A Novel Approach to Circuit-Field-Torque Coupled Time Stepping Finite Element Modeling of Electric Machines

S. L. Ho, H. L. Li, W. N. Fu, and H. C. Wong

***Abstract*—This paper presents a sub-block algorithm for the time stepping finite element solution of problems in which sets of electromagnetic field equations, circuit equations and mechanical equation are coupled together. The proposed method ensures that identical solutions are obtained for the sub-blocks by controlling the step size of the time stepping process. This new method simplifies the process of dealing with coupled-system problems and it also greatly reduces the computation time. A time stepping finite element model of an induction motor is used to demonstrate the proposed method in details.**

***Index Terms*—Electric machines, finite element models, solver.**

I. Introduction

MOST electromagnetic field problems are closely dependant on and thus coupled to their corresponding electric circuits and mechanical movement. Hence it is well known that when nonsinusoidal variables and mechanical movement are being considered collectively, the time stepping finite element method (FEM) is one of the most powerful tools to simulate the total system. Nowadays FEM equations and circuit equations can be easily coupled together in a set of equations to be solved at the same time [1]. However, difficulties will arise when the mechanical equation is coupled in since the physical position of the moving part of the system will change in each time step. Thus the mechanical position of the moving part is also an unknown in the system equations at each time step, because the coefficient matrix of the FEM equations is dependent on the mechanical position that can only be found after solving the coupled system equations. This results in FEM equations with a nonconstant coefficient matrix. The solving of such equations is very time consuming. Hence even if the FEM-circuit and mechanical coupled equations for the electric machines can be derived [2], in practice such equations cannot be used widely.

One alternative is to solve the FEM-circuit equations and mechanical equation separately [3]. Usually, at the kth step, the mechanical position θ^{k-1} is used as θ^k, then the FEM-circuit equations which have constant coefficient matrix can be solved easily. The electromagnetic torque can be computed based on the magnetic field distribution. θ^k can then be found from the mechanical equation. Such approach has indeed been used commonly in simulation studies. For example, Zhou has used this method to study three-phase and single-phase induction motors [4]. Brauer and his colleagues also used a similar method to study magnetic actuators, contactors, etc. [5]–[10].

The merit of the alternative method just described is its simplicity. However, it also has its disadvantage since θ^{k-1} is different from θ^k when solving the FEM-circuit equations at the kth step. An error will be introduced. Strictly speaking, with such method it is necessary to evaluate the coupling coefficient θ^k between the FEM-circuit equations and the mechanical equation iteratively. Many iterative steps in such nested loops are needed.

In this paper a sub-block algorithm using time stepping method to solve the afore-mentioned coupled problems is proposed. Instead of iterating between sub-blocks, the authors ensure that accurate solutions of the sub-blocks are obtained by controlling the step size of the time stepping process. The proposed method simplifies the process of dealing with coupled-system problems and it also greatly reduces the computing time needed to obtain the solution. An induction machine is used as a typical example to illustrate the method in details. The computed results are verified by comparing with the test results on an 11 kW induction motor.

II. Basic Equations of Induction Motors

In the 2-D FEM of electric machines it is assumed that the magnetic vector potential has an axial component only. For skewed rotor bar induction motors, the rotor is divided into an even number of M slices in the axial direction. In all the notation that follows m stands for the mth slice; thus $l_m = l/M$ and l is the axial length of the iron core. When $M = 1$, the machine is a nonskewed rotor bar induction machine. The field equation in the skewed rotor bar induction machines is [11]:

$$\frac{\partial}{\partial x}\left(\nu\frac{\partial A}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial A}{\partial y}\right)=-\frac{i_s}{S}-\frac{\sigma}{l_m}u_{bmn}+\sigma\frac{\partial A}{\partial t} \quad (1)$$

where

- A is the magnetic vector potential;
- i_s/S is the current density in the stator conductor;
- ν is the reluctivity of the constituent materials;
- u_{bmn} is the potential difference between the axial ends of a rotor conductor in the nth bar of the mth slice;
- σ is the conductivity of the constituent material;
- S is the total cross-sectional area of one turn on one coil side (parallel branches are considered as one turn) and

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l_m is the axial length of the mth slice conductor.

