$H_\infty$ Fixed-Lag Smoothing and Prediction for Linear Continuous-Time Systems

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Abstract

This paper addresses the $H_\infty$ fixed-lag smoothing and prediction problems for linear continuous-time systems. We present a solution to the optimal $H_2$ estimation problem for linear continuous-time systems with instantaneous and delayed measurements. It is then shown that the $H_\infty$ fixed-lag smoothing and prediction problems can be converted to the latter problem in Krein space. Therefore, the $H_2$ estimation is extended to give conditions on the existence of a $H_\infty$ fixed-lag smoother and predictor based on innovation analysis and projection in Krein space and a solution for $H_\infty$ smoother or predictor is given in terms of a Riccati differential equation and matrix differential equations.

1 Introduction

The smoothing problem is to estimate a linear combination of system states at time instant $t$ based on measurements up to $t+h$ for any $h > 0$. It is usually classified into three categories, namely the fixed-point smoothing, fixed interval smoothing and fixed-lag smoothing. Note that in the discrete-time case where the delay dynamics are finite dimensional, the smoothing problem may be converted into standard filtering through state augmentation [8] or can be solved through a recently developed direct approach [10]. However, for the continuous-time case, the delay dynamics are infinite dimensional, the smoothing problem has been proven to be difficult. The problem of $H_\infty$ fixed-interval smoothing has been solved in [6] for linear continuous-time systems and the optimal fixed interval smoother has been known to be the same as the $H_2$ optimal smoother [1]. However, for fixed point smoothing and fixed-lag smoothing, the $H_2$ and $H_\infty$ smoothing will be different. Recently, Mirkin [5] has tackled the infinite horizon $H_\infty$ fixed-lag smoothing problem using a spectral factorization approach. The finite horizon $H_\infty$ fixed lag smoothing and prediction for continuous-time systems remains open.

In this paper, we shall address the finite horizon $H_\infty$ fixed lag smoothing and prediction problems. We first discuss the $H_2$ estimation for linear continuous-time systems with instantaneous and delayed measurements. Note that the $H_2$ filtering of systems with state delays has attracted a lot of interest in the past (see, e.g. [2]-[3],[4]) where solutions that are infinite dimensional in nature have been obtained. In the present paper, through re-organized innovation, we derive a solution for the optimal $H_2$ estimator using innovation analysis and projection in Hilbert space. It is demonstrated that the $H_\infty$ estimation for linear continuous-time systems with instantaneous measurement can be converted to the $H_2$ estimation for systems with instantaneous measurement and fictitious delayed 'measurement' in Krein space [7]. Therefore, the $H_2$ estimation is extended to obtain the existence condition of the $H_\infty$ fixed lag smoother. A solution to the fixed lag smoother is given in terms of Riccati differential equation and matrix differential equations. We also present a solution to the prediction problem.

Due to the space limitation, all proofs will be omitted. The detail can be found in [11].

2 Problem Statement

In this section we shall state the $H_2$ and $H_\infty$ estimation problems to be investigated in the following sections.

2.1 $H_2$ Optimal Estimation

We shall consider the linear stochastic system described by

$$\dot{x}(t) = F(t)x(t) + \Gamma(t)u(t)$$

$$y(t) = H(t)x(t) + v(t)$$

$$z_{t-h}(t) = L(t-h)x(t-h) + v_c(t)$$

where $x(t)$ is the state vector, $y(t)$ is the measurement, $z_{t-h}(t)$ is the predicted measurement, $F(t)$ is the state transition matrix, $H(t)$ is the measurement matrix, $\Gamma(t)$ is the process noise matrix, $v(t)$ is the measurement noise, $v_c(t)$ is the prediction error, and $L(t)$ is the output matrix.

