I_1| = 2$. Let $N_H(x_2) - \{u_0\} = Z$. By (1) and (6), $Z \neq \emptyset$. By the choice of u_0, we have $N_Z(a_1) = \emptyset$. By Lemma 1, $N_Z(a_2) = \emptyset$. By Lemmas 2 and 3, $N_Z(u_0) = N_Z(v_0) = N_Z(v_0') = \emptyset$. If $\alpha(Z) \geq 2$ or there is some vertex $z \in Z$ such that $za_2^{+2} \notin E(G)$, then we have $\alpha(A \cup Z \cup \{u_0, v_0, v_0'\}) \geq 7$ or $\alpha(A \cup \{u_0, v_0, v_0', z, a_2^{+2}\}) \geq 7$, a contradiction. Hence we may assume that Z is a clique and $Z \subseteq N(a_2^{+2})$. Thus, by Lemma 2, we have $|Z| \leq 2$. Let $z_1 \in Z$. If $x_2 a_2^+ \in E(G)$, then $v_0 a_2^+ x_2 z_1 a_2^{+2} \overrightarrow{C} b_1 v_0$ is a C_{m+1}, and hence $x_2 a_2^+ \notin E(G)$. By Lemma 1, $x_2 a_2^{+2} \notin E(G)$. By (6), $a_1 x_2 \notin E(G)$. Thus we have $a_1, a_2^+, a_2^{+2} \notin N_C(x_2)$. By (1), we have $|Z| \geq 3$, which contradicts $|Z| \leq 2$. Hence, we may assume that $N_H(a_1^+) \cap N_H(a_2^+) = \emptyset$.

Let $N_H(a_i^+) = Z_i$ for $i = 1, 2$. If $\alpha(Z_1) \geq 2$ and $\alpha(Z_2) \geq 2$, then by Lemmas 3 and 4, we have $\alpha(Z_1 \cup Z_2 \cup A \cup \{u_0\}) \geq 7$, a contradiction. Thus, either Z_1 or Z_2 is a clique. Assume without loss of generality that Z_2 is a clique. We now show that H contains no (u_0, v_0)-paths of order l with $l = 3$ or 4. If not, we have $|Z_2| \leq m - 5$ for otherwise G contains a C_{m+1}. Let $|Z_2| = t$. Since $l \leq 4$, we have $l + t - 1 \leq m - 2$, which implies that $a_2^{+(l+t-1)} \in a_2^+ \overrightarrow{C} x_2$. Since Z_2 is a clique of order t, by Lemma 2, we have $v \notin N(a_2^+)$ for each $v \in a_2^{+l} \overrightarrow{C} a_2^{+(l+t-1)}$, and hence $d(a_2^+) \leq m - 1$, which contradicts (1). This implies that $U_1 \cap V_1 = \emptyset$ and $E(U_1, V_1) = \emptyset$. If $\alpha(U_1) \geq 3$ and $\alpha(V_1) \geq 3$, then by Lemmas 2 and 3, we have $\alpha(U_1 \cup V_1 \cup \{a_1^+\}) \geq 7$, and hence either $\alpha(U_1) \leq 2$ or $\alpha(V_1) \leq 2$. If $G[U_1]$ or $G[V_1]$ has a hamiltonian path, then by Lemmas 4 and 7, we have $\alpha(U_2 \cup A \cup \{u_0, v_0\}) \geq 7$ or $\alpha(V_2 \cup A \cup \{u_0, v_0\}) \geq 7$, a contradiction. Thus, by Lemma 9, either $G[U_1]$ or $G[V_1]$ is the disjoint union of two complete graphs. Suppose $G[U_1]$ is the disjoint union of two complete graphs. If $|I_2| \geq 3$, then $v_0 a_2^{+2} \notin E(G)$ by Lemma 1 and $N_{U_2}(a_2^{+2}) = \emptyset$ by Lemma 2. Thus, by Lemma 8, we have $\alpha(U_2 \cup \{u_0, v_0, a_2^{+2}\}) \geq 7$. If $|I_2| = 2$, then we have $|I_1| \geq 3$ since $m \geq 7$. If $a_1^{+2} \neq b_1^{-2}$ or $a_1^{+2} = b_1^{-2}$ and $a_1^{+2} v_0 \notin E(G)$, then since $d_C(v_0) \leq 2$, by Lemmas 2 and 8, we see that either $\alpha(U_2 \cup \{u_0, v_0, a_1^{+2}\}) \geq 7$ or $\alpha(U_2 \cup \{u_0, v_0, b_1^{-2}\}) \geq 7$. If $a_1^{+2} = b_1^{-2}$ and $a_1^{+2} v_0 \in E(G)$, then $|I_1| = 5$ and hence $m = 9$. If $N(b_1) \cap U_2 \neq \emptyset$, we let $z \in N_{U_2}(b_1)$ and $u_0 y z$ is a P_3 in $H - \{v_0\}$. Thus, $u_0 y z b_1 b_1^- b_1^{-2} v_0 a_2^+ a_2 x_2 u_0$ is a C_{m+1}, and hence $N(b_1) \cap U_2 = \emptyset$. Thus, by Lemma 8, we have $\alpha(U_2 \cup \{u_0, v_0, b_1\}) \geq 7$, a contradiction. Now, assume that $G[V_1]$ is the disjoint union of two complete graphs. If $v_0 b_1 \notin E(G)$, then by Lemmas 3 and 8, we have $\alpha(V_2 \cup \{u_0, v_0, b_1\}) \geq 7$. Hence we may assume that $b_1 v_0 \in E(G)$. Since $a_1^+ v_0 \notin E(G)$, we have $|I_1| \geq 3$. By the choice of u_0, we have $|I_2| = 2$. In this case, $a_1^+ = a_2^{+4}$. By Lemma 2, $N(a_2^{+4}) \cap V_2 = \emptyset$. Since $|I_1| \geq 3$, we have $a_2^{+4} = a_1^+ \neq b_1$, which implies that $a_2^{+4} v_0 \notin E(G)$ since $d_C(v_0) \leq 2$. Thus, by Lemmas 2 and 8, we have $\alpha(V_2 \cup \{u_0, v_0, a_2^{+4}\}) \geq 7$, a contradiction.

