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Codiameters of 3-Domination Critical Graphs with Toughness More Than One

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Abstract: A graph G is 3-domination-critical (3-critical, for short), if its domination number γ is 3 and the addition of any edge decreases γ by 1. In this paper, we show that every 3-critical graph with independence number 4 and minimum degree 3 is Hamilton-connected. Combining the result with those in [2], [4] and [5], we solve the following conjecture: a connected 3-critical graph G is Hamilton-connected if and only if $\tau(G) > 1$, where $\tau(G)$ is the toughness of G.

Key words: Domination-critical graph, Hamilton-connectivity

1. Introduction

Let $G = (V(G), E(G))$ be a graph. For the notations that are not defined here, we follow [2]. A graph G is said to be *t-tough* if for every cutset $S \subseteq V(G)$, $|S| \geq t\omega(G-S)$, where $\omega(G-S)$ is the number of components of $G-S$. The *toughness* of G, denoted by $\tau(G)$, is defined to be $\min\{|S|/\omega(G-S) \mid S \text{ is a cutset of } G\}$. Let $u, v \in V(G)$ be any two distinct vertices. We denote by $p(u, v)$ the length of a longest path connecting u and v. The *codiameter* of G, denoted by $d^*(G)$, is defined to be $\min\{p(u, v) \mid u, v \in V(G)\}$. A graph G of order n is said to be *Hamilton-connected* if $d^*(G) = n - 1$, i.e., every two distinct vertices are joined by a hamiltonian path. A graph G is called *k-domination critical*, abbreviated as *k-critical*, if $\gamma(G) = k$ and $\gamma(G + e) = k - 1$ holds for any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G. The concept of domination critical

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graphs was introduced by Sumner [7]. Given three vertices u, v and x such that $\{u, x\}$ dominates $V(G) - \{v\}$ but not v, we will write $[u, x] \to v$. It was observed in [7] that if u, v are any two nonadjacent vertices of a 3-critical graph G, then since $\gamma(G + uv) = 2$, there exists a vertex x such that either $[u, x] \to v$ or $[v, x] \to u$. In [2], Chen et al. posed the following.

Conjecture 1 (Chen et al. [2]). A connected 3-critical graph G is Hamilton-connected if and only if $\tau(G) > 1$.

In the same paper, they proved that the conjecture is true when $\alpha(G) \leq \delta(G)$.

Theorem 1 (Chen et al. [2]). Let G be a connected 3-critical graph with $\alpha(G) \leq \delta(G)$. Then G is Hamilton-connected if and only if $\tau(G) > 1$.

Let G is a 3-connected 3-critical graph. It is shown in [3] that $\tau(G) \geq 1$ and $\tau(G) = 1$ if and only if G belongs to a special infinite family $\mathcal{G}$ described in [3]. Since $\alpha(G) = \delta(G) = 3$ for each $G \in \mathcal{G}$, we have $\tau(G) > 1$ if $\alpha(G) \geq \delta(G) + 1$.

In [4], Chen et al. showed that the conjecture holds when $\alpha(G) = \delta(G) + 2$.

Theorem 2 (Chen et al. [4]). Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 2$. Then G is Hamilton-connected.

By a result of Favaron et al. [6] which states that $\alpha(G) \leq \delta(G) + 2$ for any connected 3-critical graph G, we see that the conjecture has only one case $\alpha(G) = \delta(G) + 1$ unsolved.

Recently, Chen et al. [5] showed that the conjecture is true for $\alpha(G) = \delta(G) + 1 \geq 5$.

Theorem 3 (Chen et al. [5]). Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 \geq 5$. Then G is Hamilton-connected.

Since $\tau(G) > 1$ implies $\delta(G) \geq 3$, the case $\alpha(G) = \delta(G) + 1 = 4$ remains open. In this paper, we will show that the conjecture is true when $\alpha(G) = \delta(G) + 1 = 4$. The main result of this paper is the following.

Theorem 4. Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 = 4$. Then G is Hamilton-connected.

Combining Theorems 1, 2, 3 and 4, we have the following.

Theorem 5. A connected 3-critical graph G is Hamilton-connected if and only if $\tau(G) > 1$.

By the main result of [3], we have the following.

Theorem 6. Let G be a 3-connected 3-critical graph. Then G is Hamilton-connected if and only if G does not belong to a special infinite family $\mathcal{G}$ described in [3].

Now, we restate a result due to Chen et al. for later use.

Theorem 7 (Chen et al. [1])**.** Let G be a 3-connected 3-critical graph of order n. Then $d^*(G) \geq n-2$.

2. Some Lemmas

Let G be a graph of order n, and x, y vertices of G such that a longest (x,y)-path is of length $n-2$. Let $P = P_{xy}$ be an (x,y)-path of length $n-2$. We denote by x_P the only vertex not in P and let $d(x_P) = k$ with

$N(x_P) = X = \{x_1, x_2, \ldots, x_k\}$,	indices following the orientation of P;
$A = X^+ = \{a_1, a_2, \ldots, a_s\}$,	where $a_i = x_i^+$, $x_i^+ \in P$ and $s \geq k-1$;
$B = X^- = \{b_t, b_{t+1}, \ldots, b_k\}$,	where $b_i = x_i^-$, $x_i^- \in P$ and $t \leq 2$; and
$P_i = a_i \overrightarrow{P} b_{i+1}$,	where $1 \leq i \leq k-1$.

Furthermore, we let $P_0 = x\overrightarrow{P}b_1$ if $x \notin X$ and $P_k = a_k\overrightarrow{P}y$ if $y \notin X$. The length of the path $x_1\overrightarrow{P}x_k$ is denoted by $s(P)$.

Definition. A vertex $v \in P_i$ $(1 \leq i \leq k)$ is called an A-vertex if $G[P_i \cup \{x_{i+1}\}]$ contains a hamiltonian (v, x_{i+1})-path and $v \in P_i$ $(0 \leq i \leq k-1)$ a B-vertex if $G[P_i \cup \{x_i\}]$ contains a hamiltonian (x_i, v)-path, where $x_{k+1} = y$ and $x_0 = x$.

From the definition, we can see that each a_i is an A-vertex and each b_i is a B-vertex. Furthermore, if $v \in P_i$ $(i \neq 0)$ and $v^+a_i \in E(G)$, then v is an A-vertex and if $v \in P_i$ $(i \neq k)$ and $v^-b_{i+1} \in E(G)$, then v is a B-vertex.

Lemma 1 (Chen et al. [5]). If $u_i \in P_i$ and $u_j \in P_j$ are two A-vertices (B-vertices, respectively) with $i \neq j$, then $x_Pu_i \notin E(G)$ and $u_iu_j \notin E(G)$. In particular, both $A \cup \{x_P\}$ and $B \cup \{x_P\}$ are independent sets.

Lemma 2 (Chen et al. [5]). Let $u_i \in P_i$, $u_j \in P_j$ be A-vertices with $i < j$, Q_i and Q_j are hamiltonian (u_i, x_{i+1})-path and (u_j, x_{j+1})-path in $G[P_i \cup \{x_{i+1}\}]$ and $G[P_j \cup \{x_{j+1}\}]$, respectively, $Q = u_i\overrightarrow{Q_i}x_{i+1}\overrightarrow{P}x_j$ and $R = u_j\overrightarrow{Q_j}x_{j+1}\overrightarrow{P}y$. If $v \in N_Q(u_i)$, then $v^- \notin N(u_j)$ and if $v \in N(u_i) \cap (x\overrightarrow{P}x_i \cup R)$, then $v^+ \notin N(u_j)$. In particular, let $a_i, a_j \in A$ with $i < j$ and $v \in N(a_i)$, then $v^- \notin N(a_j)$ if $v \in a_i\overrightarrow{P}x_j$ and $v^+ \notin N(a_j)$ if $v \in x\overrightarrow{P}x_i \cup a_j\overrightarrow{P}y$.

By the symmetry of A and B, Lemma 2 still holds if we exchange A and B.

Lemma 3 (Chen et al. [5]). Let $u, v \in a_i\overrightarrow{P}b_j$ with $j \geq i+1$ and $G[a_i\overrightarrow{P}b_j]$ contain a hamiltonian (u, v)-path. Suppose that $w \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$ and $uw \in E(G)$. Then $w^-v \notin E(G)$ if $w^- \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$ and $w^+v \notin E(G)$ if $w^+ \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$. In particular, let $a_i \in A$ and $b_j \in B$ with $j \geq i+1$. Suppose that $v \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$ and $a_iv \in E(G)$. Then, $v^-b_j \notin E(G)$ if $v^- \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$, and $v^+b_j \notin E(G)$ if $v^+ \in x\overrightarrow{P}x_i \cup x_j\overrightarrow{P}y$.

Lemma 4 (Chen et al. [5]). Let $u, u^+ \in P_i$. If $u^+a_l \in E(G)$ for some $l \geq i+1$, then $b_ju \notin E(G)$ for all $j \leq i$.

Lemma 5 (Chen et al. [2]). Let $|P_i| \geq 2$, $u, v \notin P_i$ and $\{u, v\} \succ P_i$. If $ua_i, vb_{i+1} \in E(G)$, then there exists some vertex $w \in P_i$ such that $uw, vw^+ \in E(G)$.

Lemma 6 (Chen et al. [5]). Let $i \geq 2$, $z \in P_j$ and $[a_i, z] \to x_P$. If $|A| \geq 3$ and $j \neq i-1$, then $A \cup \{z^+, x_P\}$ is an independent set if $z^+ \in P$ and $B \cup \{z^-, x_P\}$ is an independent set if $z^- \in P$.

Lemma 7. Let $|A| = |B| = 3$, $z \in P_j$ and $[x_P, z] \to a_i$. If $z^- \in P$, then $B \cup \{x_P, z^-\}$ is an independent set.