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where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^r$ is the input noise, $y(t) \in \mathbb{R}^m$ and $z_{\omega,n}(t) \in \mathbb{R}^p$ are respectively the instantaneous and delayed measurements, $v(t) \in \mathbb{R}^m$ and $v_z(t) \in \mathbb{R}^p$ are the measurement noises. The initial state $x(0)$ and $u(t)$, $v(t)$ and $v_z(t)$ are uncorrelated white noises with zero means and known covariance matrices $\{x(0), x(0)\} = P_0$, $\{u(t), u(s)\} = Q_u(t)\delta(t-s)$, $\{v(t), v(s)\} = Q_v(t)\delta(t-s)$ and $\{v_z(t), v_z(s)\} = Q_{v_z}(t)\delta(t-s)$, respectively.

The $H_2$ optimal estimation problem can be stated as:

$H_2$ Estimation Problem: Given the measurements up to the time instant $t + h$, $\{y(s), 0 \leq s \leq t + h; \ z_s(s + h), 0 \leq s \leq t\}$, find a linear mean square error estimator of $x(t)$, which is denoted as $\hat{x}(t | t + h, t)$.

### 2.2 $H_\infty$ Estimation

We consider the following linear system for the $H_\infty$ estimation problem:

\[
\begin{align*}
    x(t) &= F(t)x(t) + \Gamma(t)u(t) \quad (4) \\
    y(t) &= H(t)x(t) + v(t) \quad (5) \\
    z(t) &= L(t)x(t) \quad (6)
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^m$ represent the state, input noise, measurement output, measurement noise and the signal to be estimated, respectively. It is assumed that the input and measurement noises are deterministic signals and are from $L_2[0, T]$ where $T$ is the time-horizon of the estimation problem under investigation.

$H_\infty$ Estimation Problem: Given a scalar $\gamma > 0$, an integer $h$ and the observation $\{y(s), s \leq t + h\}$, find an estimate $\hat{z}(t | t + h)$ of $z(t)$, if it exists, such that the following inequality is satisfied:

\[
\begin{align*}
    \sup_{(\omega(\cdot), \chi(\cdot)) \neq 0} & \int_0^T \left\{ E \left[ (t | t + h) - \hat{z}(t | t + h) \right]^T E \left[ (t | t + h) - \hat{z}(t | t + h) \right] \right\} dt \\
    & - \chi^T(0)P_0^{-1}\chi(0) + \int_0^{t + h} \Gamma^T(t)u(t)dt + \int_0^{t + h} v^T(t)v(t)dt < \gamma^2
\end{align*}
\]

(7)

where $h_0 = \max\{0, h\}$.

Observe that the above problem includes three cases: (a) $H_\infty$ fixed-lag smoothing problem ($h > 0$); (b) $H_\infty$ filtering problem ($h = 0$) and (c) $H_\infty$ prediction ($h < 0$). The filtering problem has been addressed in [6].

### 3 H$2$ Optimal Estimation

In this section, we shall present a solution to the $H_2$ estimation for the system (1)-(3) involving instantaneous and delayed measurements using the projection in Hilbert space.

The key to our discussion in this section is to organize the instantaneous and delayed measurements in a proper way and introduce an associated innovation sequence.

It is easy to know that the linear space $\{y(s), 0 \leq s \leq t + h; \ z_s(s + h), 0 \leq s \leq t\}$, can be reorganized as

\[
L \{y(s), 0 \leq s \leq t; \ y(s), t < s \leq t + h\}
\]

where

\[
y_f(s) \triangleq \begin{bmatrix} y(s) \\ z_s(s + h) \end{bmatrix} = \begin{bmatrix} H(s) \\ L(s) \end{bmatrix} x(s) + v_f(s) \quad (9)
\]

\[
y(s) = H(s)x(s) + v(s), \quad (10)
\]

with

\[
v_f(s) = \begin{bmatrix} v(s) \\ v_z(s + h) \end{bmatrix} \quad (11)
\]

being a white noise of zero mean and variance matrix $Q_{y_f}(s) = \begin{bmatrix} Q_y(s) & 0 \\ 0 & Q_{v_z}(s + h) \end{bmatrix}$. It should be noted that $y_f(s)$ contains measurements of the state $x(s)$ at time instants $s$ and $s + h$.

Observe that (1) and (9) or (1) and (10) form a standard state space representation.