Up to now, we have shown that $R(C_m, K_7) \leq 6m - 5$. On the other hand, since $6K_{m-1}$ contains no C_m and its complement contains no K_7, we have $R(C_m, K_7) \geq 6m - 5$, and hence $R(C_m, K_7) = 6m - 5$. ∎

3. Proof of Theorem 6

To prove Theorem 6, we need the following lemmas in addition to Theorem 5.

Lemma 11 (Ore [7]). Let G be a graph of order n. If $\sigma_2(G) \geq n$, then G is hamiltonian.

The following lemma can be deduced from the known Ramsey numbers, see [8].

Lemma 12. $R(B_2, K_7) \leq 34$.

Lemma 13. Let G be a graph of order $7m - 6$ $(m \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_m, then $\delta(G) \geq m - 1$.

Proof. If there is some vertex v such that $d(v) \leq m - 2$, then $G' = G - N[v]$ is a graph of order at least $6m - 5$. Since $R(C_m, K_7) = 6m - 5$ for $m \geq 7$ by Theorem 5 and G' contains no C_m, we have $\alpha(G') \geq 7$. Thus, an independent set of order at least 7 in G' together with v form an independent set of order at least 8 in G, which contradicts $\alpha(G) \leq 7$. ∎

Lemma 14. Let G be a graph of order $7m - 6$ $(m \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_m, then G contains no W_{m-2}.

Proof. Suppose to the contrary that G contains a $W_{m-2} = \{w_0\} + C$, where $C = w_1 \cdots w_{m-2}$ is a cycle of length $m - 2$. Set $U = V(G) - V(W_{m-2})$. By Lemma 13, $\delta(G) \geq m - 1$. Thus we have $N_U(w_i) \neq \emptyset$ for $0 \leq i \leq m - 2$. Let $v_i \in N_U(w_i)$ and $V_i = N_U[v_i]$, where $0 \leq i \leq m - 2$. Since G contains no C_m, we have

$$N(V_i) \cap W_{m-2} = \{w_i\} \text{ for } 0 \leq i \leq m - 2, \tag{7}$$

$$V_i \cap V_j = \emptyset \text{ for } 0 \leq i < j \leq m - 2, \tag{8}$$

and

$$E(V_0, V_i) = \emptyset \text{ for } 1 \leq i \leq m - 2. \tag{9}$$

By (7), we have $d_{W_{m-2}}(v_i) = 1$, which implies that $|V_i| \geq m - 1$ for $0 \leq i \leq m - 2$ since $\delta(G) \geq m - 1$. By (8), we have $m(m-1) \leq |W_{m-2} \cup (\cup_{i=0}^{m-2} V_i)| \leq 7m - 6$, which implies that $m \leq 7$, and hence $m = 7$. In this case, $|G| = 43$. Thus by (8) we have $6 \leq |V_i| \leq 7$ for $0 \leq i \leq 5$. If there is some V_i such that $|V_i| = 7$, then $V(G) = V(W_5) \cup (\cup_{i=0}^{5} V_i)$. By (7) and (9), we have $N(V_0) \subseteq V_0 \cup \{w_0\}$. If $|V_0| = 7$, then since $\delta(G) \geq 6$, we have $\delta(G[V_0]) \geq 5$. By Lemma 11, $G[V_0]$ contains a C_7, a contradiction. If $|V_0| = 6$, then $G[V_0 \cup \{w_0\}] = K_7$ since $\delta(G) \geq 6$, also a contradiction. If $|V_i| = 6$ for $0 \leq i \leq 5$, then $V(G) - (V(W_5) \cup (\cup_{i=0}^{5} V_i))$ contains exactly one vertex, say y. By (7) and (9), we have $N(V_0) \subseteq V_0 \cup \{w_0, y\}$. Noting that $\delta(G) \geq 6$, we have $d_{V_0}(w_0) \geq 3$ or $d_{V_0}(y) \geq 3$, which implies that that either $G' = G[V_0 \cup \{w_0\}]$ or $G'' = G[V_0 \cup \{y\}]$ is a graph of order 7 with minimum degree at least 3 and has at most one vertex of degree 3. By Lemma

11, either G' or G'' contains a C_7, again a contradiction. ∎

Proof of Theorem 6. Let G be a graph of order 43. Suppose to the contrary that neither G contains a C_7 nor $\overline{G}$ contains a K_8. By Lemma 13, we have $\delta(G) \geq 6$.

