

1 An application of Turán theorem to domination in graphs\*

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3 **Abstract**

4 A function  $f : V(G) \rightarrow \{+1, -1\}$  defined on the vertices of a graph  $G$  is a *signed domi-*  
 5 *nating function* if for any vertex  $v$  the sum of function values over its closed neighborhood  
 6 is at least one. The *signed domination number*  $\gamma_s(G)$  of  $G$  is the minimum weight of a  
 7 signed dominating function on  $G$ . By simply changing “ $\{+1, -1\}$ ” in the above definition  
 8 to “ $\{+1, 0, -1\}$ ”, we can define the *minus dominating function* and the *minus domination*  
 9 *number* of  $G$ . In this note by applying Turán theorem we present sharp lower bounds on  
 10 the signed domination number for a graph containing no  $(k + 1)$ -cliques. As a result, we  
 11 generalize a previous result due to Kang et al. on the minus domination number of  $k$ -partite  
 12 graphs to graphs containing no  $(k + 1)$ -cliques and characterize the extremal graphs.

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# 1 Introduction

Let  $G = (V, E)$  be a graph with *vertex set*  $V$  and *edge set*  $E$ . The *open neighborhood* of a vertex  $v$ ,  $N_G(v)$ , is defined as the set of vertices adjacent to  $v$ , i.e.,  $N_G(v) = \{u \mid uv \in E\}$ . The *closed neighborhood* of  $v$  is  $N_G[v] = N_G(v) \cup \{v\}$ , or when clear, simply  $N[v]$ . The *degree* of  $v$  in  $G$ , denoted by  $d_G(v)$ , or  $d(v)$ , is the size of  $N_G(v)$ . We write  $\delta(G)$  for the *minimum degree* of  $G$ . If all the vertices of  $G$  have the same degree  $k$ , then  $G$  is  *$k$ -regular*, or simply *regular*. In particular, a 0-regular graph is the same as an empty graph. For  $S \subseteq V(G)$ , denote by  $G[S]$  the graph induced by  $S$ . Given two disjoint subsets  $T$  and  $S$  of vertices in  $G$ ,  $E(T, S)$  denotes the set of edges linking a vertex in  $T$  to a vertex in  $S$ , and  $e(T, S) = |E(T, S)|$ . Let  $k \geq 2$  be an integer. A graph  $G = (V, E)$  is called  *$k$ -partite* if  $V$  admits a partition into  $k$  classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. An  $k$ -partite graph in which every two vertices from different partition classes are adjacent is called *complete*. A *clique* in  $G$  is a complete subgraph of  $G$ . More precisely, a  $k$ -clique is a clique of order  $k$ , denoted by  $K_k$ . Obviously, every  $k$ -partite graph is a graph containing no  $(k + 1)$ -cliques.

If  $f : V \rightarrow \mathbf{R}$  is a weight function on the vertices of  $G$ , we denote  $f(S) = \sum_{v \in S} f(v)$  and  $f[v] = f(N[v])$ . A *signed dominating function* of a graph  $G$  is defined in [6] as a function  $f : V \rightarrow \{+1, -1\}$  such that for every vertex  $v$ ,  $f[v] \geq 1$ , and the minimum cardinality of the weight  $w(f) = \sum_{v \in V} f(v)$  over all such functions is called the *signed domination number*, denoted by  $\gamma_s(G)$ , i.e.,

$$\gamma_s(G) = \min\{f(V) : f \text{ is a signed dominating function of } G\}.$$

A *minus dominating function* is defined in [3] as a function of the form  $f : V \rightarrow \{+1, 0, -1\}$  such that  $f[v] \geq 1$  for all  $v \in V$ . The *minus domination number* for a graph  $G$  is

$$\gamma^-(G) = \min\{f(V) : f \text{ is a minus dominating function of } G\}.$$

By definitions, every signed dominating function of  $G$  is clearly a minus dominating function of  $G$ , so  $\gamma^-(G) \leq \gamma_s(G)$ . The minus and signed domination problems can be viewed as a proper

1 generalization of the classical domination problem. The literature on this topic of dominating  
2 functions is detailed in [8, 9].