Proof. Suppose to the contrary there is some b_l such that $b_lz^- \in E(G)$. If $l = j+1$, then z is a B-vertex, which contradicts Lemma 1 since $|B| = 3$ and $B - \{a_i\} \subseteq N(z)$. If $l < j+1$, then $j = 2$ or 3 for otherwise we have $a_2, a_3 \notin N(z)$ by Lemma 4. If $j = 2$ and $l = 1$, then by Lemmas 2 and 4, we have $b_2, a_3 \notin N(z)$, and if $j = 2$ and $l = 2$, then by Lemmas 3 and 4, $a_1, a_3 \notin N(z)$, a contradiction. Thus, we may assume $j = 3$. If $l = 3$, then by Lemma 3, $a_1, a_2 \notin N(z)$; if $l = 2$, then by Lemmas 2 and 3, $b_3, a_1 \notin N(z)$; and if $l = 1$, then by Lemma 2, $b_2, b_3 \notin N(z)$, a contradiction. If $l > j+1$, then since $b_1z \in E(G)$, by Lemma 2 we have $j = 0$. If $l = 2$, then by Lemma 2 and 3, $b_3, a_1 \notin N(z)$ and if $l = 3$, then by Lemma 3, $a_1, a_2 \notin N(z)$, a contradiction. Since $|A| = 3$ and $A - \{a_i\} \subseteq N(z)$, by Lemma 1 we have $z \notin A$, which implies $z^-x_P \notin E(G)$. Thus, $B \cup \{x_P, z^-\}$ is an independent set. ∎

Now, let G be a 3-critical graph, $\alpha(G) = \delta(G) + 1$ and $v_0 \in V(G)$ with $d(v_0) = \delta(G) = 3$. Suppose $N(v_0) = \{v_1, v_2, v_3\}$ and $I = \{v_0, w_1, w_2, w_3\}$ is an independent set. The following lemma restates a lemma due to Sumner and Blitch [7], which has become of considerable utility in dealing with 3-critical graphs. In [7] they considered the case $l \geq 4$, which guarantees $P(W) \cap W = \emptyset$. For the cases $l = 2$ and $l = 3$, Lemma 8 can be easily verified since G is a 3-critical graph.

Lemma 8. Let G be a connected 3-critical graph and U an independent set of $l \geq 2$

vertices. Then there exists an ordering $u_1, u_2, \cdots, u_l$ of the vertices of U and a sequence $P(U) = (y_1, y_2, \cdots, y_{l-1})$ of $l-1$ distinct vertices such that $[u_i, y_i] \to u_{i+1}$, $1 \le i \le l-1$.

The next lemma is a useful consequence of Lemma 8.

Lemma 9 (Favaron et al. [6]). Let U be an independent set of $l \ge 3$ vertices of a 3-critical graph G such that $U \cup \{v\}$ is independent for some $v \notin U$. Then the sequence $P(U)$ defined in Lemma 8 is contained in $N(v)$.

Since I is an independent set of order 4, by Lemmas 8 and 9, we may assume without loss of generality that $[w_i, v_i] \to w_{i+1}$ for $i = 1, 2$.

Lemma 10 (Chen et al. [5]). If $[v_0, z] \to w_i$ for $i \ne 3$, then we have $z \notin N(v_0)$ and if $[v_0, v_l] \to w_3$, then $l = 2$.

Lemma 11 (Chen et al. [5]). If $[v_0, v_2] \to w_3$, then we have $v_1, v_2, w_3 \notin N(v_3)$ and $w_1, w_2 \in N(v_3)$.

Lemma 12. Let G be 3-critical, $X = \{x_1, x_2, x_3\} = \{x_i, x_j, x_l\}$ and $\{x_P, a_i, u, v\}$ a maximum independent set. If $[x_P, x_l] \to a_i$, then we have $x_l x_i \in E(G)$, $x_i, x_l \notin N(x_j)$ and $\{x_l, x_j\} \subseteq N(u) \cap N(v)$.

Proof. Let $U = \{a_i, u, v\} = \{u_1, u_2, u_3\}$. By Lemmas 8 and 9, we may assume that $[u_m, x_{q_m}] \to u_{m+1}$ for $m = 1, 2$. Let $X - \{x_{q_1}, x_{q_2}\} = \{x_{q_3}\}$. If $[x_P, x_l] \to a_i$, then by Lemma 10, we have $a_i = u_3$ and $x_l = x_{q_2}$. Since $[u_1, x_{q_1}] \to u_2$, we have $x_{q_1} a_i \in E(G)$. By Lemma 11, $x_{q_3} a_i \notin E(G)$. Thus, since $x_i \in X$ and $x_i a_i \in E(G)$, we have $x_{q_1} = x_i$ and $x_{q_3} = x_j$, that is, $[u_1, x_i] \to u_2$ and $[u_2, x_l] \to a_i$. In this case, we have $x_i x_l \in E(G)$ and by Lemma 11, we have $x_i, x_l \notin N(x_j)$ and $\{x_l, x_j\} \subseteq N(u) \cap N(v)$. ∎

The following two lemmas can be extracted from [2].

Lemma 13 (Chen et al. [2]). Suppose that P is a longest (x, y)-path such that $|X \cap \{x, y\}|$ is as small as possible and that for this path, $d(x_P) = k \ge 4$. If G is 3-critical, then there exists an independent set I such that either $\{x_P\} \cup A \subseteq I$ or $\{x_P\} \cup B \subseteq I$ and $|I| \ge k + 1$.

Lemma 14 (Chen et al. [2]). Let G be a 3-connected 3-critical graph of order n, $x, y \in V(G)$ and $p(x, y) = n - 2$. Suppose that P is a longest (x, y)-path such that $d(x_P)$ is as large as possible and subject to this, $|X \cap \{x, y\}|$ is as small as possible. If $d(x_P) = 3$, $\{x, y\} \subseteq X$ and P_i is a clique for $i = 1, 2$, then $a_1 b_3 \notin E(G)$, and if $a_2 b_2 \in E(G)$, then $n = 8$ and $\alpha(G) = 3$.

3. Proof of Theorem 4

Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 = 4$. We still use the notations given in Section 3. Suppose to the contrary that G is not Hamilton-connected. By Theorem 7, there are two vertices x, y such that $p(x, y) = n - 2$. Among all the longest (x, y)-paths, we choose P such that

(a) $d(x_P)$ is as large as possible;
(b) subject to (a), $|\{x, y\} \cap N(x_P)|$ is as small as possible;
(c) subject to (a) and (b), $s(P)$ is as small as possible.

Choose an orientation such that $|A| \geq |B|$. Assume without loss of generality that the orientation is from x to y. Since $\alpha(G) = \delta(G) + 1 = 4$, by the choice of P and Lemma 13, we have $d(x_P) = 3$.

We consider the following two cases separately.

Case 1. $|A| = 3$

Let $U = N[x_P] \cup A$. If $|A| = 3$, then by Lemmas 8 and 9, we may assume that $[a_{i_l}, x_{j_l}] \to a_{i_{l+1}}$ for $l = 1, 2$. Thus, noting that $|A| = 3$, we have

$$d_U(x_i) \geq \delta = 3 \text{ for any } x_i \in X. \tag{1}$$

If $[a_3, b_3] \to x_P$, then $b_2a_3, a_1b_3 \in E(G)$ by Lemma 1. In this case, we have $|P_2| \geq 2$ and hence $d(x_3) \geq 4$ by (1). Thus, $Q = x\overrightarrow{P}x_1x_Px_2\overrightarrow{P}b_3a_1\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is an (x, y)-path of length $n - 2$ with $x_Q = x_3$, which contradicts the choice of P and hence

$$[a_3, b_3] \to x_P \text{ is impossible.} \tag{2}$$

Claim 1. Let $z \in P_j$ and $[x_P, z] \to a_i$. If $z^+ \in P$, then $A \cup \{x_P, z^+\}$ is an independent set.

Proof. If $|B| = 3$, then since $B - \{a_i\} \subseteq N(z)$, by Lemma 1 we have $z \notin B$. If $|B| = 2$ and $z = b_2$, then we must have $a_2 = b_3 = a_i$. Since $P_3 \subseteq N(z)$, by Lemmas 1 and 2 we have $N(a_i) \cap P_3 = \emptyset$. Thus, by the choice of P, we have $N(a_i) = X$, which contradicts $\tau(G) > 1$ since $\omega(G - X) \geq 3$. If $|B| = 2$ and $z = b_3$, then $a_1 = b_2 = a_i$. Since $P_3 \subseteq N(z)$, by Lemmas 1 and 3 we have $N(a_i) \cap P_3 = \emptyset$. If $a_ix_3 \in E(G)$, then by the choice of P, we have $N(a_i) = X$, which contradicts $\tau(G) > 1$. If $x_3a_i \notin E(G)$, then $P' = xx_Px_2\overrightarrow{P}y$ is an (x, y)-path of length $n-2$ such that $s(P') < s(P)$, a contradiction. Therefore, we have $z \notin B$ and hence $z^+x_P \notin E(G)$. Thus, by Lemma 1, we need only to show $A \cup \{z^+\}$ is an independent set.

Suppose to the contrary there is some a_l such that $a_l z^+ \in E(G)$. If $l = j$, then z is an A-vertex, which contradicts Lemma 1 since $|A| = 3$ and $A - \{a_i\} \subseteq N(z)$. If $l < j$, then by Lemmas 2 and 3, we have $a_{j+1}, b_j \notin N(z)$, which implies $j = 3$. If $l = 1$, then by Lemma 3, we have $b_2, b_3 \notin N(z)$ and if $l = 2$, then by Lemmas 2 and 3 we have $a_1, b_3 \notin N(z)$, a contradiction. Thus we have $l > j$.

If $|B| = 3$, then since $b_1 z \in E(G)$, by Lemma 4 we have $j = 0$. Thus, if $l = 1$, then by Lemma 3 we have $b_2, b_3 \notin N(z)$; if $l = 2$, then by Lemmas 2 and 3, we have $a_1, b_3 \notin N(z)$; and if $l = 3$, then by Lemma 2, we have $a_1, a_2 \notin N(z)$, a contradiction. Thus, we have $|B| = 2$.