Before starting to prove Theorem 6, we first establish the following claims.

Claim 7. G contains no $K_1 + P_5$.

Proof. Suppose that G contains $K_1 + P_5$, say, $P = v_1 \cdots v_5$ and $V(P) \subseteq N(v_0)$. Let $U = V(G) - \{v_i \mid 0 \leq i \leq 5\}$ and $N_U(v_i) = U_i$ for $0 \leq i \leq 5$. Because of $\delta(G) \geq 6$, we have $U_i \neq \emptyset$ for $0 \leq i \leq 5$.

If $U_2 \cap U_4 \neq \emptyset$, then we let $v_6 \in U_2 \cap U_4$, $X = \{v_i \mid 0 \leq i \leq 6\}$ and $Y = V(G) - X$. Set $Y_i = N_Y(v_i)$, $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $0 \leq i \leq 6$. If $v_3v_6 \in E(G)$, then G contains a C_7. By Lemma 14, $v_1v_5 \notin E(G)$. Thus, noting that $|X| = 7$ and $\delta(G) \geq 6$, we have $Y_i \neq \emptyset$ for $i = 1, 3, 5, 6$. Since G contains no C_7, it is easy to check that $Y_i \cap Y_j = \emptyset$ for $i = 1, 3, 5, 6$ and $j \neq i$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 3, 5, 6\}$ and $i \neq j$. Thus we have $|Z_i| \geq 5$ for $i = 1, 3, 5, 6$. For the same reason, we have $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $i, j \in \{1, 3, 5, 6\}$ and $i \neq j$. By Lemma 14, we have $\alpha(Z_i) \geq 2$ for $i = 1, 3, 5, 6$. Thus we have $\alpha(Z_1 \cup Z_3 \cup Z_5 \cup Z_6) \geq 8$, a contradiction. Hence we have $U_2 \cap U_4 = \emptyset$.

If $U_0 \cap U_4 \neq \emptyset$, we let $v_6 \in U_0 \cap U_4$, $X = \{v_i \mid 0 \leq i \leq 6\}$ and $Y = V(G) - X$. Set $Y_i = N_Y(v_i)$, $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $0 \leq i \leq 6$. Since $U_2 \cap U_4 = \emptyset$, we have $Y_2 \cap Y_4 = \emptyset$. If $Y_2 \cap Y_0 = \emptyset$, then since G contains no C_7, it is easy to see that $Y_i \cap Y_j = \emptyset$ for $i = 1, 2, 5, 6$ and $j \neq i$, which implies that $|Z_i| \geq 5$ for $i = 1, 2, 5, 6$. By Lemma 14, $\alpha(Z_i) \geq 2$ for $i = 1, 2, 5, 6$. Since G contains no C_7, we have $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $i, j \in \{1, 2, 5, 6\}$ and $i \neq j$. Thus, we have $\alpha(Z_1 \cup Z_2 \cup Z_5 \cup Z_6) \geq 8$, a contradiction. Hence, we may assume $Y_2 \cap Y_0 \neq \emptyset$, say $v_7 \in Y_2 \cap Y_0$. Let $X' = X \cup \{v_7\}$ and $Y' = V(G) - X'$. Set $Y'_i = N_{Y'}(v_i)$, $z'_i \in Y'_i$ and $N_{Y'}(z'_i) = Z'_i$ for $0 \leq i \leq 7$. Since G contains no C_7, $\{v_1, v_5, v_6, v_7\}$ is an independent set. Thus we have $Y'_i \neq \emptyset$ for $i = 1, 5, 6, 7$. In this case, it is easy to see that $Y'_i \cap Y'_j = \emptyset$ for $i = 1, 5, 6, 7$ and $j \neq i$ since otherwise G contains a C_7. This implies that $|Z'_i| \geq 5$ for $i = 1, 5, 6, 7$. By Lemma 14, $\alpha(Z'_i) \geq 2$. Since G contains no C_7, we have $Z'_i \cap Z'_j = \emptyset$ and $E(Z'_i, Z'_j) = \emptyset$ for $i, j \in \{1, 5, 6, 7\}$ and $i \neq j$, which implies that $\alpha(Z'_1 \cup Z'_5 \cup Z'_6 \cup Z'_7) \geq 8$, again a contradiction. Thus we have $U_0 \cap U_4 = \emptyset$. By the symmetry of U_2 and U_4, we have $U_0 \cap U_2 = \emptyset$. Therefore, $U_0 \cap (U_2 \cup U_4) = \emptyset$.