3 The decision problems for the signed domination [3] and the minus domination [7] of a  
4 graph have been showed to be NP-complete, even when the graph is restricted to a bipartite  
5 graph or a chordal graph. Hence, it is of interest to determine upper bounds on the domination  
6 parameters of a graph. In 1999, Dunbar et al. [5] posed a conjecture on lower bounds of the  
7 minus domination number for bipartite graphs. Later, this conjecture was proved by several  
8 separate subsets of the authors [13, 15, 18, 21]. In [12], Kang et al. further extended the result  
9 to  $k$ -partite graphs for  $k \geq 2$ . Other results and progress on the research for signed and minus  
10 domination of graphs can be found in [1, 2, 4, 10, 11, 14, 16, 17, 19].

11 Turán's theorem [20], proved in 1941, is one of the fundamental results in extremal graph  
12 theory. In this paper we investigate an application of Turán theorem to the signed domination  
13 in graph theory. By making use of Turán theorem, we present a lower bound on the signed  
14 domination number of graphs containing no  $(k + 1)$ -cliques in terms of its order and minimum  
15 degree. As a special case of the above result, we give a lower bound on the minus domination  
16 number of graphs containing no  $(k + 1)$ -cliques and characterize the extremal graphs attaining  
17 this bound, which extend and strengthen a recent result due to Kang et al. [12] on the minus  
18 domination number of  $k$ -partite graphs.

## 19 **2 Main results**

20 A graph  $G \not\supseteq H$  on  $n$  vertices with the largest possible number of edges is called *extremal* for  
21  $n$  and  $H$ ; its number of edges is denoted by  $\text{ex}(n, H)$ . The unique complete  $k$ -partite graphs  
22 on  $n \geq k$  vertices whose partition sets differ in size by at most 1 are called *Turán graphs*; we  
23 denote them by  $T^k(n)$  and their number of edges by  $t_k(n)$ . Clearly,  $T^k(n) = K_n$  for all  $n \leq k$ .  
24 The following Turán theorem from extremal graph theory is well-known, we will make use of it  
25 in our proof.

**Theorem 1** (Turán theorem, 1941 [20]) *For any integer  $k \geq 1$ , if  $G = (V, E)$  is a graph on  $n$  vertices and  $\text{ex}(n, K_{k+1})$  edges containing no  $(k + 1)$ -cliques, then  $G$  is a  $T^k(n)$  and*

$$|E| = t_k(n) \leq \frac{k-1}{2k}n^2$$

1 *with equality if and only if  $k$  divides  $n$ .*

2 We start with presenting a sharp lower bound on the signed domination number for graphs  
3 on  $n$  vertices containing no  $(k + 1)$ -cliques.

**Theorem 2** *For any integer  $k \geq 2$ , let  $G = (V, E)$  be a graph of order  $n$  with no  $(k + 1)$ -cliques and  $c = \lceil \delta^*/2 \rceil + 1$  where  $\delta^* = \max\{2, \delta(G)\}$ . Then*

$$\gamma_s(G) \geq \frac{k}{k-1} \left( -c + \sqrt{c^2 + 4\frac{k-1}{k}nc} \right) - n$$

4 *and this bound is sharp.*

5 **Proof.** Let  $f : V \rightarrow \{+1, -1\}$  be a signed dominating function on  $G$  with  $f(V(G)) = \gamma_s(G)$   
6 and let  $P$  and  $M$  be the sets of vertices in  $V$  that are assigned the value  $+1$  and  $-1$ , respectively,  
7 under  $f$ . Then  $n = |P| + |M|$ . For convenience, let  $|P| = p$  and  $|M| = m$ . For each vertex  
8  $v \in M$ ,  $v$  is adjacent to at least  $c$  ( $\geq 2$ ) vertices in  $P$  since  $f[v] \geq 1$ , i.e.  $|N_G(v) \cap P| \geq c$ .  
9 Hence,

$$e(P, M) = \sum_{v \in M} |N_G(v) \cap P| \geq c|M| = cm = c(n - p). \quad (1)$$