Since the stator current is usually unknown, the electric circuit equation of the stator winding should be coupled in:

$$e + R_1 i_s + L_\sigma \frac{\mathrm{d}i}{\mathrm{d}t} = v_s \tag{2}$$

where

v_s is the impressed voltage;
i_s is the phase current;
R_1 is the stator resistance of each phase and
L_σ is the end winding inductance.

The total induced electromotive force e is [1]:

$$e = \frac{l_M}{S} \sum_{m=1}^{M} \left(\iint_{\Omega_m^+} \frac{\partial A}{\partial t}\, d\Omega - \iint_{\Omega_m^-} \frac{\partial A}{\partial t}\, d\Omega \right) \tag{3}$$

where Ω^+ and Ω^- are, respectively, the cross-sectional areas of the "go" and "return" side of the phase conductors of the coils.

Substituting (3) into (2), the stator circuit equation is:

$$\frac{l_M}{S} \sum_{m=1}^{M} \left(\iint_{\Omega_m^+} \frac{\partial A}{\partial t}\, d\Omega - \iint_{\Omega_m^-} \frac{\partial A}{\partial t}\, d\Omega \right) + R_1 i_s + L_\sigma \frac{di_s}{dt} = v_s \tag{4}$$

The rotor bar current in the nth bar of the mth slice is:

$$i_{bmn} = \sigma \left(\iint_{\Omega_{mn}} \left(-\frac{\partial A}{\partial t} + \frac{u_{bmn}}{l_M} \right) \mathrm{d}\Omega \right) \tag{5}$$

Since the end rings connect all the rotor bars electrically, another circuit relationships of i_{bmn} and u_{bmn} can be introduced according to the rotor end network as described before by the authors [11].

By FEM discretization of (1) and coupling the stator circuit equations and the rotor cage circuit equations together, the following set of equations results [11]:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ 0 & C_{22} & 0 & 0 \\ 0 & 0 & C_{33} & C_{34} \\ 0 & 0 & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} A \\ i_s \\ u \\ i_r \end{bmatrix} + \begin{bmatrix} D_{11} & 0 & 0 & 0 \\ D_{21} & D_{22} & 0 & 0 \\ D_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \frac{\partial A}{\partial t} \\ \frac{\partial i_s}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial i_r}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 \\ P_2 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

where the first line of the sub-matrix is the magnetic field equations; the second line is the stator circuit equations; the third line is the branch equations of the rotor bars; the last line is the mesh-current equations of the rotor network. The unknowns $[A]$ are the magnetic vector potentials; $[i_s]$ are the stator phase currents; $[u]$ are the voltages of the rotor bars on each slice and $[i_r]$ are the mesh-currents in the rotor network. $[P_2]$ is the vector associated with the voltage applied in the terminal of stator windings.

The mechanical equation governing the rotor motion is:

$$J_m \frac{d\omega}{dt} = T_e - T_f \tag{7}$$

where

J_m is the moment of inertia;
ω is the rotor speed;
T_e is the electromagnetic torque and T_f is the load torque.

Maxwell Stress Tensor is used for torque computation.

III. Sub-Block Algorithm for the Solution

To obtain the solution in the time domain, the Backward Euler's method is used to discretize the time variable. If the solution at the $(k-1)$th step is known, then at the kth step:

$$\omega^k = \omega^{k-1} + \frac{T^k}{J_m} \Delta t \tag{8}$$

$$\theta^k = \theta^{k-1} + \omega^k \Delta t \tag{9}$$

$$\begin{bmatrix} C_{11}^k + \frac{D_{11}^k}{\Delta t} & C_{12}^k & C_{13}^k & 0 \\ \frac{D_{21}^k}{\Delta t} & C_{22}^k + \frac{D_{22}^k}{\Delta t} & 0 & 0 \\ \frac{D_{31}^k}{\Delta t} & 0 & C_{33}^k & C_{34}^k \\ 0 & 0 & C_{43}^k & C_{44}^k + \frac{D_{44}^k}{\Delta t} \end{bmatrix} \begin{bmatrix} A^k \\ i_s^k \\ u^k \\ i_r^k \end{bmatrix} = \begin{bmatrix} 0 \\ P_2^k \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\Delta t} \begin{bmatrix} D_{11}^k & 0 & 0 & 0 \\ D_{21}^k & D_{22}^k & 0 & 0 \\ D_{31}^k & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44}^k \end{bmatrix} \begin{bmatrix} A^{k-1} \\ i_s^{k-1} \\ u^{k-1} \\ i_r^{k-1} \end{bmatrix} \tag{10}$$

where θ is the position of the rotor; step size $\Delta t = t^k - t^{k-1}$. The coefficient matrix can become symmetrical after some transformation [11].

Coupling (8), (9) and (10) together, the set of the total system equations will have the form of:

$$\left[C^k + \frac{D^k}{\Delta t} \right] \begin{bmatrix} A^k \\ i_s^k \\ u^k \\ i_r^k \\ \omega^k \\ \theta^k \end{bmatrix} = \left[\frac{D^k}{\Delta t} \right] \begin{bmatrix} A^{k-1} \\ i_s^{k-1} \\ u^{k-1} \\ i_r^{k-1} \\ \omega^{k-1} \\ \theta^{k-1} \end{bmatrix} + [P^k] \tag{11}$$

In general the FEM rotor mesh moves in accordance with rotor movement. At the kth step, the position of the rotor mesh can be determined by θ^k. During the iteration process in finding the solution for the set of nonlinear equations, the rotor position of the rotor mesh will be updated again and again. $[C^k]$, which is dependent on the rotor meshes, is no longer constant and this will give rise to programming difficulties.

The proposed method is based on the following formula which is used to estimate the local truncation error. To a general differential equation of initial problems with unknown X:

$$\frac{\partial X}{\partial t} = f(X, t) \tag{12}$$

Using Taylor's expansion at t^k, the local truncation error of solution X when using the Backward Euler's method at the kth step is [12]

$$\varepsilon^* = -\frac{(\Delta t)^2}{2}\frac{\partial^2 X(\xi)}{\partial t^2} \approx \frac{(X_\#^k - X_*^k)}{2} \tag{13}$$

where $t^{k-1} \le \xi \le t^k$; $X_\#$ is the approximate solution using the Euler's method at the kth step:

$$X_\#^k = X^{k-1} + (\Delta t) f(X^{k-1}, t^{k-1}) \tag{14}$$

and X_* is the approximate solution using Backward Euler's method at the kth step:

$$X_*^k = X^{k-1} + (\Delta t) f(X_*^k, t^k) \tag{15}$$

The essence of the proposed method is to divide the total systems of equations into two sub-blocks. The first block consists of the mechanical equation only. The second block includes the FEM equations, the stator circuit equations and the rotor circuit equations. At the kth step an explicit method (such as the Euler's method here) is firstly used for the first block and hence its solution can be obtained directly from the $(k-1)$th solution. For the second block an implicit method (such as the Backward Euler's method here) is used. In order to ensure that the solutions are sufficiently accurate, the first block will be solved again by using the implicit method.

Therefore, instead of using (8), one could use the Euler's method to obtain an initial guess of ω^k as follows:

$$\omega^{k(0)} = \omega^{k-1} + \frac{T^{k-1}}{J_m}\Delta t \tag{16}$$

then θ^k can still be obtained using (9).

Consequently one could use the Backward Euler's method to compute the electromagnetic field according to (10).

During the process of solving (10), the position of the rotor mesh is fixed. After solving (10), one obtains the distribution of magnetic vector potential A^k, hence the electromagnetic torque T^k can be obtained and ω^k is computed again by using the Backward Euler's method according to (8).