Since G contains no C_7, we have $U_i \cap U_j = \emptyset$ for $i = 1, 5$ and $j \neq i$, and $U_3 \cap (U_2 \cup U_4) = \emptyset$. Thus, noting that $U_2 \cap U_4 = U_0 \cap (U_2 \cup U_4) = \emptyset$, we have $U_i \cap U_j = \emptyset$ for $i \in \{1, 2, 4, 5\}$ and $j \neq i$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 1, 2, 4, 5$, then we have $|V_i| \geq 5$. By Lemma 14, $\alpha(V_i) \geq 2$. Since G contains no C_7, we see that V_1, V_2, V_4 and V_5 are pairwise disjoint and there are no edges between any two of them. Thus we have $\alpha(V_1 \cup V_2 \cup V_4 \cup V_5) \geq 8$, a contradiction. ∎

Claim 8. G contains no W_5^-.

Proof. Suppose that G contains a W_5^-, say, $C = v_1 \cdots v_5$ and $W_5^- = \{v_0\} + C - \{v_0v_1\}$. Let $U = V(G) - \{v_i \mid 0 \le i \le 5\}$ and $U_i = N_U(v_i)$ for $0 \le i \le 5$. Since $\delta(G) \ge 6$, we have $U_i \ne \emptyset$. Noting that G contains no C_7, we have $U_i \cap U_j = \emptyset$ for $i \in \{0, 1, 3, 4\}$ and $j \ne i$, and $E(U_i, U_j) = \emptyset$ for $i, j \in \{0, 1, 3, 4\}$ and $i \ne j$. Take $u_i \in U_i$ and set $V_i = N_U(u_i)$ for $i = 0, 1, 3, 4$, then we have $|V_i| \ge 5$. By Lemma 14, $\alpha(V_i) \ge 2$ for $i = 0, 1, 3, 4$. Since G contains no C_7, we have $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $i, j \in \{0, 1, 3, 4\}$ and $i \ne j$, which implies that $\alpha(V_0 \cup V_1 \cup V_3 \cup V_4) \ge 8$, a contradiction. ∎

Claim 9. G contains no W_4.

Proof. Suppose that G contains a W_4, say $C = v_1 \cdots v_4$ is a cycle and $V(C) \subseteq N(v_0)$. Let $U = V(G) - \{v_i \mid 0 \le i \le 4\}$ and set $U_i = N_U(v_i)$ for $0 \le i \le 4$. By Claim 7, $U_0 \cap U_i = \emptyset$ for $1 \le i \le 4$. By Claim 8, $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_4 = U_4 \cap U_1 = \emptyset$. If $U_1 \cap U_3 \ne \emptyset$, then $U_2 \cap U_4 = \emptyset$ for otherwise $av_1v_0v_2bv_4v_3$ is a C_7, where $a \in U_1 \cap U_3$ and $b \in U_2 \cap U_4$. By symmetry, we may assume that $U_1 \cap U_3 = \emptyset$. Let $u_i \in U_i$ for $0 \le i \le 4$. Since $\delta(G) \ge 6$, we have $|U_i| \ge 2$. Thus we can choose u_2 such that $u_2 \ne u_4$. Set $V_i = N_U(u_i)$ for $i = 0, 1, 3$. By the arguments above, we have $|V_i| \ge 5$ for $i = 0, 1, 3$. By Lemma 14, $\alpha(V_i) \ge 2$ for $i = 0, 1, 3$. Since G contains no C_7, we have $u_2u_4 \notin E(G)$ and $u_2, u_4 \notin V_0 \cup V_1 \cup V_3$. For the same reason, we have $E(\{u_2, u_4\}, V_0 \cup V_1 \cup V_3) = \emptyset$, $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $i, j \in \{0, 1, 3\}$ and $i \ne j$, which implies that $\alpha(\{u_2, u_4\} \cup V_0 \cup V_1 \cup V_3) \ge 8$, a contradiction. ∎

Claim 10. G contains no K_4.

Proof. Suppose that G contains a K_4, say $S = \{v_1, v_2, v_3, v_4\}$ is a clique. Set $U = V(G) - S$ and $U_i = N_U(v_i)$ for $1 \le i \le 4$. Since $\delta(G) \ge 6$, we have $|U_i| \ge 3$.

If there are U_i and U_j with $i \ne j$ such that $U_i \cap U_j \ne \emptyset$, we assume without loss of generality that $v_5 \in U_3 \cap U_4$. Let $X = S \cup \{v_5\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \le i \le 5$. By Claim 7, we have $(Y_3 \cup Y_4) \cap (Y_1 \cup Y_2 \cup Y_5) = \emptyset$. By Claim 8, $Y_5 \cap (Y_1 \cup Y_2) = \emptyset$. Since G contains no C_7, we have $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 3, 5\}$ and $j \ne i$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 1, 2, 3, 5$. Since $\delta(G) \ge 6$, we may choose u_1 such that $u_1 \ne u_2$. By the arguments above, we have $|Z_i| \ge 4$ for $i = 1, 2, 3, 5$. By Claim 9, $\alpha(Z_i) \ge 2$. Because G contains no C_7, we see that $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $i, j \in \{1, 2, 3, 5\}$ and $i \ne j$, which implies that $\alpha(Z_1 \cup Z_2 \cup Z_3 \cup Z_5) \ge 8$, a contradiction. Hence we have $U_i \cap U_j = \emptyset$ for $1 \le i < j \le 4$.