If $j = 2$, then $l = 3$. By Lemma 4 we have $b_2 z \notin E(G)$, which implies $a_1 = b_2 = a_i$. Let $Q = x x_P x_2 \overrightarrow{P} y$. Obviously, $|Q| = n - 1$ and $x_Q = a_1$. By the choice of P, we have $d(a_1) = 3$. If $N(a_1) \cap P_3 \neq \emptyset$, say $v \in N(a_1) \cap P_3$, then the (x, y)-path $x x_P x_3 \overleftarrow{P} z^+ a_3 \overrightarrow{P} v^- z \overleftarrow{P} a_1 v \overrightarrow{P} y$ is hamiltonian, and hence $N(a_1) \cap P_3 = \emptyset$. If $a_1 x_3 \in E(G)$, then since $d(a_1) = 3$, we have $N(a_1) = X$, which contradicts $\tau(G) > 1$. Thus, Q is an (x, y)-path of length $n - 2$ with $s(Q) < s(P)$, which contradicts the choice of P. If $j = 1$ and $l = 2$, then by Lemma 3 we have $b_3 z \notin E(G)$, which implies $a_2 = b_3 = a_i$. This contradicts Lemma 1 since $z b_2 \in E(G)$, which implies z^+ is a B-vertex. If $j = 1$ and $l = 3$, then by Lemma 2 we have $z a_2 \notin E(G)$, which implies $a_2 = a_i$. If $N(a_2) \cap P_3 \neq \emptyset$, say $v \in N(a_2) \cap P_3$, then the (x, y)-path $x \overrightarrow{P} z v^- \overleftarrow{P} a_3 z^+ \overrightarrow{P} x_2 x_P x_3 \overleftarrow{P} a_2 v \overrightarrow{P} y$ is hamiltonian, and hence $N(a_2) \cap P_3 = \emptyset$. If $a_2 = b_3$, then we have $d(a_2) = 3$ and $x a_2 \in E(G)$ for otherwise we can choose $R = x \overrightarrow{P} x_2 x_P x_3 \overrightarrow{P} y$ replacing P. In this case, we have $N(a_2) = X$, which contradicts $\tau(G) > 1$. Thus we may assume $a_2 \neq b_3$. Let $S = x \overrightarrow{P} z a_2^+ \overrightarrow{P} x_3 x_P x_2 \overleftarrow{P} z^+ a_3 \overrightarrow{P} y$. Then S is an (x, y)-path of length $n - 2$ with $x_S = a_2$. Noting that $N(a_2) \cap P_3 = \emptyset$, by the choice of P, we have $d(a_2) = 3$ and $x a_2 \in E(G)$. In this case, $N(a_2) = \{x_1, x_2, a_2^+\}$. Since $a_2 \neq b_3$, we have $a_2^+ \neq x_3$ and hence $s(S) < s(P)$, a contradiction. Thus, we have $a_l z^+ \notin E(G)$ for any $a_l \in A$, and hence $A \cup \{x_P, z^+\}$ is an independent set. ∎

Claim 2. Let $v \in P_i$, where $1 \leq i \leq 3$. If $a_i v^+ \in E(G)$, then $a_i v \in E(G)$.

Proof. Since $v^+ a_i \in E(G)$, v is an A-vertex. If $a_i v \notin E(G)$, then by Lemma 1, $A \cup \{x_P, v\}$ is an independent set of order 5, a contradiction. ∎

Claim 3. If $z \in P_1$ and $[a_2, z] \to x_P$, then $B \cup \{x_P, z^-\}$ is an independent set.

Proof. If $z^- b_2 \in E(G)$, then z is a B-vertex. By Lemma 1, $z b_3 \notin E(G)$, which implies $a_2 b_3 \in E(G)$. By Claim 2, $P_2 \subseteq N[a_2]$. If $a_1 x_2 \in E(G)$, then z is an A-vertex, which contradicts Lemma 1 since $z a_3 \in E(G)$. If $a_1 x_3 \in E(G)$, then the (x, y)-path $x \overrightarrow{P} x_1 x_P x_2 \overrightarrow{P} x_3 a_1 \overrightarrow{P} z^- b_2 \overleftarrow{P} z a_3 \overrightarrow{P} y$ is hamiltonian. Thus, we have $x_2, x_3 \notin N(a_1)$. Since

$x_Pa_3 \notin E(G)$, there is some vertex w such that $[x_P, w] \to a_3$ or $[a_3, w] \to x_P$. If $[a_3, w] \to x_P$, then by Lemma 6 we have $w \in P_2$ or $w = y$. Since $P_2 \subseteq N[a_2]$, we see that each vertex of $P_2 - \{b_3\}$ is an A-vertex. Thus, if $w \in P_2$, then we have $w = b_3$, which contradicts (2), and hence we have $w = y$. If $[x_P, w] \to a_3$, then since $x_2, x_3 \notin N(a_1)$, we have $w \notin X$ by Lemma 12. Thus, by Claim 1, we have $w = y$. In both cases, $y \neq a_3$ and $a_1y \in E(G)$. By Lemma 4, $zy^- \notin E(G)$ and hence $a_2y^- \in E(G)$. Thus, $R = x\overrightarrow{P}x_1x_Px_3\overleftarrow{P}a_2y^-\overleftarrow{P}a_3z\overrightarrow{P}b_2z^-\overleftarrow{P}a_1y$ is an (x, y)-path of length $n - 2$ with $x_R = x_2$. Since $z \in P_1$ and $|A| = 3$, we have $|P_1| \geq 2$. By (1), $d(x_R) = d(x_2) \geq 4$, which contradicts the choice of P. Therefore, $z^-b_2 \notin E(G)$.

If $z^-b_3 \in E(G)$, then by Lemma 1 we have $a_2x_3 \notin E(G)$ since $a_1z \in E(G)$, which implies z^- is an A-vertex. If $a_2x_1 \in E(G)$, then $x\overrightarrow{P}x_1a_2\overrightarrow{P}b_3z^-\overleftarrow{P}a_1z\overrightarrow{P}x_2x_Px_3\overrightarrow{P}y$ is a hamiltonian (x, y)-path. Thus, we have $x_1, x_3 \notin N(a_2)$. Since $z^-b_2 \notin E(G)$, we have $z \neq b_2$. If $a_1b_2 \in E(G)$, then by Claim 2, z is an A-vertex, which contradicts Lemma 1 since $za_3 \in E(G)$, and hence $a_1b_2 \notin E(G)$. Thus, there is some vertex w such that $[a_1, w] \to b_2$ or $[b_2, w] \to a_1$. It is easy to see $w \neq x_P$. Thus, in order to dominate x_P, we have $w \in X$. If $[a_1, w] \to b_2$, then $w \neq x_2$. Noting that $x_1, x_3 \notin N(a_2)$, we can see that $w \neq x_1, x_3$. Thus, we have $[b_2, w] \to a_1$. Obviously, $w \neq x_1$. If $w = x_2$, then $x_2b_3 \in E(G)$. By Lemma 3, $a_2b_2 \notin E(G)$. Since $a_2x_3 \notin E(G)$, we have $zx_3 \in E(G)$. If $b_2a_3 \in E(G)$, then the (x, y)-path $x\overrightarrow{P}z^-b_3\overleftarrow{P}x_2x_Px_3z\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is hamiltonian, and hence $b_2a_3 \notin E(G)$. Thus, $A \cup \{b_2, x_P\}$ is an independent set of order 5, a contradiction. Hence, $w \neq x_2$, which implies $w = x_3$, that is, $[b_2, x_3] \to a_1$. In this case, $a_2b_2 \in E(G)$ since $a_2x_3 \notin E(G)$. By Lemma 5, there is some vertex $u \in P_2$ such that $b_2u, u^+x_3 \in E(G)$. Thus, the (x, y)-path $x\overrightarrow{P}z^-b_3\overleftarrow{P}u^+x_3x_Px_2\overrightarrow{P}ub_2\overleftarrow{P}za_3\overrightarrow{P}y$ is hamiltonian, a contradiction. Hence, $z^-b_3 \notin E(G)$.

Since $za_3 \in E(G)$, we have $b_1z^- \notin E(G)$ by Lemma 4 if $|B| = 3$ and $z \notin A$ by Lemma 1, which implies $z^-x_P \notin E(G)$. Thus $B \cup \{x_P, z^-\}$ is an independent set. ∎

Claim 4. If $z \in P_2$ and $[a_3, z] \to x_P$, then $B \cup \{x_P, z^-\}$ is an independent set.

Proof. Since $za_1 \in E(G)$, we have $b_2z^- \notin E(G)$ by Lemma 3. Since $z \in P_2$ and $za_1 \in E(G)$, by Lemma 1, $|P_2| \geq 2$. By (1), $d(x_3) \geq 4$ and $d(x_1) \geq 4$ if $|B| = 3$. If $z^-b_1 \in E(G)$ or $z^-b_3 \in E(G)$, then by Lemma 2, we have $zb_2 \notin E(G)$, and hence $b_2a_3 \in E(G)$. Thus, $Q = x\overrightarrow{P}b_1z^-\overleftarrow{P}x_2x_Px_3\overleftarrow{P}za_1\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is an (x, y)-path of length $n - 2$ with $x_Q = x_1$ if $z^-b_1 \in E(G)$ and $R = x\overrightarrow{P}x_1x_Px_2\overrightarrow{P}z^-b_3\overleftarrow{P}za_1\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is an (x, y)-path of length $n - 2$ with $x_R = x_3$ if $z^-b_3 \in E(G)$, which contradicts the choice of P. Hence, we have $z^-b_1, z^-b_3 \notin E(G)$. Since $za_1 \in E(G)$, by Lemma 1 we have $z \notin A$, and hence $z^-x_P \notin E(G)$. Thus, $B \cup \{x_P, z^-\}$ is an independent set. ∎

Since $|A| = 3$, by Lemma 10, there are some vertices a_i with $i \geq 2$ and $z \notin X$ such that $[x_P, z] \to a_i$ or $[a_i, z] \to x_P$. If $|B| = 3$, then by Lemma 7 and Claim 1, we have $[a_i, z] \to x_P$. By Lemma 6, we have $z \in P_{i-1}$. Thus, by Claims 3 and 4, we see $B \cup \{x_P, z^-\}$ is an independent set of order 5, a contradiction. Hence we have $|B| = 2$.

Claim 5. If $[x_P, y] \to a_i$, then $B \cup \{x_P, y^-\}$ is an independent set.