### 3.1 Innovation sequence and Riccati equation

First, we introduce the following stochastic sequence:

\[
w(s, t) \triangleq y(s) - \hat{y}(s | s, t), s > t \quad (12)
\]

\[
w(t, t) \triangleq y_f(t) - \hat{y}_f(t | t, t) \quad (13)
\]

where $\hat{y}_f(t | t, t)$ is the linear mean square (LLMS) estimate of $y_f(t)$ given $\{y_f(r), 0 \leq r < t\}$ and $\hat{y}(s | s, t)$ is the LLMS estimate of $y(s)$ given $\{y(r), 0 \leq r < t\}$ and $\{y(r), t < r < s\}$. It is clear that $w(t, t)$ is the innovation associated with the standard Kalman filtering based on observation $y_f(t)$. Then we have the following relationships

\[
w(s, t) = y(s) - H(s)\hat{x}(s | s, t)
\]

\[
= H(s)e(s, t) + v(s), s > t
\]

(14)

\[
w(t, t) = y_f(t) - \begin{bmatrix} H(t) \\ L(t) \end{bmatrix} \hat{x}(t | t, t)
\]

\[
= \begin{bmatrix} H(t) \\ L(t) \end{bmatrix} e(t, t) + v_f(t)
\]

(15)

where

\[
e(s, t) = x(s) - \hat{x}(s | s, t), s \geq t
\]

while $\hat{x}(s | s, t)$ is defined similarly to $\hat{y}(s | s, t)$ and $\hat{y}_f(t | t, t)$. The following lemma shows that $\{w(\cdot, \cdot)\}$ is indeed an innovation sequence.
Lemma 1

\[ \{w(s,t), 0 \leq s \leq t; w(s,t), t < s \leq t+h \} \tag{17} \]

is the innovation which is an uncorrelated white noise and spans the same linear space as

\[ L \{y_f(s), 0 \leq s \leq t; y(s), t < s \leq t+h \} \tag{18} \]

Let \( P_f^\Delta \triangleq (x(r), e(s,t)); r,s \geq t \) be the cross-covariance matrix of the state \( x(r) \) with the state estimation error \( e(s,t) \), where \( e(s,t) \) is as defined in (14). Then, from (14) and (15), the innovation covariance matrix

\[ Q_w(s,t) = \begin{cases} Q_v(s), & s > t \\ Q_v(t), & s = t \end{cases} \tag{19} \]

Theorem 1 The cross-covariance matrix \( P^\Delta_{s,t} \), \( r,s \geq t \) can be calculated as follows:

1) For \( r = s = t \), \( P^\Delta_{s,t} \) is the solution to the following standard Riccati equation

\[
\frac{d P^\Delta_{s,t}}{dt} = F(t)P^\Delta_{s,t} + P^\Delta_{s,t}F^T(t) - K^\Delta_{s,t}Q_v(t)\left[K^\Delta_{s,t}\right]^T + \Gamma(t)Q_v(t)\Gamma^T(t), \quad P^\Delta_{s,t}(0) = P_0 \tag{20}
\]

with

\[ K^\Delta_{s,t} = P^\Delta_{s,t}\left[\begin{array}{c} H(t) \\ L(t) \end{array}\right]^T \tag{21} \]

2) For \( r > s > t \), \( P^\Delta_{s,t} \) is the unique solution of the linear matrix differential equation

\[
\frac{d P^\Delta_{s,t}}{ds} = F(s)P^\Delta_{s,t} + P^\Delta_{s,t}F^T(s) - K^\Delta_{s,t}Q_v(s)\left[K^\Delta_{s,t}\right]^T + \Gamma(s)Q_v(s)\Gamma^T(s) \tag{22}
\]

where the initial condition \( P^\Delta_{s,t}(t) \) is calculated in (20) and

\[ K^\Delta_{s,t} = P^\Delta_{s,t}H^T(s)Q_v^{-1}(s) \tag{23} \]

3) For \( r > s > t \), \( P^\Delta_{s,t} \) is the unique solution of the linear matrix differential equation

\[
\frac{d P^\Delta_{s,t}}{dr} = \Phi(r)P^\Delta_{s,t} \tag{24}
\]

with the initial value \( P^\Delta_{s,t} \) as calculated in (22).