Take $u_i \in U_i$ for $1 \le i \le 4$. Set $T = \{u_1, u_2, u_3, u_4\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $1 \le i \le 4$. If $\Delta(G[T]) \ge 2$, then G contains a C_7, and hence we may assume that $\Delta(G[T]) \le 1$. Thus, noting that $U_i \cap U_j = \emptyset$ for $1 \le i < j \le 4$, we have $|V_i| \ge 4$ for $1 \le i \le 4$. By Claim 9, $\alpha(V_i) \ge 2$. Since G contains no C_7, it is easy to see that $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $1 \le i < j \le 4$, which implies that $\alpha(\cup_{i=1}^4 V_i) \ge 8$, a contradiction. ∎

Claim 11. G contains no K_1+P_4.

Proof. Suppose that G contains K_1+P_4, say $P=v_1v_2v_3v_4$ is a path and $V(P)\subseteq N(v_0)$. Set $S=\{v_i \mid 0\le i\le 4\}$, $U=V(G)-S$ and $U_i=N_U(v_i)$ for $0\le i\le 4$.

If $U_3\cap U_4\neq\emptyset$, we let $v_5\in U_3\cap U_4$. Set $X=S\cup\{v_5\}$, $Y=V(G)-X$ and $Y_i=N_Y(v_i)$ for $0\le i\le 5$. Since G contains no C_7, we have $Y_1\cap Y_i=\emptyset$ for $i\neq 1$, $Y_2\cap Y_i=\emptyset$ for $i\neq 0,2$ and $Y_4\cap Y_i=\emptyset$ for $i\neq 3,4$. For the same reason, we have $E(Y_i,Y_j)=\emptyset$ for $i,j\in\{1,2,4\}$ and $i\neq j$. Let $z_i\in Y_i$ and $Z_i=N_Y(z_i)$ for $i=1,2,4$. By the arguments above, we have $|Z_1|\ge 5$ and $|Z_i|\ge 4$ for $i=2,4$. Note that G contains no C_7, we see that $E(\{v_5\},Z_1\cup Z_2\cup Z_4)=\emptyset$, Z_1, Z_2 and Z_4 are pairwise disjoint and there is no edges between any two of them. By Claims 7, 9 and 10, we have $\alpha(Z_1)\ge 3$ and $\alpha(Z_i)\ge 2$ for $i=2,4$, which implies that $\alpha(\{v_5\}\cup Z_1\cup Z_2\cup Z_4)\ge 8$, a contradiction. Hence we have $U_3\cap U_4=\emptyset$. By symmetry, $U_1\cap U_2=\emptyset$. Thus we have $U_1\cap U_2=U_3\cap U_4=\emptyset$.

If $U_2\cap U_4\neq\emptyset$, we let $v_5\in U_2\cap U_4$. Set $X=S\cup\{v_5\}$, $Y=V(G)-X$ and $Y_i=N_Y(v_i)$ for $0\le i\le 5$. Since G contains no C_7, we have $Y_i\cap Y_j=\emptyset$ for $i=1,5$ and $j\neq i$. Let $z_i\in Y_i$ and $Z_i=N_Y(z_i)$ for $i=1,5$, then by the arguments above, we have $|Z_i|\ge 5$ for $i=1,5$. By Claims 7, 9 and 10, we have $\alpha(Z_i)\ge 3$ for $i=1,5$. By Claim 10, $v_2v_4\notin E(G)$. If $Z_1\cap Z_5\neq\emptyset$ or $E(Z_1,Z_5)\neq\emptyset$ or $E(\{v_2,v_4\},Z_1\cup Z_5)\neq\emptyset$, then G contains a C_7, a contradiction. Thus we have $\alpha(\{v_2,v_4\}\cup Z_1\cup Z_5)\ge 8$, a contradiction. Hence we have $U_2\cap U_4=\emptyset$. By symmetry, $U_1\cap U_3=\emptyset$. Thus we have $U_1\cap U_3=U_2\cap U_4=\emptyset$.