Proof. Since $|A| = 3$ and $A - \{a_i\} \subseteq N(y)$, by Lemma 1 we have $y \neq a_3$, which implies $y^- x_P \notin E(G)$. If $a_i \neq a_1$, then by Lemma 3, we have $b_2, b_3 \notin N(y^-)$. If $a_i = a_1$, then we have $b_3, a_2 \in N(y)$. By Lemmas 2 and 3, we have $b_2, b_3 \notin N(y^-)$. Thus, $B \cup \{x_P, y^-\}$ is an independent set. ∎

Claim 6. If $[a_2, z] \to x_P$, then $z = y$.

Proof. By Lemma 6, we have $z \in P_1$ or $z = y$. If $z \neq y$, then $z \in P_1$. Since $x_P a_3 \notin E(G)$, there is some vertex w such that $[x_P, w] \to a_3$ or $[a_3, w] \to x_P$. If $w = y$, then by Lemma 6 or Claim 5, $B \cup \{x_P, y^-\}$ is an independent set. If $z^- y^- \in E(G)$, then the (x, y)-path $x_1 x_P x_2 \overleftarrow{P} z a_1 \overrightarrow{P} z^- y^- \overleftarrow{P} a_2 y$ is hamiltonian, and hence $z^- y^- \notin E(G)$. Thus, by Claim 3, we can see that $B \cup \{x_P, y^-, z^-\}$ is an independent set of order 5, and hence $w \neq y$. If $[x_P, w] \to a_3$, then by Claim 1, we have $w \in \{x_1, x_2\}$. By Lemma 12, we have $a_1 x_2 \in E(G)$. By Claim 2, z is an A-vertex, which contradicts Lemma 1 since $z a_3 \in E(G)$. Thus, we have $[a_3, w] \to x_P$. By Lemma 6, we have $w \in P_2$. By Claim 4, $B \cup \{x_P, w^-\}$ is an independent set. Noting that z^- and w^- are A-vertices, we have $z^- w^- \notin E(G)$ by Lemma 1. Thus, by Claim 3, $B \cup \{x_P, w^-, z^-\}$ is an independent set of order 5, a contradiction. ∎

Claim 7. If $[a_2, y] \to x_P$ or $[x_P, y] \to a_2$, then $a_3 y, a_1 b_2, a_2 b_3 \in E(G)$.

Proof. By Lemma 1, $a_3 y \in E(G)$. Thus, y^- is an A-vertex. By Lemma 6 or Claim 5, $B \cup \{x_P, y^-\}$ is an independent set. If $a_1 b_2 \notin E(G)$ or $a_2 b_3 \notin E(G)$, then $a_2 b_2 \in E(G)$ for otherwise $\{x_P, a_1, b_2, a_2, y^-\}$, or $\{x_P, b_3, b_2, a_2, y^-\}$ is an independent set and $a_1 b_3 \in E(G)$ for otherwise $\{x_P, a_1, b_2, b_3, y^-\}$, or $\{x_P, b_3, a_1, a_2, y^-\}$ is an independent set, which contradicts $\alpha(G) = 4$. Thus, by Lemmas 1 and 3, we have

$$a_1, b_3 \notin N(x_2) \text{ and } a_2, b_2 \notin N(x_1) \cup N(x_3). \tag{3}$$

If $a_1 b_2 \notin E(G)$, then there is some vertex w such that $[a_1, w] \to b_2$ or $[b_2, w] \to a_1$. Obviously, $w \neq x_P$. Thus, in order to dominate x_P, we have $w \in X$. By (3), we have $[b_2, x_3] \to a_1$. By Lemma 5, there is some vertex $v \in P_2$ such that $b_2 v, x_3 v^+ \in E(G)$, which implies the (x, y)-path $x_1 x_P x_2 \overrightarrow{P} v b_2 \overleftarrow{P} a_1 b_3 \overleftarrow{P} v^+ x_3 \overrightarrow{P} y$ is hamiltonian, and hence $a_1 b_2 \in E(G)$. If $a_2 b_3 \notin E(G)$, then there is some vertex u such that $[a_2, u] \to b_3$ or

$[b_3, u] \to a_2$. Clearly, $u \neq x_P$, and hence $u \in X$. By (3), we have $[a_2, x_1] \to b_3$. By Lemma 5, there is some vertex $v \in P_1$ such that $x_1 v, a_2 v^+ \in E(G)$, which implies the (x, y)-path $x_1 v \overleftarrow{P} a_1 b_3 \overleftarrow{P} a_2 v^+ \overrightarrow{P} x_2 x_P x_3 \overrightarrow{P} y$ is hamiltonian, and hence $a_2 b_3 \in E(G)$. ▮

Claim 8. If $[x_P, z] \to a_2$ and $z \in \{x_1, x_3\}$, then $a_1 b_2, a_2 b_3 \in E(G)$.

Proof. By Lemma 12, we have $a_1 x_3 \in E(G)$. By Lemma 3, we have $b_2, b_3 \notin N(a_3)$. If $a_1 b_2 \notin E(G)$ or $a_2 b_3 \notin E(G)$, then $a_1 b_3 \in E(G)$ for otherwise $\{x_P, a_1, b_2, b_3, a_3\}$, or $A \cup \{x_P, b_3\}$ is an independent set of order 5. Thus by Lemmas 2 and 3, we have $b_2 \notin N(x_1) \cup N(x_3)$, which contradicts $z \in \{x_1, x_3\}$. ▮

Claim 9. If $[x_P, z] \to a_2$ and $z \in \{x_1, x_3\}$, then $a_3 y \in E(G)$.

Proof. Since $x_P a_3 \notin E(G)$, there is some vertex w such that $[x_P, w] \to a_3$ or $[a_3, w] \to x_P$. If $[x_P, w] \to a_3$, then since $z \in X$, by Lemma 10 we have $w \notin X$. By Claim 1, $w = y$. If $[a_3, w] \to x_P$, then by Lemma 6, $w \in P_2$ or $w = y$. If $w \in P_2$, then by Claims 2 and 8, we have $w = b_3$, which contradicts (2). Thus, we have $w = y$ in both cases. By Lemma 6 or Claim 5, $B \cup \{x_P, y^-\}$ is an independent set. If $a_3 y \notin E(G)$, then since $z \in X$, by Lemma 10, there is some vertex $u \in V(G) - N[x_P]$ such that $[x_P, u] \to a_1$ or $[a_1, u] \to x_P$. Since $a_3 y \notin E(G)$, by Claim 1, we can see that $[x_P, u] \to a_1$ is impossible. Thus, we have $[a_1, u] \to x_P$. If $u \in B$, say $u = b_i$, then since $b_i a_3, a_1 y^- \in E(G)$, by Lemma 5 there is some vertex $v \in P_3 - \{y\}$ such that $b_i v, a_1 v^+ \in E(G)$, which contradicts Lemma 3. Thus, in order to dominate a_3, we have $u \in P_3 - \{y\}$ by Claims 2 and 8. Since $a_2 u \in E(G)$, by Lemma 2, $a_3 u^+ \notin E(G)$. If $a_1 u^+ \in E(G)$ or $a_2 u^+ \in E(G)$, then by Lemma 3, $b_3 u \notin E(G)$, which implies $a_1 b_3 \in E(G)$. Thus, by Lemmas 2 and 3, we have $b_2 \notin N(x_1) \cup N(x_3)$, which contradicts $z \in \{x_1, x_3\}$. Hence, $a_1, a_2 \notin N(u^+)$, which implies $A \cup \{x_P, u^+\}$ is an independent set of order 5, a contradiction. Thus, we have $a_3 y \in E(G)$. ▮

Since $x_P a_2 \notin E(G)$, there is some vertex z such that $[x_P, z] \to a_2$ or $[a_2, z] \to x_P$. If $[a_2, z] \to x_P$, then $z = y$ by Claim 6. By Claim 7, we have $a_3 y, a_1 b_2, a_2 b_3 \in E(G)$. If $[x_P, z] \to a_2$, then by Claim 1, we have $z \in \{x_1, x_3, y\}$. Thus, by Claims 7, 8 and 9, we have $a_3 y, a_1 b_2, a_2 b_3 \in E(G)$. Hence, by Claim 2, we have

$$P_i \subseteq N[a_i] \text{ for } i = 1, 2, 3. \tag{4}$$

If $z = y$, then by Lemma 1 and (4), we have $P_3 \subseteq N[y]$. If $z \neq y$, then by Claims 1 and 6, we have $[x_P, z] \to a_2$ and $z \in \{x_1, x_3\}$. Since $x_P a_3 \notin E(G)$, there is some vertex u such that $[x_P, u] \to a_3$ or $[a_3, u] \to x_P$. If $u \neq y$, then Lemma 10 and Claim 1, we have $[a_3, u] \to x_P$. By Lemma 6, we have $u \in P_2$. By (4), we have $u = b_3$, which contradicts (2). If $u = y$, then by Lemma 6, $B \cup \{y^-\}$ is an independent set. Since $a_1 x_P \notin E(G)$, there is some vertex w such that $[a_1, w] \to x_P$ or $[x_P, w] \to a_1$. Since $z \in X$, by Lemma 10, $w \notin X$. If $w = y$, then by Lemma 1 and (4), $P_3 \subseteq N[y]$. If $w \neq y$, then by Claim 1, we have $[a_1, w] \to x_P$. In order to dominate a_2, a_3, we have $w \in B$, which is impossible since $\{a_1, w\} \not\succ y^-$. Therefore, we have

$$P_3 \subseteq N[y]. \tag{5}$$

Let w be a vertex such that $[x_P, w] \to a_3$ or $[a_3, w] \to x_P$. If $z \in X$, then by Lemma 10, Claim 1 and (4), we have $[a_3, w] \to x_P$. By Lemma 6, we have $w \in P_2$ or $w = y$. By (2) and (4), we have $w = y$. If $z \notin X$, then by Claims 1 and 6, we have $z = y$. Thus, we have