4) For \( t < r < s \), \( P^\Delta_{s,t} \) is the unique solution of the linear matrix differential equation

\[
\frac{d P^\Delta_{s,t}}{ds} = P^\Delta_{s,t}A^T(s,t) \tag{25}
\]

where the initial condition \( P^\Delta_{s,t}(r) \) is calculated in (22) and

\[ A(s,t) = F(s)\left[\begin{array}{c} L(s) - K^\Delta(s)H(s) \end{array}\right] \tag{26} \]

\( K^\Delta_{s,t} \) is as (23).

3.2 Optimal estimate \( \hat{x}(t | t+h,t) \) with \( h \geq 0 \)

Based on the discussion in the previous subsection, the following result is obtained immediately.

Theorem 2 Consider the system (1)-(3). The optimal estimator is given by

\[
\hat{x}(t | t+h,t) = \hat{x}(t | t,t) + \int_t^{t+h} P^\Delta_{t+h,t}H^T(s)Q_v^{-1}(s) [y(s) - H(s)\hat{x}(s | s,t)] ds \tag{27}
\]

where \( \hat{x}(t | t,t) \) is the standard Kalman filtering estimate as given by

\[
\hat{x}(t | t,t) = F(t)\hat{x}(t | t,t) + K^\Delta_{t,t}w(t) \\
= \left[ F(t) - K^\Delta_{t,t}\left[\begin{array}{c} H(t) \\ L(t) \end{array}\right] \right] \hat{x}(t | t,t) + K^\Delta_{t,t}\left[\begin{array}{c} y(t) \\ e(t) \end{array}\right] \tag{28}
\]

with the initial estimate \( \hat{x}(0 | 0,0) \) and \( \hat{x}(s | s,t), s > t \) is given by

\[
\frac{d}{ds} \hat{x}(s | s,t) = F(s)\hat{x}(s | s,t) + K^\Delta_{s,t}w(s) \\
= \left[ F(s) - K^\Delta_{s,t}H(s) \right] \hat{x}(s | s,t) + K^\Delta_{s,t}y(s), \tag{29}
\]

with the initial condition \( \hat{x}(t | t,t) \). In the above, \( K^\Delta_{s,t} \) is as (21) and \( K^\Delta_{s,t}, t < s \) is as (23). The matrix \( P^\Delta_{s,t}, t \leq s \) and \( P^\Delta_{s,t}, t < s \) are given in Theorem 1.

4. \( H_\infty \) Estimation

In this section, we shall present solutions to the \( H_\infty \) fixed-lag smoothing and prediction problems. We first demonstrate that the \( H_\infty \) smoothing and prediction can be converted to the \( H_2 \) ones in Krein space. The innovation and projection method of the last section is then extended to give a solution to the \( H_\infty \) fixed-lag smoothing and prediction.

To start with, we introduce the following cost function in association with the system (4)-(6)

\[
J_\infty(t+h,0) \triangleq x^T(0)P_0^{-1}x(0) + \int_0^{t+h} u^T(s)u(s) ds \\
+ \int_0^{t+h} [y(s) - H(s)x(s)]^T [y(s) - H(s)x(s)] ds \\
- \gamma^{-2} \int_0^{t+h} \hat{z}(s | s+h) - L(s)x(s)]^T [\hat{z}(s | s+h) - L(s)x(s)] ds \tag{30}
\]
In the case of fixed-lag smoothing \((h > 0)\) and thus \(h_0 = h\), we define
\[
\tilde{z}(s \mid s + h) = 0, \quad x(s) = 0, \quad L(s) = 0, \quad \text{for } s < 0.
\]
Then, (30) can be rewritten as
\[
\begin{align*}
J_s(t + h) & = x^T(0)P_0^{-1}x(0) + \int_0^{T+h} u^T(s)u(s)ds \\
&+ \int_0^{T+h} \left[ y_c(s) - H_L(s)\tilde{x}(s) \right]^T \left[ \begin{array}{cc}
I_m & 0 \\
0 & -\gamma^2 I_p
\end{array} \right] \left[ y_c(s) - H_L(s)\tilde{x}(s) \right]ds
\end{align*}
\]
where
\[
y_c(s) = \left[ \begin{array}{c}
y(s) \\
\bar{z}(s-h \mid s)
\end{array} \right], \quad \tilde{x}(s) = \left[ \begin{array}{c}
x(s) \\
x(s-h)
\end{array} \right],
\]
\[
H_L(s) = \left[ \begin{array}{cc}
H(s) & 0 \\
0 & L(s-h)
\end{array} \right]
\]
(31)