By the arguments above, we have $(U_1\cup U_4)\cap(U_2\cup U_3)=\emptyset$. By Claim 7, $U_0\cap(U_1\cup U_4)=\emptyset$. By Claim 8, $U_1\cap U_4=\emptyset$. Thus, we have $U_i\cap U_j=\emptyset$ for $i=1,4$ and $j\neq i$. Let $u_i\in U_i$ for $1\le i\le 4$. Since $\delta(G)\ge 6$, we may choose u_2,u_3 such that $u_2\neq u_3$. Set $V_i=N_U(u_i)$ for $i=1,4$, then we have $|V_i|\ge 5$ for $i=1,4$. By Claims 7, 9 and 10, we have $\alpha(V_i)\ge 3$ for $i=1,4$. If $u_2u_3\in E(G)$ or $\{u_2,u_3\}\cap(V_1\cup V_4)\neq\emptyset$ or $E(\{u_2,u_3\},V_1\cup V_4)\neq\emptyset$, then G contains a C_7, a contradiction. For the same reason, we have $V_1\cap V_4=\emptyset$ and $E(V_1,V_4)=\emptyset$, which implies that that $\alpha(\{u_2,u_3\}\cup V_1\cup V_4)\ge 8$, a contradiction. ∎

Claim 12. G contains no B_3.

Proof. Assume that G contains a B_3, say, $v_1v_2\in E(G)$ and $v_3,v_4,v_5\in N(v_1)\cap N(v_2)$. Set $U=V(G)-\{v_i \mid 1\le i\le 5\}$ and $U_i=N_U(v_i)$ for $1\le i\le 5$.

If $U_3\cap U_4\neq\emptyset$, we assume $v_6\in U_3\cap U_4$. Set $X=\{v_i \mid 1\le i\le 6\}$, $Y=V(G)-X$ and $Y_i=N_Y(v_i)$ for $1\le i\le 6$. Since G contains no C_7, we see that $Y_5\cap Y_i=\emptyset$ for $i\neq 5$ and $Y_i\cap Y_j=\emptyset$ for $i=3,4$ and $j\neq 3,4$. Thus we can take $z_i\in Y_i$ for $3\le i\le 5$ such that $z_3\neq z_4$. Note that G contains no C_7, $z_iz_j\notin E(G)$ for $3\le i<j\le 5$. Set $Z_i=N_Y(z_i)$ for $3\le i\le 5$. By the arguments above, we have $|Z_5|\ge 5$ and $|Z_i|\ge 4$ for $i=3,4$. By Claims 7, 9 and 10, we have $\alpha(Z_5)\ge 3$ and $\alpha(Z_i)\ge 2$ for

$i = 3, 4$. If $E(\{v_6\}, \cup_{i=3}^5 Z_i) \neq \emptyset$, then G contains a C_7, a contradiction. For the same reason, we have $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $3 \leq i < j \leq 5$. Thus we get that $\alpha(\{v_6\} \cup (\cup_{i=3}^5 Z_i)) \geq 8$, a contradiction. Hence we have $U_3 \cap U_4 = \emptyset$. By symmetry, we have $U_i \cap U_j = \emptyset$ for $3 \leq i < j \leq 5$.

By Claim 11, we have $(U_1 \cup U_2) \cap (U_3 \cup U_4 \cup U_5) = \emptyset$, which implies that $U_i \cap U_j = \emptyset$ for $i = 3, 4, 5$ and $j \neq i$. Let $u_i \in U_i$ and $N_U(u_i) = V_i$ for $i = 3, 4, 5$. Since $\delta(G) \geq 6$, by the arguments above, we have $|V_i| \geq 5$ for $i = 3, 4, 5$. By Claims 7, 9 and 10, we have $\alpha(V_i) \geq 3$. Thus, noting that G contains no C_7, we have $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $3 \leq i < j \leq 5$, which implies that $\alpha(\cup_{i=3}^5 V_i) \geq 9$, a contradiction. ∎

Claim 13. G contains no W_4^-.

Proof. Suppose G contains a W_4^-, say, $W_4^- = \{v_5\} + C - \{v_1v_5\}$, where $C = v_1v_2v_3v_4$ is a cycle. Set $S = \{v_i \mid 1 \leq i \leq 5\}$, $U = V(G) - S$ and $U_i = N_U(v_i)$ for $1 \leq i \leq 5$.

If $U_1 \cap U_5 \neq \emptyset$, we let $v_6 \in U_1 \cap U_5$. Set $X = S \cup \{v_6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leq i \leq 6$. Since G contains no C_7, we have $E(Y_4, Y_6) = \emptyset$ and $Y_i \cap Y_j = \emptyset$ for $i = 4, 6$ and $j \neq i$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 4, 6$. By the arguments above, we have $|Z_i| \geq 5$. By Claims 7, 9 and 10, we have $\alpha(Z_i) \geq 3$ for $i = 4, 6$. Because G contains no C_7, we have $Z_4 \cap Z_6 = \emptyset$, $E(Z_4, Z_6) = \emptyset$ and $E(\{v_1, v_5\}, Z_4 \cup Z_6) = \emptyset$, which implies that $\alpha(\{v_1, v_5\} \cup Z_4 \cup Z_6) \geq 8$, a contradiction. Thus we have $U_1 \cap U_5 = \emptyset$. By the symmetry of U_3 and U_5, we have $U_1 \cap U_3 = \emptyset$, and hence $U_1 \cap (U_3 \cup U_5) = \emptyset$.