$$\text{either } w = y \text{ or } z = y. \tag{6}$$

By (6), we have $y \neq a_3$, which implies $y^- x_P \notin E(G)$. Let v be a vertex such that $[x_P, v] \to y^-$ or $[y^-, v] \to x_P$. By Lemma 6, Claim 5 and (6), $B \cup \{x_P, y^-\}$ is an independent set. By (4), y^- is an A-vertex. Thus, by Lemma 1 and (4), we have $N(y^-) \cap P_i = \emptyset$ for $i = 1, 2$. If $[y^-, v] \to x_P$, then we must have $v = y$, which implies $\{x_P, y\} \succ V(G)$ by (5), a contradiction. Thus, we have $[x_P, v] \to y^-$. By (4), we have $v \in X$. If $y^- = a_3$, then by Lemma 12, we have $N(a_3) \cap \{x_1, x_2\} = \emptyset$, which implies $d(a_3) = 2$, a contradiction. Thus, we have $y^- \neq a_3$. In this case, $y^- \notin A$. By Lemmas 8 and 9, we may assume $[a_{i_l}, x_{j_l}] \to a_{i_{l+1}}$ for $l = 1, 2$ and $X - \{x_{j_1}, x_{j_2}\} = \{x_{j_3}\}$. This implies $v = x_{j_3}$. Since y^- is an A-vertex, we have $y^- a_{i_1} \notin E(G)$ or $y^- a_{i_2} \notin E(G)$, which implies either $y^- x_{j_1} \in E(G)$ or $y^- x_{j_2} \in E(G)$. Thus, since $x_{j_1} x_{j_2} \in E(G)$, we can see that either $\{x_{j_1}, x_{j_3}\} \succ V(G)$ or $\{x_{j_2}, x_{j_3}\} \succ V(G)$, a contradiction.

Case 2. $|A| = 2$

In this case, our main idea is to prove that P_i is a clique for $i = 1, 2$. In order to do this, we first show that either $a_1 b_2 \in E(G)$ or $a_2 b_3 \in E(G)$ and then $a_1 b_2, a_2 b_3 \in E(G)$.

If $|P_i| = 1$ for some $i \in \{1, 2\}$, then by the choice of P, we have $N(a_i) = X$, which contradicts $\tau(G) > 1$. Thus, we have $|P_i| \geq 2$ for $i = 1, 2$, which implies $b_2^-, a_2^+ \notin X$. Noting that $a_2, b_2 \in N(x_2)$, by the choice of P, we see that

$$\text{there is no } (x, y)\text{-path } Q \text{ such that } x_Q = a_2 \text{ or } b_2. \tag{7}$$

Claim 10. If $a \in P_1$ is an A-vertex, then $a a_2^+ \notin E(G)$, and if $b \in P_2$ is a B-vertex, then $b b_2^- \notin E(G)$.

Proof. Let Q be a hamiltonian (a, x_2)-path in $G[P_1 \cup \{x_2\}]$. If $a a_2^+ \in E(G)$, then

$R = x_1 x_P x_2 \overleftarrow{Q} a a_2^+ \overrightarrow{P} x_3$ is an (x, y)-path of length $n-2$ with $x_R = a_2$, which contradicts (7). As for the latter part, the proof is similar. ∎

Claim 11. If $a \in P_2$ is an A-vertex and $aa_1^+ \in E(G)$, then $N(a_1) = \{x_1, x_3, a_1^+\}$. Similarly, if $b \in P_1$ is a B-vertex and $bb_3^- \in E(G)$, then $N(b_3) = \{x_1, x_3, b_3^-\}$.

Proof. Let Q be a hamiltonian (a, x_3)-path in $G[P_2 \cup \{x_3\}]$. If $aa_1^+ \in E(G)$, then $R = x_1 x_P x_2 \overleftarrow{P} a_1^+ a \overrightarrow{Q} x_3$ is an (x, y)-path of length $n - 2$ with $x_R = a_1$. By the choice of P, we have $d(a_1) = 3$ and $x_1, x_3 \in N(a_1)$, which implies $N(a_1) = \{x_1, x_3, a_1^+\}$. As for the latter part, the proof is similar. ∎

Let $a \in P_1 - \{b_2\}$ and $b \in P_2 - \{a_2\}$. Suppose P' is an (a, b_2^-)-path with $V(P') = P_1 - \{b_2\}$ and P'' an (a_2^+, b)-path with $V(P'') = P_2 - \{a_2\}$. We have the following two claims.

Claim 12. If $(N(x_1) \cup N(x_3)) \cap \{b_2^-, a_2^+\} \neq \emptyset$, then $ab \notin E(G)$.

Proof. By symmetry, we may assume $N(x_1) \cap \{b_2^-, a_2^+\} \neq \emptyset$. If $ab \in E(G)$, then $Q = x_1 b_2^- \overleftarrow{P'} ab \overleftarrow{P''} a_2^+ a_2 x_2 x_P x_3$ is an (x, y)-path of length $n - 2$ with $x_Q = b_2$ if $x_1 b_2^- \in E(G)$, and $R = x_1 a_2^+ \overrightarrow{P''} ba \overrightarrow{P'} b_2^- b_2 x_2 x_P x_3$ is an (x, y)-path of length $n - 2$ with $x_R = a_2$ if $x_1 a_2^+ \in E(G)$, which contradicts (7). ∎

Claim 13. If $v \in P_2$ and $av \in E(G)$, then $v^+, v^- \notin N(b_2^-)$ and if $u \in P_1$ and $bu \in E(G)$, then $u^+, u^- \notin N(a_2^+)$.

Proof. If $v^+ b_2^- \in E(G)$, then $Q = x_1 x_P x_2 \overrightarrow{P} va \overrightarrow{P'} b_2^- v^+ \overrightarrow{P} x_3$ is an (x, y)-path of length $n - 2$ with $x_Q = b_2$ and if $v^- b_2^- \in E(G)$, then $R = x_1 x_P x_2 \overrightarrow{P} v^- b_2^- \overleftarrow{P'} av \overrightarrow{P} x_3$ is an (x, y)-path of length $n - 2$ with $x_R = b_2$, which contradicts (7). As for the latter part, the proof is similar. ∎

Claim 14. If $a_2 b_2 \in E(G)$ and $[a_1, x_2] \to b_3$, then $P_2 - \{b_3\} \subseteq N(x_2)$ and $N(b_3) = \{x_1, x_3, b_3^-\}$.

Proof. If $v \in P_2$ and $a_1 v \in E(G)$, then by Lemma 5, there is some vertex $u \in a_2 \overrightarrow{P} v$ such that $ux_2, u^+ a_1 \in E(G)$. Thus, $x_1 x_P x_2 u \overleftarrow{P} a_2 b_2 \overleftarrow{P} a_1 u^+ \overrightarrow{P} x_3$ is a hamiltonian (x, y)-path, and hence $N(a_1) \cap P_2 = \emptyset$, which implies $P_2 - \{b_3\} \subseteq N(x_2)$. On the other hand, since $Q = x_1 \overrightarrow{P} b_2 a_2 \overrightarrow{P} b_3^- x_2 x_P x_3$ is an (x, y)-path of length $n - 2$ with $x_Q = b_3$, by the choice of P, we have $d(b_3) = 3$ and $x_1 \in N(b_3)$, which implies $N(b_3) = \{x_1, x_3, b_3^-\}$. ∎

Claim 15. If $a_1 b_2, a_2 b_3 \notin E(G)$, then either $a_1 b_3 \in E(G)$ or $a_2 b_2 \in E(G)$.

Proof. Otherwise, $\{x_P, a_1, a_2, b_2, b_3\}$ is an independent set of order 5 by Lemma 1, a contradiction. ∎

Assume $a_1b_2, a_2b_3 \notin E(G)$. Let z be a vertex such that $[a_1, z] \to b_2$ or $[b_2, z] \to a_1$. Obviously, $z \neq x_P$. In order to dominate x_P, we have $z \in X$. It is easy to check that there are four cases: $[a_1, x_1] \to b_2$, $[a_1, x_3] \to b_2$, $[b_2, x_2] \to a_1$ or $[b_2, x_3] \to a_1$, and at least one of the four cases occurs.

If $[a_1, x_1] \to b_2$, then by Lemma 1, $x_1a_2 \in E(G)$. By Lemma 3, $a_1b_3 \notin E(G)$. By Claim 15, $a_2b_2 \in E(G)$. By Lemma 3, $a_1, b_3 \notin N(x_2)$. Consider $a_2b_3 \notin E(G)$, we can easily get that $[b_3, x_3] \to a_2$. Thus, consider $a_1b_3 \notin E(G)$, we have $[a_1, x_2] \to b_3$ or $[b_3, x_2] \to a_1$. Since $[a_1, x_1] \to b_2$ and $[b_3, x_3] \to a_2$, by symmetry, we may assume that $[a_1, x_2] \to b_3$. By Claim 14, $P_2 - \{b_3\} \subseteq N(x_2)$ and $N(b_3) = \{x_1, x_3, b_3^-\}$. Thus, we have $P_1 \subseteq N(x_3)$ since $[b_3, x_3] \to a_2$. Since $[a_1, x_1] \to b_2$ and $a_1 \notin N(x_2)$, we have $x_1x_2 \in E(G)$. Therefore, we have $\{x_2, x_3\} \succ V(G)$, a contradiction. Hence, $[a_1, x_1] \to b_2$ is impossible. By symmetry, $[b_3, x_3] \to a_2$ is impossible.