For the case of prediction \((h < 0)\) and thus \(h_0 = 0\), we assume
\[
y(s) = 0, \quad x(s) = 0, \quad H(s) = 0, \quad \text{for } s < 0.
\]
In this situation, (30) can be rewritten as
\[
\begin{align*}
J_s(t + h) & = x^T(0)P_0^{-1}x(0) + \int_0^{T+h} u^T(s)u(s)ds \\
&+ \int_0^{T+h} \left[ y_c(s) - H_L(s)\bar{z}(s) \right]^T \left[ \begin{array}{cc}
I_m & 0 \\
0 & -\gamma^2 I_p
\end{array} \right] \left[ y_c(s) - H_L(s)\bar{z}(s) \right]ds
\end{align*}
\]
(34)

where \(y_c(s), \bar{z}(s), H_L(s)\) are as in (32) and (33).

### 4.1 Stochastic system in Krein space

To derive an \(H_m\) estimator, the following stochastic system associated with (4)-(6) is introduced:
\[
\begin{align*}
\dot{x}(t) & = \Phi(t)x(t) + \Gamma(t)v(t) \\
y(t) & = H(t)x(t) + v(t) \\
\tilde{z}(t - h \mid t) & = L(t-h)x(t) + v_c(t)
\end{align*}
\]
(35-37)

where \(u(t), \Gamma(t)\) and \(v_c(t)\) are uncorrelated white noises, with zero means and covariance matrixes \(Q_u(t), Q_{\Gamma}(t)\delta(t-s)\), \(Q_v(t), Q_{v_c}(t)\delta(t-s)\), \(Q_{v\Gamma}(t), Q_{v_c\Gamma}(t)\delta(t-s)\)

where
\[
Q_u(t) = I, \quad Q_{\Gamma}(t) = I_m, \quad Q_{v_c}(t) = -I_p\gamma^2,
\]
\[
Q_{v\Gamma}(t) = \left[ \begin{array}{cc}
I_m & 0 \\
0 & -I_p\gamma^2
\end{array} \right].
\]
(38)

Observe that the introduced stochastic system (35)-(37) is of the same form as that of (1)-(3) for which the \(H_2\) smoothing has been considered. In the following, we’ll adopt similar notations as defined in the last section.

Observe that the difference between the stochastic model (35)-(37) and the stochastic model (1)-(3) is that the latter involves a fictitious 'measurement' \(\tilde{z}\) with error covariance being negative definite. Therefore, in the sequel the innovation and projection associated with the latter system will be discussed in Krein space [7] rather than in the usual Hilbert space. For detail of Krein space, readers are referred to [7].

We now introduce the innovation of the observation \(y_c(t)\) as
\[
w_c(t) = y_c(t) - \tilde{y}_c(t \mid t)
\]
(39)

where \(y_c(t)\) is as in (32),
\[
\tilde{y}_c(t \mid t) = \left[ \begin{array}{c}
y(t) \\
\tilde{z}(t-h \mid t)
\end{array} \right]
\]
(40)

and \(\tilde{y}_c(t \mid t)\), if it exists, is the projection of \(y_c(t)\) on the linear Krein space \(L\{y_c(s), s \le t\}\). In view of (32), \(L\{y_c(s), s \le t\}\) is equivalent to
\[
L\{y_j(s), 0 \le s \le t-h; \quad y(s), t-h < s \le t\}
\]
(41)

for \(h \ge 0\) or
\[
L\{y_j(s), 0 \le s \le t; \quad \tilde{z}(s \mid s+h), t < s \le t-h\}
\]
(42)

for \(h < 0\), where
\[
y_j(s) = \left[ \begin{array}{c}
y(s) \\
\tilde{z}(s \mid s+h)
\end{array} \right]
\]
(43)