If $U_1 \cap U_4 \neq \emptyset$, we let $v_6 \in U_1 \cap U_4$. Set $X = S \cup \{v_6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leq i \leq 6$. Since G contains no C_7, we have $E(Y_3, Y_6) = \emptyset$ and $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$. By Claim 11, $Y_3 \cap (Y_2 \cup Y_4) = \emptyset$. By Claim 12, $Y_3 \cap Y_5 = \emptyset$. If $Y_3 \cap Y_1 \neq \emptyset$, then G contains a C_7, a contradiction. Thus we have $Y_3 \cap Y_i = \emptyset$ for $i \neq 3$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 3, 6$, then by the arguments above, we have $|Z_i| \geq 5$. By Claims 7, 9 and 10, we have $\alpha(Z_i) \geq 3$ for $i = 3, 6$. By Claim 10, $v_2v_4 \notin E(G)$. Thus, noting that G contains no C_7, we have $Z_3 \cap Z_6 = \emptyset$, $E(Z_3, Z_6) = \emptyset$ and $E(\{v_2, v_4\}, Z_3 \cup Z_6) = \emptyset$, which implies that $\alpha(\{v_2, v_4\} \cup Z_3 \cup Z_6) \geq 8$, a contradiction. Thus we have $U_1 \cap U_4 = \emptyset$. By the symmetry of U_2 and U_4, we have $U_1 \cap U_2 = \emptyset$, and hence $U_1 \cap (U_2 \cup U_4) = \emptyset$.

By the arguments above, $U_1 \cap U_i = \emptyset$ for $i \neq 1$. By Claim 11, $U_3 \cap (U_2 \cup U_4) = \emptyset$. By Claim 12, $U_3 \cap U_5 = \emptyset$. Thus we have $U_3 \cap U_i = \emptyset$ for $i \neq 3$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 1, 3$, then by the arguments above, we have $|V_i| \geq 5$. By Claims 7, 9 and 10, we have $\alpha(V_i) \geq 3$ for $i = 1, 3$. By Claim 10, $v_2v_4 \notin E(G)$. Thus, noting that G contains no C_7, we have $V_1 \cap V_3 = \emptyset$, $E(V_1, V_3) = \emptyset$ and $E(\{v_2, v_4\}, V_1 \cup V_3) = \emptyset$, which implies that $\alpha(\{v_2, v_4\} \cup V_1 \cup V_3) \geq 8$, a contradiction. ∎

Claim 14. G contains no B_2.

Proof. Suppose to the contrary that G contains a B_2, say, $v_1v_2v_3v_4$ is a cycle with diagonal v_2v_4. Set $U = V(G) - \{v_1, v_2, v_3, v_4\}$ and $U_i = N_U(v_i)$ for $1 \leq i \leq 4$.

If $E(U_1, U_3) \neq \emptyset$, we assume $v_5 \in U_1$, $v_6 \in U_3$ and $v_5v_6 \in E(G)$. Let $X = \{v_i \mid 1 \leq$

$i \leq 6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leq i \leq 6$. Since G contains no C_7, it is easy to get that $Y_i \cap Y_j = \emptyset$ for $i = 2, 4$ and $j \neq i$, and $Y_5 \cap (Y_1 \cup Y_6) = \emptyset$. Let $z_i \in Y_i$ for $i = 2, 4, 5$, $Z_2 = N_Y(z_2) - \{z_4\}$, $Z_4 = N_Y(z_4) - \{z_2\}$ and $Z_5 = N_Y(z_i)$. Then by the arguments above, we have $|Z_i| \geq 4$ for $i = 2, 4, 5$. By Claims 7, 9 and 10, we have $\alpha(Z_i) \geq 2$ for $i = 2, 4, 5$. Noting that G contains no C_7, we see that $E(\{v_1, v_3\}, Z_2 \cup Z_4 \cup Z_5) = \emptyset$, $(Z_2 \cup Z_4) \cap Z_5 = \emptyset$ and there is no edge between any two of the three Z_2, Z_4 and Z_5. By Claim 10, $v_1v_3 \notin E(G)$. If $Z_2 \cap Z_4 = \emptyset$, then by the arguments above, we have $\alpha(\{v_1, v_3\} \cup Z_2 \cup Z_4 \cup Z_5) \geq 8$, and hence we may assume that $Z_2 \cap Z_4 \neq \emptyset$. Since $E(Z_2, Z_4) = \emptyset$, we see that $Z_2 \cap Z_4$ is an independent set. If $|Z_2 \cap Z_4| \geq 4$, then $\alpha(\{v_1, v_3\} \cup (Z_2 \cap Z_4) \cup Z_5) \geq 8$, and hence we may assume that $|Z_2 \cap Z_4| \leq 3$. In this case, we have $Z_2' = Z_2 - (Z_2 \cap Z_4) \neq \emptyset$ and $Z_4' = Z_4 - (Z_2 \cap Z_4) \neq \emptyset$. If $|Z_2 \cap Z_4| \geq 2$, then noting that $E(Z_2, Z_4) = \emptyset$, we have $\alpha(Z_2 \cup Z_4) \geq 4$, which implies that $\alpha(\{v_1, v_3\} \cup (Z_2 \cup Z_4) \cup Z_5) \geq 8$, a contradiction. If $|Z_2 \cap Z_4| = 1$, then we have $|Z_i'| \geq 3$ for $i = 2, 4$. By Claim 10, $\alpha(Z_i') \geq 2$ for $i = 2, 4$. Obviously, $Z_2' \cap Z_4' = \emptyset$. Thus we have $\alpha(\{v_1, v_3\} \cup Z_2' \cup Z_4' \cup Z_5) \geq 8$, again a contradiction. Hence we have $E(U_1, U_3) = \emptyset$.