If $[a_1, x_3] \to b_2$, then $a_2x_3 \in E(G)$, which implies b_3 is an A-vertex. By Lemma 1, $a_1b_3 \notin E(G)$. By Claim 15, $a_2b_2 \in E(G)$. By Lemma 3, $a_1, b_3 \notin N(x_2)$. Consider $a_2b_3 \notin E(G)$, we have $[a_2, x_1] \to b_3$ or $[b_3, x_1] \to a_2$. If $[a_2, x_1] \to b_3$, then $x_1b_3 \notin E(G)$. In this case, consider $a_1a_2 \notin E(G)$, we have $[a_2, x_3] \to a_1$. Thus, by Claim 11, $a_1^+a_2, a_1^+b_3 \notin E(G)$. Now, consider $a_1^+a_2 \notin E(G)$. It is not difficult to check that there is no vertex w such that $[a_1^+, w] \to a_2$ or $[a_2, w] \to a_1^+$, a contradiction. If $[b_3, x_1] \to a_2$, then consider $a_1b_3 \notin E(G)$, we have $[a_1, x_2] \to b_3$ or $[b_3, x_2] \to a_1$. Since $[a_1, x_3] \to b_2$ and $[b_3, x_1] \to a_2$, by symmetry, we may assume that $[a_1, x_2] \to b_3$. By Claim 14, $P_2 - \{b_3\} \subseteq N(x_2)$ and $N(b_3) = \{x_1, x_3, b_3^-\}$. Since $[b_3, x_1] \to a_2$, we have $P_1 \subseteq N(x_1)$. Since $[a_1, x_3] \to b_2$ and $a_1x_2 \notin E(G)$, we have $x_2x_3 \in E(G)$. Thus, we have $\{x_1, x_2\} \succ V(G)$, a contradiction. Hence, $[a_1, x_3] \to b_2$ is impossible. By symmetry, $[b_3, x_1] \to a_2$ is impossible.

If $[b_2, x_2] \to a_1$, then $x_2b_3 \in E(G)$. By Lemma 3, $a_2b_2 \notin E(G)$. By Claim 15, $a_1b_3 \in E(G)$. By Lemmas 2 and 3, $a_2, b_2 \notin N(x_1) \cup N(x_3)$. In this case, it is not difficult to see that there is no vertex w such that $[a_2, w] \to b_3$ or $[b_3, w] \to a_2$, a contradiction. Thus, $[b_2, x_2] \to a_1$ is impossible. By symmetry, $[a_2, x_2] \to b_3$ is impossible.

If $[b_2, x_3] \to a_1$, then by Lemma 5, there is some vertex $u \in P_2$ such that $ub_2, u^+x_3 \in E(G)$. If $a_1b_3 \in E(G)$, then $x_1x_Px_2\overrightarrow{P}ub_2\overleftarrow{P}a_1b_3\overleftarrow{P}u^+x_3$ is a hamiltonian (x, y)-path, and hence $a_1b_3 \notin E(G)$. By Claim 15, $a_2b_2 \in E(G)$. By Lemma 3, $a_1, b_3 \notin N(x_2)$. Consider $a_2b_3 \notin E(G)$. Since $[b_3, x_3] \to a_2$, $[b_3, x_1] \to a_2$ and $[a_2, x_2] \to b_3$ are impossible, we have $[a_2, x_1] \to b_3$. If $a_1^+a_2^+ \in E(G)$, then $Q = x_1x_Px_2a_2b_2\overleftarrow{P}a_1^+a_2^+\overrightarrow{P}x_3$ is an (x, y)-path of length $n - 2$ with $x_Q = a_1$, which contradicts the choice of P since $a_1x_3 \notin E(G)$. By Claim 11, $a_1^+a_2 \notin E(G)$. Consider $a_1^+a_2 \notin E(G)$, we have

$[a_1^+, x_1] \to a_2$ or $[a_1^+, x_3] \to a_2$. If $[a_1^+, x_1] \to a_2$, then $a_1^+ b_3, x_1 a_2^+ \in E(G)$, which implies $R = x_1 a_2^+ \overrightarrow{P} b_3 a_1^+ \overrightarrow{P} b_2 a_2 x_2 x_P x_3$ is an (x, y)-path of length $n-2$ with $x_R = a_1$, a contradiction. Hence, we have $[a_1^+, x_3] \to a_2$. Since $[b_2, x_3] \to a_1$ and $[a_2, x_1] \to b_3$, by symmetry, we have $[b_3^-, x_1] \to b_2$. Thus, $x_1 b_2, a_2 x_3 \notin E(G)$. Now, consider $a_1 b_3 \notin E(G)$, we have $[a_1, x_2] \to b_3$ or $[b_3, x_2] \to a_1$. By symmetry, we may assume that $[a_1, x_2] \to b_3$. By Claim 14, $x_1 b_3 \in E(G)$, which contradicts $[a_2, x_1] \to b_3$. Therefore, $[b_2, x_3] \to a_1$ is impossible.

It follows from the argument above that either $a_1 b_2 \in E(G)$ or $a_2 b_3 \in E(G)$.

Since $a_1 b_2 \in E(G)$ or $a_2 b_3 \in E(G)$, by symmetry, we may assume $a_1 b_2 \in E(G)$. If $a_2 b_3 \notin E(G)$, then there is some vertex z such that $[a_2, z] \to b_3$ or $[b_3, z] \to a_2$. Obviously, $z \neq x_P$ and hence $z \in X$. It is not difficult to see that there are four cases: $[a_2, x_1] \to b_3$, $[a_2, x_2] \to b_3$, $[b_3, x_1] \to a_2$ or $[b_3, x_3] \to a_2$, and at least one of the four cases occurs.

In order to prove $a_2 b_3 \in E(G)$, we need the following four claims.

Claim 16. If $a_2 b_3 \notin E(G)$, then $P_1 \subseteq N[a_1]$ and $N(b_3) \cap P_1 = \emptyset$.

Proof. If $[a_2, x_1] \to b_3$ or $[b_3, x_1] \to a_2$, then since $a_1 b_2 \in E(G)$, we have $b_2^- x_1 \in E(G)$ by Lemma 1 and Claim 10. By Claim 12, $a_1 b_3 \notin E(G)$. By Claim 10, $b_2^- b_3 \notin E(G)$. If $a_1 b_2^- \notin E(G)$, then $\{a_1, b_2^-, a_2, b_3, x_P\}$ is an independent set of order 5, and hence $a_1 b_2^- \in E(G)$. If $P_1 \not\subseteq N[a_1]$, then since $a_1 b_2 \in E(G)$, there is some vertex $v \in P_1 - \{b_2^-, b_2\}$ such that $a_1 v \notin E(G)$ and $a_1 v^+ \in E(G)$. Clearly, v is an A-vertex. By Claim 12, $v b_3 \notin E(G)$. Thus, $\{a_1, v, a_2, b_3, x_P\}$ is an independent set of order 5, and hence $P_1 \subseteq N[a_1]$. Thus, by Lemma 1 and Claim 12, we have $N(b_3) \cap P_1 = \emptyset$.

If $[b_3, x_3] \to a_2$, then since $b_2 x_3 \in E(G)$, we have $a_1 b_3 \notin E(G)$ by Lemma 3. If $P_1 \not\subseteq N[a_1]$, we let $v \in P_1 - \{b_2\}$ such that $a_1 v \notin E(G)$ and $a_1 v^+ \in E(G)$. Clearly, v is an A-vertex. By Lemma 3, $v b_3 \notin E(G)$. Thus, $\{a_1, v, a_2, b_3, x_P\}$ is an independent set of order 5, and hence $P_1 \subseteq N[a_1]$. By Lemmas 1 and 3, we have $N(b_3) \cap P_1 = \emptyset$.

If $[a_2, x_2] \to b_3$, then $x_2 a_1 \in E(G)$, and hence b_2 is an A-vertex. By Lemma 1, $b_2 a_2 \notin E(G)$. If $N(a_2) \cap P_1 \neq \emptyset$, then since $a_1 a_2 \notin E(G)$, there is some vertex $u \in P_1$ such that $u^- a_2 \notin E(G)$ and $u a_2 \in E(G)$. Obviously, $u^- x_2 \in E(G)$. This contradicts Lemma 3, since $a_1 b_2 \in E(G)$ implies there is a (u, u^-)-path P' with $V(P') = V(P_1)$. Thus, $N(a_2) \cap P_1 = \emptyset$, and hence $P_1 \subseteq N(x_2)$. If $P_1 \not\subseteq N[b_2]$, then since $a_1 b_2 \in E(G)$, there is some vertex $u \in P_1$ such that $u^- b_2 \in E(G)$ and $u b_2 \notin E(G)$. Obviously, u is a B-vertex. Thus, $\{u, b_2, a_2, b_3, x_P\}$ is an independent set of order 5, a contradiction. Hence, $P_1 \subseteq N[b_2]$. By Lemma 1, $N(b_3) \cap a_1^+ \overrightarrow{P} b_2 = \emptyset$. If $a_1 b_3 \in E(G)$, then by Claims 10 and 13, we have $b_2 a_2^+ \notin E(G)$ and $a_2^+, b_2^- \notin N(x_1) \cup N(x_3)$. Thus, consider

$a_2b_2 \notin E(G)$, we cannot find a vertex w such that $[a_2, w] \to b_2$ or $[b_2, w] \to a_2$, and hence $a_1b_3 \notin E(G)$, which implies $N(b_3) \cap P_1 = \emptyset$. If $v \in P_1$ and $a_1v \notin E(G)$, then noting that $N(a_2) \cap P_1 = \emptyset$, $\{a_1, v, a_2, b_3, x_P\}$ is an independent set of order 5, and hence $P_1 \subseteq N[a_1]$. ∎

Claim 17. Let $z \in P_2$, $Q_1 = a_2\overrightarrow{P}z^-$ and $Q_2 = z^+\overrightarrow{P}b_3$. If $a_2b_3 \notin E(G)$ and $a_1, b_2^-, b_3 \in N(z)$, then Q_i is a clique for $i = 1, 2$ and $E(Q_1, Q_2) = \emptyset$.