10 On the other hand, for each vertex  $v \in P$ ,  $|N_G(v) \cap M| \leq |N_G(v) \cap P|$ , and so

$$e(M, P) = \sum_{v \in P} |N_G(v) \cap M| \leq \sum_{v \in P} |N_G(v) \cap P| = \sum_{v \in P} d_{G[P]}(v). \quad (2)$$

11 Since  $G$  contains no  $(k + 1)$ -cliques, so does  $G[P]$ . Applying Turán theorem, together with  
12 inequalities (1) and (2), we have

$$c(n - p) \leq e(P, M) \leq \frac{k-1}{k}p^2, \quad (3)$$

or equivalently,

$$\frac{k-1}{k}p^2 + cp - cn \geq 0.$$

Hence,

$$p \geq \left( -c + \sqrt{c^2 + 4\frac{k-1}{k}cn} \right) / 2 \left( \frac{k-1}{k} \right).$$

1 Therefore,

$$\gamma_s(G) = 2p - n \geq \frac{k}{k-1} \left( -c + \sqrt{c^2 + 4\frac{k-1}{k}cn} \right) - n.$$

2 That the bound is sharp may be seen as follows: For positive integers  $k, s \geq 2$ , let  $F_1$  be the  
3 Turán graph  $T^k(k s)$ , that is,  $F_1$  is a complete  $k$ -partite graph of order  $k s$  with equal partition  
4 sets  $V_1, V_2, \dots, V_k$  and  $|V_i| = s$  for  $i = 1, \dots, k$ . Let  $F_2$  be a  $(s-2)$ - or  $(s-3)$  ( $\geq 0$ )-regular  
5  $k$ -partite graph of order  $k(k-1)s$  with equal partition sets  $U_1, U_2, \dots, U_k$  and  $|U_i| = (k-1)s$   
6 for  $i = 1, \dots, k$ . Let  $F(k, s)$  be a family of graphs obtained from the disjoint union of  $F_1$  and  $F_2$   
7 by joining each vertex of  $V_i$  with all the vertices of  $U_i$  for each  $i = 1, \dots, k$ . Let  $X_i = V_i \cup U_{i+1}$   
8 where  $i+1 \pmod k$ . Then every one of  $F(k, s)$  be a  $k$ -partite graph of order  $n = k^2 s$  with equal  
9 partition sets. Note that for all  $i$ , each vertex of  $U_i$  in  $F(k, s)$  has minimum degree  $2(s-1)$  or  
10  $2(s-1) - 1$ . For each graph  $H \in F(k, s)$ , we assign to each vertex of  $F_1$  the value  $+1$  and to  
11 each vertex of  $F_2$  the value  $-1$ . It is easy to check that  $f[v] = 1$  for each vertex  $v \in V(F_1)$  and  
12  $f[v] = 1$  or  $2$  for each vertex  $v \in V(F_2)$ , so we produce a signed dominating function  $f$  of  $H$   
13 with weight

$$\begin{aligned} w(f) = f(V(H)) &= p - m = V(F_1) - V(F_2) \\ &= ks - k(k-1)s = ks(2-k) \\ &= \frac{k}{k-1} \left( -s + \sqrt{s^2 + 4\frac{k-1}{k}ns} \right) - n. \end{aligned}$$

Note that  $s \geq 2$ . If  $s = 2$ , then  $\delta^* = 2$ , and thus  $c = s$ ; if  $s \geq 3$ , then  $\delta^* = \delta(G) = 2(s - 1)$  or  $2(s - 1) - 1$ , and thus  $c = s$ . Consequently,

$$\gamma_s(G) = \frac{k}{k-1} \left( -c + \sqrt{c^2 + 4 \frac{k-1}{k} nc} \right) - n.$$