According to (13), the difference between $\omega^{k(0)}$ and ω^k is an indicator of the discretization error of the Backward Euler's method. It can also be expressed in per unit value as:

$$\varepsilon = \left|\frac{\omega^k - \omega^{k(0)}}{\omega_N}\right| \times 100\% \tag{17}$$

where ω_N is the rated speed of the motor.

Since the error is dependent on the step size Δt, Δt will be adjusted automatically to ensure the error is sufficiently small. The strategy of adjusting the step size automatically, by including the prediction of the step size before the field computation at each time step, has been described in [12].

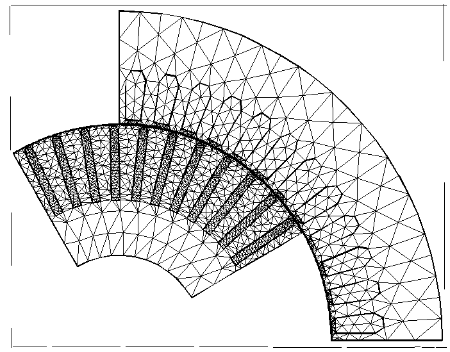

Fig. 1. FEM mesh of one slice of 11 kW motor after a rotation of 30° (2523 nodes, 4439 elements).

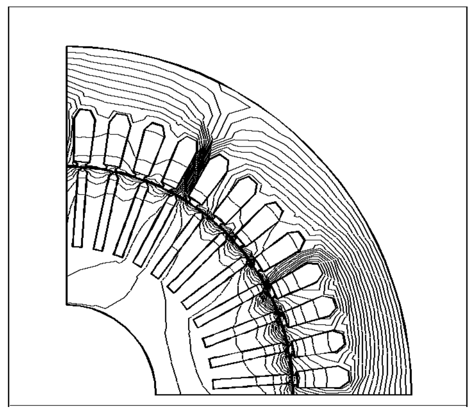

Fig. 2. Computed flux distribution at starting process.

In each time step, the Newton–Raphson method coupled with the incomplete Cholesky-conjugate gradient (ICCG) algorithm is used to solve the set of large nonlinear field equations (10). Only the nonzero elements in the upper triangular coefficient matrix need to be stored in the program.

IV. Example

The proposed method has been used to simulate the starting operation of an 11 kW induction motor with skewed rotor bars (380 V, 50 Hz, Δ connected, 4 poles, 48 slots in stator, 44 slots in rotor, and a skewing of 1.2 rotor slot pitch). In the multi-slice FEM model the motor in the axial direction is divided into 4 slices. The FEM mesh of one slice is shown in Fig. 1. The set of system equations at each time step has a total of 10 744 unknowns.

The programs are run on a personal computer Pentium/366 MHz. The solution of the system equation at each time step requires about 0.52 min. of CPU time on average using the Newton–Raphson method coupled with the ICCG method.

The applied voltages at the stator winding terminals are:

$$\begin{aligned} v_A &= V_m \cos(\omega t - \pi/3) \\ v_B &= V_m \cos(\omega t - \pi/3 - 2\pi/3) \\ v_C &= V_m \cos(\omega t - \pi/3 - 4\pi/3) \end{aligned}$$

where $V_m = 380\sqrt{2}$ V. At $t = 0$, these voltages are applied to the motor terminals suddenly. The computed magnetic flux

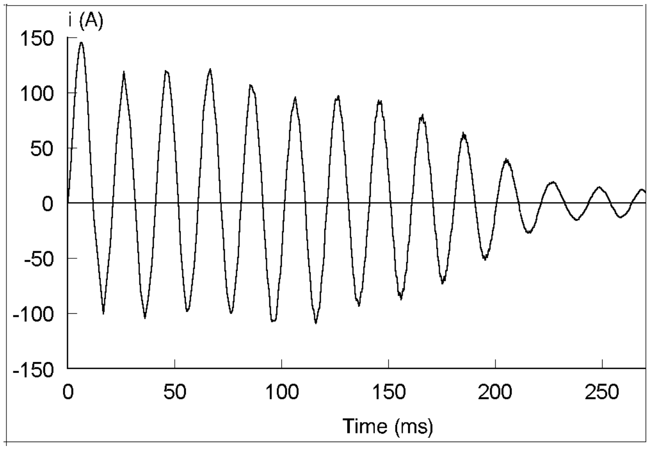

Fig. 3. Computed stator phase current at the starting process using the 2-D multi-slice FEM model.