Proof. By Lemma 1 and Claim 10, z is neither an A-vertex nor a B-vertex. Thus, $z \in P_2 - \{a_2, b_3\}$. By Claim 13, $a_1, b_2^- \notin N(z^+) \cup N(z^-)$. If $Q_2 \nsubseteq N[b_3]$, then since $zb_3 \in E(G)$, there is some vertex $v \in Q_2$ such that $vb_3 \notin E(G)$ and $v^-b_3 \in E(G)$. Obviously, v is a B-vertex. If $z^-v \in E(G)$ or $b_3z^- \in E(G)$, then z is a B-vertex, and hence $v, b_3 \notin N(z^-)$. Thus, by Claim 10, we can see that $\{b_2^-, z^-, v, b_3, x_P\}$ is an independent set of order 5, and hence $Q_2 \subseteq N[b_3]$. In this case, we have $N(z^-) \cap Q_2 = \emptyset$ for otherwise z is a B-vertex. If there are two vertices $u, v \in Q_2$ such that $uv \notin E(G)$, then since u and v are B-vertices, by Claim 10 we can see that $\{b_2^-, z^-, u, v, x_P\}$ is an independent set of order 5, and hence Q_2 is a clique. If $N(a_2) \cap Q_2 \neq \emptyset$, then since Q_2 is a clique, it is easy to see that z is an A-vertex. Thus, $N(a_2) \cap Q_2 = \emptyset$. If $a_2z^- \notin E(G)$, then $\{a_1, a_2, z^-, z^+, x_P\}$ is an independent set of order 5, and hence $a_2z^- \in E(G)$. If $Q_1 \nsubseteq N[a_2]$, then since $a_2z^- \in E(G)$, there is some vertex $v \in Q_1$ such that $va_2 \notin E(G)$ and $a_2v^+ \in E(G)$. Clearly, v is an A-vertex. If $vz^+ \in E(G)$, then z is an A-vertex, a contradiction. Thus, $\{a_1, a_2, v, z^+, x_P\}$ is an independent set of order 5, and hence $Q_1 \subseteq N[a_2]$. In this case, $N(z^+) \cap Q_1 = \emptyset$ for otherwise z is an A-vertex. If $u, v \in Q_1$ and $uv \notin E(G)$, then $\{a_1, u, v, z^+, x_P\}$ is an independent set of order 5, and hence Q_1 is a clique. If $v_i \in Q_i$ for $i = 1, 2$ and $v_1v_2 \in E(G)$, then $v_1 \neq a_2, z^-$, and hence $x_2a_2\overrightarrow{P}v_1^-z^-\overleftarrow{P}v_1v_2\overrightarrow{P}b_3v_2^-\overleftarrow{P}z$ is a hamiltonian (x_2, z)-path in $G[P_2 \cup \{x_2\}]$, which implies z is a B-vertex, a contradiction. Thus, we have $E(Q_1, Q_2) = \emptyset$. ∎

Claim 18. If $a_2b_3 \notin E(G)$, then for any $z \in P_2$, both $[x_P, z] \to a_2$ and $[a_2, z] \to x_P$ are impossible.

Proof. Suppose to the contrary that there is some vertex $z \in P_2$ such that $[x_P, z] \to a_2$ or $[a_2, z] \to x_P$. If $[x_P, z] \to a_2$, then $z \neq b_3$. If $[a_2, b_3] \to x_P$, then by Lemmas 1 and 5, there is some vertex $u \in P_1$ such that $ub_3, u^+a_2 \in E(G)$, which contradicts Lemma 3. Thus, we have $z \neq b_3$ in both cases. Let $P' = a_2\overrightarrow{P}z^-$ and $P'' = z^+\overrightarrow{P}b_3$. Since $a_1b_2 \in E(G)$, by Lemma 1, we have $b_2^-a_2 \notin E(G)$. Thus, $a_1, b_2^-, b_3 \in N(z)$. Since $za_1 \in E(G)$, by Lemma 3 and Claim 13, we have $b_2, b_2^- \notin N(z^-)$. By Lemma 1 and Claim 16, we have $a_1\overrightarrow{P}b_2^- \subseteq N(z)$. By Claim 17, $P'' \subseteq N(z)$. Since $\{b_2^-, a_2, b_3, x_P\}$ is a maximum independent set, by Lemma 10, there is some vertex $u \in \{b_2^-, b_3\}$ and a vertex $w \in V(G) - N[x_P]$ such that $[u,w] \to x_P$ or $[x_P, w] \to u$. If $[x_P, w] \to u$, then $w \neq z$. If $u = b_2^-$, then since $wb_3 \in E(G)$, by Claims 16 and 17, we have $w \in P''$ which is impossible since $wa_2 \notin E(G)$. If $u = b_3$, then $w \notin P''$ by Claim 17. Since $b_2 z^- \notin E(G)$, we have $w \neq b_2, z^-$. Thus, by Lemma 1 and Claims 16 and 17, we see that $w \notin P_1 \cup P_2$, a contradiction. Hence, we have $[u,w] \to x_P$. If $u = b_2^-$, then in order to dominate a_2 and b_3, we have $w = z$ by Lemma 1 and Claims 16 and 17. If $u = b_3$, then in order to dominate P_1 and P', it is easy to see that $w = z$ by Lemma 1 and Claims 16 and 17. In both cases, we have $P' \subseteq N(z)$ by Lemma 1 and Claim 17. Thus, we have $[a_2, z] \to x_P$. If $b_2 z \in E(G)$, then we have $\{x_P, z\} \succ V(G)$. If $b_2 z \notin E(G)$, then $a_2 b_2 \in E(G)$. By Lemma 3, $b_2^-, b_3 \notin N(x_2)$. If $z^- x_2 \in E(G)$, then $x_1 x_P x_2 z^- \overleftarrow{P} a_2 b_2 \overleftarrow{P} a_1 z \overrightarrow{P} x_3$ is a hamiltonian (x,y)-path. Thus, $b_2^-, z^-, b_3 \notin N(x_2)$. Noting that $\{b_2^-, z^-, b_3, x_P\}$ is an independent set, by Lemmas 8 and 9, we have $x_1, x_3 \in N(x_2)$, which implies $\{x_2, z\} \succ V(G)$, a contradiction. ▮

Claim 19. If $a_2 b_3 \notin E(G)$ and $x_1, x_3, b_2 \notin N(a_2)$, then $[x_P, a_2^+] \to a_1$ is impossible.

Proof. If $[x_P, a_2^+] \to a_1$, then $a_2^+ b_3 \in E(G)$. If $a_2^+ \overrightarrow{P} b_3 \not\subseteq N[b_3]$, then there is some vertex $v \in a_2^+ \overrightarrow{P} b_3$ such that $v^- b_3 \in E(G)$ and $vb_3 \notin E(G)$. Clearly, v is a B-vertex. By Claim 10, $v, b_3 \notin N(b_2^-)$. By Lemma 1 and Claim 16, $a_2 \notin N(b_2^-)$. If $a_2 v \in E(G)$, then it is easy to see that a_2^+ is a B-vertex, which contradicts Lemma 1 since $b_2 a_2^+ \in E(G)$. Thus, $\{b_2^-, a_2, v, b_3, x_P\}$ is an independent set of order 5, a contradiction. Hence, we have $a_2^+ \overrightarrow{P} b_3 \subseteq N[b_3]$, which implies $N(a_2) \cap a_2^+ \overrightarrow{P} b_3 = \{a_2^+\}$. Thus, noting that $x_1, x_3, b_2 \notin N(a_2)$, by Lemma 1 and Claim 16, we have $d(a_2) = 2$, a contradiction. Hence, $[x_P, a_2^+] \to a_1$ is impossible. ▮

We now begin to prove $a_2 b_3 \in E(G)$. Suppose to the contrary that $a_2 b_3 \notin E(G)$.

Since $x_P a_2 \notin E(G)$, there is some vertex z such that $[x_P, z] \to a_2$ or $[a_2, z] \to x_P$. By Claim 16, we have $z \notin P_1$. By Claim 18, we have $z \notin P_2$. Thus, we have $z \in X$. In this case, we have $[x_P, x_1] \to a_2$ or $[x_P, x_3] \to a_2$.

If $[a_2, x_1] \to b_3$, then $[x_P, x_1] \to a_2$ is impossible. If $[x_P, x_3] \to a_2$, then by Lemma 12, we have $x_1 x_3 \notin E(G)$, which is impossible since $a_2 x_3 \notin E(G)$ and $[a_2, x_1] \to b_3$. Thus, $[a_2, x_1] \to b_3$ is impossible. If $[a_2, x_2] \to b_3$, we let $\{i, j\} = \{1, 3\}$. If $[x_P, x_i] \to a_2$, then by Lemma 12, we have $x_2 x_i \in E(G)$. Since $[a_2, x_2] \to b_3$, we have $x_2 x_j \in E(G)$ or $a_2 x_j \in E(G)$, which implies $\{x_i, x_2\} \succ V(G)$ or $\{x_i, a_2\} \succ V(G)$, a contradiction. Thus, $[a_2, x_2] \to b_3$ is impossible. Therefore, we have $[b_3, x_1] \to a_2$ or $[b_3, x_3] \to a_2$.

By Claim 16, $\{x_P, a_1, a_2, b_3\}$ is a maximum independent set. Since $[x_P, x_1] \to a_2$ or $[x_P, x_3] \to a_2$, by Lemma 12, we have $x_1, x_3 \in N(a_1) \cap N(b_3)$ and $x_1, x_3 \notin N(a_2)$. If $[x_P, b_2] \to b_3$, then since $[b_3, x_1] \to a_2$ or $[b_3, x_3] \to a_2$, we have $\{b_2, x_3\} \succ V(G)$ or

$\{b_2, x_1\} \succ V(G)$ by Lemma 1, a contradiction. Obviously, $[x_P, b_3] \rightarrow b_2$ is impossible. Thus, there is some vertex $u \in X$ such that $[b_2, u] \rightarrow b_3$ or $[b_3, u] \rightarrow b_2$. Since $\{b_3, x_i\} \not\succ a_2$ for $i = 1, 3$ and $x_1, x_3 \in N(b_3)$, we have $[b_2, x_2] \rightarrow b_3$.

Since $[x_P, x_1] \rightarrow a_2$ or $[x_P, x_3] \rightarrow a_2$, by Lemma 12, we have $x_2x_3 \notin E(G)$ or $x_1x_2 \notin E(G)$. Noting that $[b_2, x_2] \rightarrow b_3$, by Lemma 1, we have $b_2x_3 \in E(G)$ or $b_2x_1 \in E(G)$. Thus, if $a_2b_2 \in E(G)$, then we have $\{b_2, x_1\} \succ V(G)$ or $\{b_2, x_3\} \succ V(G)$, and hence $a_2b_2 \notin E(G)$. Thus, we have $x_1, x_3, b_2 \notin N(a_2)$.