1  $\square$

2 As a somewhat weak case of Theorem 2, we can easily extend a result due to Wang [18] on  
 3 the signed domination number for bipartite graphs to graphs containing no  $(k + 1)$ -cliques and  
 4 characterize the extremal graphs achieving this bound. For this purpose, we define a family  
 5  $\mathcal{H}(k, s)$  of graphs as follows:

6 Let  $p, m, k$  and  $s$  be positive integers satisfying the following conditions:

7 (i)  $p = ks$ ,  $s$  is even if  $k = 2$ ;  $s \geq 1$  if  $k \geq 3$ .

8 (ii)  $m = (k - 1)p^2/2k = \frac{1}{2}k(k - 1)s^2$ .

9 Let  $F_1$  be the Turán graph  $T^k(ks)$  as described in the proof of Theorem 2. Let  $F_3$  be an  
 10 empty graph of order  $m$ . Let  $H(k, s)$  be the family of graphs in which each graph is obtained  
 11 from the disjoint union of  $F_1$  and  $F_3$  by adding edges as follows: if  $k \geq 3$ , we join each vertex of  
 12  $F_1$  with exactly  $(k - 1)s$  vertices of  $F_3$ , and join each vertex of  $F_3$  with exactly 2 vertices of  $F_1$   
 13 (since  $2m = (k - 1)p^2/k = k(k - 1)s^2$ , such an addition of edges is possible); if  $k = 2$ , we can  
 14 partition  $V(F_3)$  into two subsets  $U_1$  and  $U_2$  with  $|U_1| = |U_2| = s^2/2$ , then join each vertex of  $V_i$   
 15 with exactly  $s$  vertices of  $U_i$  and join each vertex of  $U_i$  with exactly 2 vertices of  $V_i$  for  $i = 1, 2$ .  
 16 By our construction, every one of  $H(k, s)$  contains no  $(k + 1)$ -cliques, and each vertex of  $F_1$  in  
 17  $H(k, s)$  has degree  $2(k - 1)s$  while each vertex of  $F_3$  in  $H(k, s)$  has degree 2. An example of  
 18 the graphs  $H(3, 2)$  is shown in Fig. 1. Let  $\mathcal{H}(k, s) = \cup H(k, s)$ , where  $p, m, k$  and  $s$  take over  
 19 all integers satisfying conditions (i) and (ii).

**Theorem 3** For any integer  $k \geq 2$ , let  $G = (V, E)$  be a graph of order  $n$  with no  $(k + 1)$ -cliques,

then

$$\gamma_s(G) \geq \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k}n} \right) - n,$$

1 where equality holds if and only if  $G \in \mathcal{H}(k, s)$ .

**Proof.** We define

$$g(x) = \frac{k}{k-1} \left( -x + \sqrt{x^2 + 4\frac{k-1}{k}nx} \right) - n.$$

It is easy to check that  $g'(x) > 0$  when  $x, n \geq 1$ , so  $g(x)$  is a strictly monotone increasing function when  $x \geq 1$ . Note that  $c \geq 2$ , hence

$$\gamma_s(G) \geq g(c) \geq g(2) = \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k}n} \right) - n.$$

2 The first part of the corollary follows.

3 Next we characterize the extremal graphs achieving this lower bound. First, suppose that  
 4  $\gamma_s(G) = 2k/(k-1) \left( -1 + \sqrt{1 + 2(k-1)n/k} \right) - n$  holds. Then  $c = 2$  and all the equalities hold  
 5 in (1), (2) and (3). Hence, we obtain

$$2|M| = e(P, M) = \sum_{v \in P} |N_G(v) \cap M| = \sum_{v \in P} |N_G(v) \cap P| = \sum_{v \in P} d_{G[P]}(v) = \frac{k-1}{k} p^2. \quad (4)$$