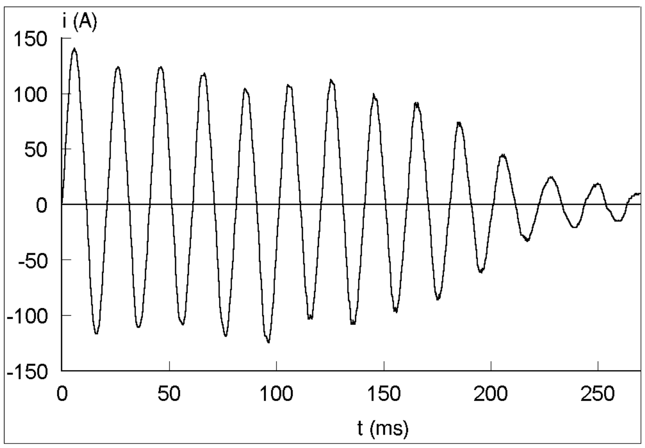

Fig. 4. Measured stator phase current at the starting process.

distribution at one snapshot during the starting process is shown in Fig. 2. The computed stator phase current (phase A) with inertia load is shown in Fig. 3, while the measured current (phase A) is shown in Fig. 4.

The result computed using the proposed method shows very good correlation with the test data. The computed electromagnetic torque is shown in Fig. 5.

When using the proposed sub-block method with a maximum allowable error ε as defined in (17) being 0.1%, then by integrating from $t = 0$ to $t = 280$ ms, the solution process needs 6735 steps. The minimum step size is 0.02 ms while the maximum step size is 0.16 ms. It can be shown that in a fixed step size method (the alternative method as described before that has no iteration between two blocks) with the same allowable error in each step, the computation requires about 14 000 steps. That means the CPU time of the proposed method can be reduced by a factor of two.

V. Conclusion

When an electric circuit—electromagnetic field problem is coupled with the mechanical problem together, and if the system of equations are required to be solved by the time stepping method, one could separate the mechanical equation in a sub-block, and put the circuit—electromagnetic equations in another sub-block. By utilizing the fact that the solution of explicit algorithm and the solution of implicit algorithm are close if the time step size is small enough, one could effectively and simply solve the set of the total system equations without introducing additional iteration between two blocks. The merits of the proposed method are its programming simplicity and the reduction in computing time.

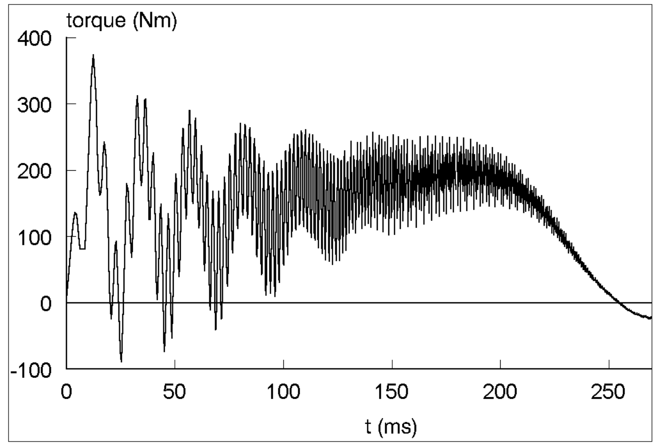

Fig. 5. Computed torque during the starting process using the 2-D multi-slice FEM model.

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