Thus, \(w_c(t)\) can be rewritten as
\[
w_c(t) = \left[ \begin{array}{c}
y(t) \\
\tilde{z}(t-h \mid t)
\end{array} \right] - H_L(t) \left[ \begin{array}{c}
\tilde{z}(t-h \mid t) \\
\tilde{z}(t-h \mid t)
\end{array} \right]
\]
(44)

where
- For \(h \ge 0\), \(\tilde{z}(t \mid t-h)\) and \(\tilde{z}(t-h \mid t-h)\) are the projections of \(x(t)\) and \(x(t-h)\) on the linear Krein space of (41).
- For \(h < 0\), \(\tilde{z}(t \mid t-h)\) and \(\tilde{z}(t-h \mid t-h)\) are the projections of \(x(t)\) and \(x(t-h)\) on the linear Krein space of (42).

### 4.2 Sufficient and necessary condition for the existence of an \(H_m\) estimator

The following result presents a necessary and sufficient condition for the existence of an \(H_m\) estimator.
Consider the system (4)-(6) and the associated performance criterion (7). Then, for a given scalar $\gamma > 0$, an estimator $\hat{z}(t | t + h)$ that achieves (7) exists and for $h < 0$, $\hat{x}(t | t + h, t)$ exists for $-h \leq t \leq T$ and $\hat{x}(t | t + h, t)$ exists for $0 \leq t \leq T$, where $x(t)$ and $\hat{x}(t | t + h, t)$ are respectively the projections of $x(t + h)$ and $\hat{x}(t | t + h, t)$ on the linear Krein space of $x(t)$.

In the above, $Q,w$ is as in (38), the matrix $P_{a}(s)$ is given in (48), $P_{a}(s)$ and $P_{b}(s)$ are obtained by (22) and (25), respectively, with covariance matrices $Q_{w}$ and $Q_{w}$ as given in (38),

$$\frac{d}{ds}P_{a}(s) = F(s)P_{a}(s) + F^{T}(s)P_{b}(s) + \Gamma(s)^{T}(s)$$

with the initial condition $P_{a}(0) = P_{0}$.

A. Smoother $(h > 0)$

When $h > 0$, the projections $\hat{x}(t | t + h, t)$ and $\hat{x}(t | t + h, t)$ in Krein space are similar to the case of $H_{2}$ estimation discussed in the last Section except that the former involves an indefinite covariance matrix. The projections $\hat{x}(t | t + h, t)$ and $\hat{x}(t | t + h, t)$ in Krein space are obtained directly from Theorem 2 as

$$\hat{x}(t | t + h, t) = \hat{x}(t | t, t) + \int_{t}^{t+h} P_{a}(s)H^{T}(s)\{y(s) - H(s)\hat{x}(s | s, s, t)\}ds$$

and

$$\frac{d}{ds}\hat{x}(s | s, t) = \{F(s) - P_{a}(s)H^{T}(s)H(s)\}\hat{x}(s | s, t) + P_{a}(s)H^{T}(s)\}y(s), \hat{x}(s | s, t) |_{s=t} = \hat{x}(t | t, t)$$

The projections of $\hat{x}(t | t + h, t)$ and $\hat{x}(t | t + h, t)$ with $h < 0$ are different from the case of $h > 0$, but they are dual to each other. So the projections for $h < 0$ can be easily obtained by a similar discussion. In fact, note that for $h < 0$, $\hat{x}(t + h | t + h, t)$ and $\hat{x}(t | t + h, t)$ are respectively the projections of states $x(t + h)$ and $\hat{x}(t | t + h, t)$ on the linear Krein space of (46). The re-organized innovation associated with (46) can be given as

$$\{w(s, s), 0 \leq s \leq t + h; w(t + h, s) |_{t + h < s \leq t}\}$$

where $w(s, s)$ has a similar definition as in the case of $h > 0$ and $w(t + h, s) |_{s > t + h}$ is defined as