If $E(U_1 \cup U_3, U_2 \cup U_4) \neq \emptyset$, we assume without loss of generality that $v_5 \in U_3$, $v_6 \in U_4$ and $v_5v_6 \in E(G)$. Let $X = \{v_i \mid 1 \leq i \leq 6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leq i \leq 6$. Since G contains no C_7, we have $Y_1 \cap Y_i = \emptyset$ for $i \neq 1$, $Y_3 \cap (Y_2 \cup Y_5) = \emptyset$ and $Y_6 \cap (Y_2 \cup Y_4 \cup Y_5) = \emptyset$. By Claim 11, $Y_3 \cap Y_4 = \emptyset$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 1, 3, 6$. Since $v_3v_6 \notin E(G)$ by Claim 11 and $\delta(G) \geq 6$, we have $|Y_i| \geq 2$ for $i = 3, 6$. Thus we may choose z_3 such that $z_3 \neq z_6$. By the arguments above, we have $|Z_1| \geq 5$ and $|Z_i| \geq 4$ for $i = 3, 6$. By Claims 7, 9 and 10, we have $\alpha(Z_1) \geq 3$ and $\alpha(Z_i) \geq 2$ for $i = 3, 6$. Noting that G contains no C_7, we see that $E(\{v_5\}, Z_1 \cup Z_3 \cup Z_6) = \emptyset$, Z_1, Z_3, Z_6 are pairwise disjoint and there is no edge between any two of them. This implies that $\alpha(\{v_5\} \cup Z_1 \cup Z_3 \cup Z_6) \geq 8$, a contradiction. Hence we have $E(U_1 \cup U_3, U_2 \cup U_4) = \emptyset$.

By Claims 11, 12 and 13, we have $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 4$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $1 \leq i \leq 3$. Obviously, $|V_i| \geq 5$ for $1 \leq i \leq 3$. Since G contains no C_7, $E(U_1, U_3) = \emptyset$ and $E(U_1 \cup U_3, U_2 \cup U_4) = \emptyset$, we have $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $1 \leq i < j \leq 3$. By Claims 7, 9 and 10, we have $\alpha(V_i) \geq 3$ for $1 \leq i \leq 3$, which implies that $\alpha(\cup_{i=1}^{3} V_i) \geq 9$, again a contradiction. ∎

We now begin to prove Theorem 6.

If there is some vertex v such that $d(v) \leq 8$, then $G' = G - N[v]$ is a graph of order at least 34. By Lemma 12 and Claim 14, we have $\alpha(G') \geq 7$, which implies that $\alpha(G) \geq 8$, a contradiction. Hence we have $\delta(G) \geq 9$.

Let $v_0 \in V(G)$. Since $d(v_0) \geq 9$, $G[N(v_0)]$ contains at least two edges for otherwise we have $\alpha(N(v_0)) \geq 8$. By Claim 14, $G[N(v_0)]$ contains no P_3. Thus, $G[N(v_0)]$ contains two independent edges, say $v_1v_2, v_3v_4 \in E(G[N(v_0)])$. Set $U = V(G) - \{v_i \mid 0 \leq i \leq 4\}$ and $N_U(v_i) = U_i$ for $1 \leq i \leq 4$. By Claim 11, we have $E(\{v_1, v_2\}, \{v_3, v_4\}) = \emptyset$, which implies that $|U_i| \geq 7$ for $1 \leq i \leq 4$. By Claim 14, we have $\alpha(U_i) \geq 4$ for $1 \leq i \leq 4$. Since G contains no C_7, we have $E(U_1 \cup U_2, U_3 \cup U_4) = \emptyset$. Thus, if $U_1 \cap U_3 = \emptyset$ or $U_2 \cap U_4 = \emptyset$, then we have $\alpha(U_1 \cup U_3) \geq 8$ or $\alpha(U_2 \cup U_4) \geq 8$, and hence we may assume $a \in U_1 \cap U_3$ and $b \in U_2 \cap U_4$. By Claim 14, we have $a \neq b$, which implies that $av_1v_0v_2bv_4v_3$ is a C_7 in G, a contradiction.

By the arguments above, we have $R(C_7, K_8) \leq 43$. On the other hand, since $7K_6$ contains no C_7 and its complement contains no K_8, we have $R(C_7, K_8) \geq 43$ and hence $R(C_7, K_8) = 43$. ∎

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