The equality chain implies that

$$|E(G[P])| = \frac{k-1}{2k} p^2 \text{ and } m = |M| = \frac{k-1}{2k} p^2.$$

6 Note the fact that  $G[P]$  contains no  $(k+1)$ -cliques. Applying Turán theorem,  $G[P]$  is a  
 7 complete  $k$ -partite graph with equal partition classes, and so  $k$  divides  $p$ . Let  $p = ks$ . Then  
 8  $G[P]$  is isomorphic to some  $F_1$ . The equality  $\sum_{v \in P} |N_G(v) \cap M| = \sum_{v \in P} |N_G(v) \cap P|$  implies  
 9 that each vertex  $v$  of  $P$  has exactly  $(k-1)s$  neighbors in  $M$ . Hence  $d_G(v) = 2(k-1)s \geq 2$  as  
 10  $k \geq 2$ . By definition, each vertex of  $M$  has degree at least 2. Then  $\delta(G) \geq 2$ . However, since  
 11  $c = \lceil \delta^*/2 \rceil + 1 = \lceil \delta(G)/2 \rceil + 1 = 2$ , it follows that  $\delta(G) = 2$ . The equality chain (4) implies  
 12 that each vertex of  $M$  in  $G$  is exactly adjacent to two vertices of  $P$  and has minimum degree  
 13 2. Hence  $M$  is an independent set of vertices in  $G$ , and so  $M$  is isomorphic to some empty

1 graph  $F_3$  of order  $m$ . So  $G$  is isomorphic to one of the family  $H(k, s)$  of graphs. It follows that  
 2  $G \in \mathcal{H}(k, s)$ .

3 On the other hand, suppose  $G \in \mathcal{H}(k, s)$ . Thus, there exist integers  $k$  and  $s$  such that  
 4  $G \in H(k, s)$ . Assigning to each vertex of  $F_1$  the value  $+1$  and to each vertex of  $F_3$  the value  
 5  $-1$ , we produce a signed dominating function  $f$  of  $G$  with weight

$$\begin{aligned} w(f) = f(V(G)) &= p - m = V(F_1) - V(F_3) = p - (k - 1)p^2/2k \\ &= \frac{2k}{k - 1} \left( -1 + \sqrt{1 + \frac{2(k - 1)}{k}n} \right) - n. \end{aligned}$$

Consequently,

$$\gamma_s(G) = \frac{2k}{k - 1} \left( -1 + \sqrt{1 + \frac{2(k - 1)}{k}n} \right) - n.$$

6  $\square$

7 Now we turn our attention to the minus domination of graphs. Kang et al. [12] presented a  
 8 sharp lower bound on the minus domination number for  $k$ -partite graphs below, which solved  
 9 and strengthened a conjecture on the minus domination number for bipartite graphs posed by  
 10 Dunbar et al. [5].

**Theorem 4** (Kang et al. [12]) *If  $G = (V, E)$  is a  $k$ -partite graph of order  $n$  with  $k \geq 2$ , then*

$$\gamma^-(G) \geq \frac{2k}{k - 1} \left( -1 + \sqrt{1 + \frac{2(k - 1)}{k}n} \right) - n$$

11 *and this bound sharp.*

12 Note that every  $k$ -partite graph is a graph containing no  $(k + 1)$ -cliques. By Theorem 3, we  
 13 can further generalize and strengthen the above result to graphs containing no  $(k + 1)$ -cliques.

**Theorem 5** *For any integer  $k \geq 2$ , let  $G = (V, E)$  be a graph of order  $n$  with no  $(k + 1)$ -cliques,  
 then*

$$\gamma^-(G) \geq \frac{2k}{k - 1} \left( -1 + \sqrt{1 + \frac{2(k - 1)}{k}n} \right) - n,$$



1 where equality holds if and only if  $G \in \mathcal{H}(k, s)$ .