$$w(t + h, s) = \hat{x}(s | s + h) - L(s)\hat{x}(s | t + h, s)$$

where $\hat{x}(s | s + h)$ is the projection of the state $x(s)$ on the linear Krein space of (46). It is then easy to know that

$$w(t + h, s) = L(s)e(t + h, s) + v(s)$$

where

$$e(t + h, s) = x(s) - \hat{x}(s | t + h, s)$$
Similarly, we denote that
\[ q_{r+h}^r(x(r), \psi(t+h,s)); r,s \geq t + h \]  \hspace{1cm} (60)
Then \[ q_{r+h}^r \] can be computed by the Lemma below.

Lemma 2 
- For \( s = t + h \), \( q_{r+t}^{r+h} \) is as in (48).
- For \( s > t + h \), \( q_{r+t}^{r+h} \) satisfies
\[ \frac{d q_{r+t}^{r+h}}{ds} = F(s)q_{r+t}^{r+h} + \gamma^2 q_{r+t}^{r+h} H^T(s)H(s)q_{r+t}^{r+h} + \Gamma(s)\dot{\Gamma}(s) \]  \hspace{1cm} (61)
with the initial condition \( q_{r+t}^{r+h} \).
- For \( s > r = t + h \), \( q_{r+t}^{r+h} \) satisfies
\[ \frac{d q_{r+t}^{r+h}}{ds} = q_{r+t}^{r+h} \dot{A}(t+h,s) \]  \hspace{1cm} (62)
with the initial condition \( q_{r+t}^{r+h} \) and
\[ A(t+h,s) = F(s)[\alpha + \gamma^2 q_{r+t}^{r+h} L^T(s)L(s)] \].

Lemma 3 For \( h < 0 \), the projections of \( \tilde{x}(t+h|t+h,t) \) and \( \tilde{x}(t+h|t+h,t) \) are given by
\[ \tilde{x}(t+h|t+h,t) = \tilde{x}(t+h|t+h,t) - \gamma^2 \int_{t+h}^{t} q_{r+t}^{r+h} L^T(s)\tilde{z}(s|s+h) - L(s)\tilde{z}(s|s+h)ds \]  \hspace{1cm} (63)
and
\[ \frac{d \tilde{z}(s|t+h,s)}{ds} = -\gamma^2 q_{r+t}^{r+h} L^T(s)\tilde{z}(s|s+h) + [F(s) + \gamma^2 q_{r+t}^{r+h} L^T(s)L(s)]\tilde{z}(s|t+h,s) \]  \hspace{1cm} (64)
where \( t+h \leq s \leq t \) and the initial value \( \tilde{x}(t+h|t+h,t) \) is as (51) and the matrices \( q_{r+t}^{r+h} \) and \( q_{r+t}^{r+h} \) are given by (61)-(62).

Theorem 5 Consider the system (4)-(6) and the associated performance criterion (7). Given scalar \( \gamma > 0 \) and \( h < 0 \), if there exists a bounded solution \( q_{r+t}^{r+h} \) to the Riccati equation (48) over \( 0 \leq r \leq N \) and an unique bounded solution \( q_{r+t}^{r+h} \) to (61) over \( t + h \leq s \leq t \) and \( 0 \leq t \leq N \). Then the multi-step prediction \( \tilde{x}(t+h|t+h) \) that achieves (7) is solvable and the predictor \( \tilde{x}(t+h|t+h) \) is
\[ \tilde{x}(t+h|t+h) = L(t)\tilde{x}(t+h|t+h) \]  \hspace{1cm} (65)
where \( \tilde{x}(t+h|t+h) \) is computed by (64) and (51).

5 Conclusion
This paper has addressed the \( H_{\infty} \) fixed lag smoothing and prediction problems for linear continuous-time systems. It was shown that the \( H_{\infty} \) estimation is in fact an \( H_2 \) one for continuous-time systems with delayed measurement in Krein space. Applying innovation analysis and projection theory in Krein space, the \( H_2 \) estimation result was extended to give the existence condition and a solution to the \( H_{\infty} \) smoother and predictor.

References