By Claim 10, $a_1a_2^+ \notin E(G)$. Since $x_1, x_3, b_2 \notin N(a_2)$, by Claim 19, $[x_P, a_2^+] \rightarrow a_1$ is impossible. Obviously, $[x_P, a_1] \rightarrow a_2^+$ is impossible. Thus, there is some vertex $w \in X$ such that $[a_1, w] \rightarrow a_2^+$ or $[a_2^+, w] \rightarrow a_1$. Since $[b_2, x_2] \rightarrow b_3$, we have $x_2b_3 \notin E(G)$. Thus, noting that $\{a_1, x_i\} \not\succ a_2$ for $i = 1, 3$, $\{a_1, x_2\} \not\succ b_3$ and $x_1, x_3 \in N(a_1)$, we have $[a_2^+, x_2] \rightarrow a_1$. If $[x_P, x_1] \rightarrow a_2$, then by Lemma 12, we have $x_1x_2 \in E(G)$ and $x_2x_3 \notin E(G)$. In this case, we have $a_2^+x_3 \in E(G)$, which implies $\{x_1, a_2^+\} \succ V(G)$, a contradiction. If $[x_P, x_3] \rightarrow a_2$, then by Lemma 12, we have $x_2x_3 \in E(G)$ and $x_1x_2 \notin E(G)$. In this case, we have $a_2^+x_1 \in E(G)$, which implies $\{x_3, a_2^+\} \succ V(G)$, again a contradiction. Thus, we have $a_2b_3 \in E(G)$.

Up to now, we have shown that $a_1b_2, a_2b_3 \in E(G)$. In the following, we will show that P_i is a clique for $i = 1, 2$. If $P_i \not\subseteq N[a_i]$, then since $a_ib_{i+1} \in E(G)$, there is some vertex $u \in P_i$ such that $a_iu \notin E(G)$ and $a_iu^+ \in E(G)$. We let $u_i \in P_i$ be such a vertex if $P_i \not\subseteq N[a_i]$, where $i = 1, 2$.

If $P_1 \not\subseteq N[a_1]$, then $\{a_1, u_1, a_2, x_P\}$ is an independent set. By Lemma 9, we have $[a_1, x_1] \rightarrow a_2$, $[a_1, x_3] \rightarrow a_2$, $[a_2, x_2] \rightarrow a_1$ or $[a_2, x_3] \rightarrow a_1$.

If $P_2 \not\subseteq N[a_2]$, then $\{a_1, u_1, a_2, u_2, x_P\}$ is an independent set of order 5, a contradiction. By Lemma 1, we have $N(a_1) \cap (P_2 - \{b_3\}) = \emptyset$ and $N(u_1) \cap (P_2 - \{b_3\}) = \emptyset$. We now show $a_1, u_1 \notin N(b_3)$. By Claim 10, we have $b_2^-b_3 \notin E(G)$, and hence we may assume $u_1 \neq b_2^-$. If $[a_1, x_1] \rightarrow a_2$ or $[a_1, x_3] \rightarrow a_2$, we have $x_1a_2^+ \in E(G)$ or $x_3a_2^+ \in E(G)$. By Claim 13, we have $a_1, u_1 \notin N(b_3)$. If $[a_2, x_3] \rightarrow a_1$, then $b_2^-x_3 \in E(G)$. By Claim 13, $a_1, u_1 \notin N(b_3)$. If $[a_2, x_2] \rightarrow a_1$, then since $x_2b_2^-, a_1b_2 \in E(G)$, b_2 is an A-vertex. By Lemma 1, $a_2b_2 \notin E(G)$. Since $\{b_2, a_2, b_3, x_P\}$ is an independent set, by Lemma 9, we have $[a_2, x_1] \rightarrow b_2$, $[a_2, x_3] \rightarrow b_2$, $[b_2, x_1] \rightarrow a_2$ or $[b_2, x_3] \rightarrow a_2$. This implies $(N(x_1) \cup N(x_3)) \cap \{b_2^-, a_2^+\} \neq \emptyset$. By Claim 13, $a_1, u_1 \notin N(b_3)$. Thus, we have

$$N(a_1) \cap P_2 = \emptyset \text{ and } N(u_1) \cap P_2 = \emptyset. \tag{8}$$

Let $a \in \{a_1, u_1\}$ and $w \in V(G) - N[x_P]$. If $[x_P, w] \rightarrow a$ or $[a, w] \rightarrow x_P$, then by (8), we have $w \in P_1$ and $P_2 \subseteq N(w)$. Thus, by Lemma 1, w is neither an A-vertex nor a B-vertex. Obviously, $|P_1| \geq 3$. Since $a_1b_2 \in E(G)$, it is easy to see that

$$G[P_1] \text{ contains a hamiltonian } (w, w^+)\text{-path}. \tag{9}$$

If $[a_1, x_1] \to a_2$ or $[a_1, x_3] \to a_2$, then since w is not an A-vertex, we have $a_1w^+ \notin E(G)$, and hence $x_1w^+ \in E(G)$ or $x_3w^+ \in E(G)$. If $x_1w^+ \in E(G)$, then since $a_1b_2 \in E(G)$, we see that w is a B-vertex, a contradiction. If $x_3w^+ \in E(G)$, then by (9) and Lemma 3, we have $wb_3 \notin E(G)$, which contradicts $P_2 \subseteq N(w)$. If $[a_2, x_2] \to a_1$, then since $a_2w, b_2x_2 \in E(G)$, by Lemma 5, there is some vertex $v \in w\overrightarrow{P}b_2$ such that $va_2, v^+x_2 \in E(G)$, which contradicts Lemma 3 since $a_1b_2 \in E(G)$, which implies $G[P_1]$ contains a hamiltonian (v, v^+)-path. Since $a_2, b_2 \in N(w)$, by (9) and Lemma 3, we have $w^+x_3, w^+a_2 \notin E(G)$, which implies $[a_2, x_3] \to a_1$ is impossible. Thus, for any $a \in \{a_1, u_1\}$ and $w \in V(G) - N[x_P]$, both $[x_P, w] \to a$ and $[a, w] \to x_P$ are impossible, which contradicts Lemma 10 since $\{a_1, u_1, a_2, x_P\}$ is an independent set. Therefore, we have $P_1 \subseteq N[a_1]$.

If $P_2 \not\subseteq N[a_2]$, then since $P_1 \subseteq N[a_1]$, by symmetry, we have $P_2 \subseteq N[b_3]$. Thus, u_2 is both an A-vertex and a B-vertex. By Lemma 1, $P_1 \cap N(u_2) = \emptyset$. Since $x_Pa_2 \notin E(G)$, there is some vertex w such that $[x_P, w] \to a_2$ or $[a_2, w] \to x_P$. If $[a_2, w] \to x_P$, then $w \notin P_1$ for otherwise $\{a_2, w\} \not\succ u_2$. Thus, we have $w \in P_2$. Since $P_2 \subseteq N[b_3]$, by Lemma 1, we have $a_2b_2, wb_2^- \in E(G)$, which contradicts Lemma 3. Thus, we have $[x_P, w] \to a_2$. If $w \in P_1$, then $wu_2 \notin E(G)$ and if $w \in P_2$, then $wb_2 \notin E(G)$. Thus, we have $w \in \{x_1, x_3\}$. If $[x_P, x_1] \to a_2$, then $x_1x_2 \in E(G)$ by Lemma 12. In this case, we have $\{x_1, b_3\} \succ V(G)$. If $[x_P, x_3] \to a_2$, then by Lemma 12, we have $x_2x_3 \in E(G)$ and $x_1x_2 \notin E(G)$. Thus, we have $x_2, a_2, a_2^+ \notin N(x_1)$ for otherwise $\gamma(G) = 2$. Since $b_2u_2 \notin E(G)$, there is some vertex v such that $[b_2, v] \to u_2$ or $[u_2, v] \to b_2$. Obviously, $v \neq x_P$, and hence $v \in X$. Since $[x_P, x_3] \to a_2$ implies $b_2, b_3 \in N(x_3)$, we have $v \neq x_3$. Since $\{u_2, x_1\} \not\succ a_2$ and $\{b_2, x_1\} \not\succ a_2^+$, we have $v \neq x_1$, and hence $v = x_2$, which implies $[b_2, x_2] \to u_2$. Since $x_1x_2 \notin E(G)$, we have $x_1b_2 \in E(G)$. If $a_2b_2 \in E(G)$, then $\{x_3, b_2\} \succ V(G)$, and hence $a_2b_2 \notin E(G)$. Now, consider $x_Pu_2 \notin E(G)$. Since $\{a_1, a_2, u_2\}$ is an independent set and $[x_P, x_3] \to a_2$, by Lemma 10, there is some vertex $u \in V(G) - N[x_P]$ such that $[x_P, u] \to u_2$ or $[u_2, u] \to x_P$. Since $N(a_2) \cap P_1 = \emptyset$ and $N(u_2) \cap P_1 = \emptyset$, we have $u \in P_2$ in both cases. This is impossible since $\{u_2, u\} \not\succ b_2$. Thus, we have $P_2 \subseteq N[a_2]$.

By symmetry, we have $P_i \subseteq N(a_i) \cap N(b_{i+1})$ for $i = 1, 2$. If P_1 is not a clique, then there are two vertices $u, v \in P_1 - \{a_1, b_2\}$ such that $uv \notin E(G)$. Obviously, u and v are both A-vertices and B-vertices. Thus, $(N(u) \cup N(v)) \cap P_2 = \emptyset$. Since $\{u, v, a_2, x_P\}$ is an independent set, by Lemma 10, there is some $w \in V(G) - N[x_P]$ and a vertex in $\{u, v\}$, say u, such that $[u, w] \to x_P$ or $[x_P, w] \to u$. It is easy to see that such a vertex w does not exist, and hence P_1 is a clique. By symmetry, P_2 is a clique.

Since P_i is a clique for $i = 1, 2$, by Lemmas 1 and 14, we have $E(P_1, P_2) \subseteq \{a_2b_2\}$. If $a_2b_2 \notin E(G)$, then X is a 3-cutset such that $\omega(G-X) = 3$, which contradicts $\tau(G) > 1$. If $a_2b_2 \in E(G)$, then by Lemma 14, we have $\alpha(G) = 3$, again a contradiction.

The proof of Theorem 4 is complete. ∎

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