**Proof.** Let  $f : V \rightarrow \{+1, 0, -1\}$  be a minus dominating function on  $G$  with  $f(V(G)) = \gamma^-(G)$  and let  $Q$  be the set of vertices in  $V$  that are assigned the value 0. Further, Let  $G' = G - Q$  and  $n' = |V(G')|$ . Then  $G'$  keeps to be a graph without  $(k+1)$ -cliques and  $n' \leq n$ . Clearly,  $f' = f|_{G'}$  is a signed dominating function on  $G'$ , so  $\gamma_s(G') \leq f'(V(G')) = f(V(G))$ . By Theorem 3, we have

$$\gamma^-(G) \geq \gamma_s(G') \geq \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k} n'} \right) - n'.$$

We define

$$h(x) = \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k} x} \right) - x.$$

2 It is easy to check that  $h'(x) \leq 0$  when  $x \geq 3$ , so  $h(x)$  is a strictly monotone decreasing function  
3 on the variable  $x \geq 3$ . This implies that

$$\gamma^-(G) \geq \gamma_s(G') \geq \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k} n} \right) - n$$

4 if  $n' \geq 3$ . If  $n' \leq 2$ , then each vertex in  $G'$  is assigned the value  $+1$ , so no vertices in  $G$  are  
5 assigned the value  $-1$ , and thus  $\gamma^-(G) \geq 1$ . Note that  $1 \geq \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k} n} \right) - n$  for  
6  $k \geq 2$ , so the results follows.

We further characterize the extremal graphs attaining this bound. If the equality holds, i.e.,

$$\gamma^-(G) = 2k/(k-1) \left( -1 + \sqrt{1 + 2(k-1)n/k} \right) - n,$$

7 then  $h(n') = h(n)$ . Observing the fact that  $h(x)$  is a strictly monotone function on variable  
8  $x$  when  $x \geq 3$ , we have  $n' = n$ . Hence  $Q = \emptyset$ . Thus  $f$  is also a minimum signed dominating  
9 function, i.e.,  $\gamma_s(G) = 2k/(k-1) \left( -1 + \sqrt{1 + 2(k-1)n/k} \right) - n$ . The characterization follows  
10 from Theorem 3.  $\square$

11 As an immediate consequence of Theorems 3 and 5, we obtain the following extremal result  
12 on the minus domination and signed domination of a graph containing no  $(k+1)$ -cliques.

1 **Theorem 6** For any integer  $k \geq 2$ , let  $G = (V, E)$  be a graph of order  $n$  with no  $(k+1)$ -cliques,  
2 then the following statements are equivalent.

3 (i)  $\gamma_s(G) = \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k}n} \right) - n;$

4 (ii)  $\gamma^-(G) = \frac{2k}{k-1} \left( -1 + \sqrt{1 + \frac{2(k-1)}{k}n} \right) - n;$

5 (iii)  $G \in \mathcal{H}(k, s).$

### 6 **3 Conclusion**

7 In this note, by applying the well-known Turán theorem, we obtained lower bounds on the  
8 signed and minus domination numbers of graphs containing no  $(k+1)$ -clique and characterized  
9 extremal graphs attaining these bounds. It remains to be seen whether or not Theorem 2 is  
10 also true for the minus domination number of graphs containing no  $(k+1)$ -cliques.

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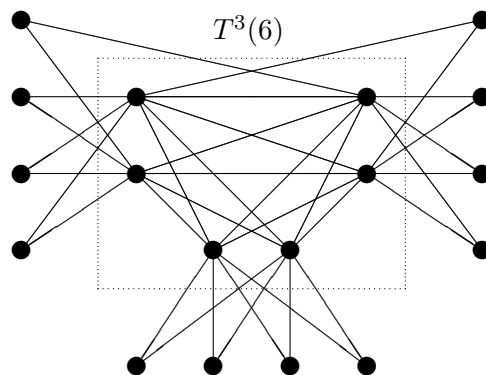


Figure 1: An example of the graphs  $H(3, 2)$ .