Parallel Machine Scheduling to Minimize the Sum of Quadratic Completion Times

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Abstract

We consider the parallel machine scheduling problem of minimizing the sum of quadratic job completion times. We first prove that the problem is strongly NP-hard. We then demonstrate by probabilistic analysis that the shortest processing time rule solves the problem asymptotically. The relative error of the rule converges in probability to zero under the assumption that the job processing times are independent random variables uniformly distributed in $(0, 1)$. We finally provide some computational results, which show that the rule is effective in solving the problem in practice.

Keywords: parallel machine scheduling, quadratic completion time, probabilistic analysis

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1 Introduction

We consider a problem of scheduling n jobs $J_1, J_2, \ldots, J_n$ on m identical parallel machines. The processing times of the n jobs are given by $p_1, p_2, \ldots, p_n$. All jobs are available at time zero. No preemption is allowed. The objective is to find a schedule π that minimizes the quadratic cost function: $Q(\pi) = \sum_{j=1}^{n} C_j(\pi)^2$, where $C_j(\pi)$ is the completion time of job J_j in schedule π. In the three-field notation of Lawler et al. [12], the problem is denoted by $P||\sum C_j^2$.

Due to its practical usefulness, parallel machine scheduling has attracted much attention of researchers and numerous results have appeared in the literature (see Cheng and Sin [6] and Lawler et al. [12]). A well-known result is that the shortest processing time (SPT) rule solves the linear cost problem: $P||\sum C_j$. Note that the SPT rule always schedules the shortest unscheduled job whenever a machine becomes idle. Also, the SPT rule solves the single machine problem $1||\sum C_j^2$ (see Townsend [19]), from which we can deduce that in any optimal schedule for the problem $P||\sum C_j^2$, the jobs on the same machine are sequenced in nondecreasing order of their processing times. However, it is unknown whether or not the SPT rule solves the problem $P||\sum C_j^2$.

Compared with the linear cost function, the quadratic one is more appropriate for cases where the later a job is finished, the greater is the cost per unit of elapsed time. On the other hand, the makespan (i.e., the maximum completion time) and the total completion time are two important objective functions for scheduling problems, but it is difficult to find a schedule that minimizes both of these two objectives. Noticing that from a mathematical point of view, the makespan is the sum of infinite powers of the completion times, one may consider using the sum of quadratic completion times as the objective function as a tradeoff between the makespan and the total completion time. The single machine problem with a quadratic cost function has attracted much attention of the scheduling research community. Besides solving $1||\sum C_j^2$, Townsend [19] further studied a branch-and-bound algorithm for the weighted version. Subsequently, many improvements and generalizations have appeared (see References [4, 8, 11, 15, 17, 18]). Alidaee [1, 2] and Fisher and Krieger [9] presented heuristics for problems with more general cost functions. In the parallel machine enviroment, several papers have considered minimizing the sum of quadratic machine completion times, i.e., the sum of the quadratic completion times of the last jobs on all machines (see References [3, 5, 13]). But no results have been reported for the sum of quadratic completion times of all jobs.

In this paper, we investigate the parallel machine problem $P||\sum C_j^2$. First, the problem is proved to be strongly NP-hard. Then, we focus on studying the SPT rule. We use probabilistic analysis to characterize the effectiveness of this rule. Probabilistic analysis is an approach that examines the performance of a heuristic when applied to random instances drawn from some distribution. Also, to supplement the analysis, we provide computational results to demonstrate the effectiveness of the *SPT* rule in practice.

The rest of this paper is organized as follows. In Section 2, we prove the strong NP-hardness of $P||\sum C_j^2$. In Section 3, we show the asymptotic optimality of the *SPT* rule by using probabilistic analysis and present the computational results. Finally, Section 4 includes some concluding remarks.

2 NP-hardness result

In this section, we prove that the problem $P||\sum C_j^2$ is strongly NP-hard by presenting a reduction from Numerical Three-Dimensional Matching (N3DM), which is known to be strongly NP-hard (see Garey and Johnson [10]).

N3DM Given three sets of positive integers $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_m\}$ with $\sum_{i=1}^m (x_i + y_i + z_i) = mb$, decide if there exist one-to-one functions ϕ and ψ on the set $\{1, 2, \ldots, m\}$ such that

$$x_i + y_{\phi(i)} + z_{\psi(i)} = b \quad (i = 1, 2, \ldots, m)\,.$$

Theorem 1 *$P||\sum C_j^2$ is strongly NP-hard.*

Proof Given an instance of N3DM, we create an instance of $P||\sum C_j^2$ with m machines and $3m$ jobs. For $i = 1, 2, \ldots, m$, jobs J_{3i-2} , J_{3i-1} , J_{3i} require processing times

$$\begin{aligned} p_{3i-2} &= x + \frac{x_i}{2}\,, \\ p_{3i-1} &= y + \frac{y_i}{2}\,, \\ p_{3i} &= z + z_i\,, \end{aligned}$$

respectively, where

$$\begin{aligned} x &= \frac{mb}{2}\,, \\ y &= m\left(x + \frac{b}{2}\right)\,, \\ z &= 2m\left(x + y + \frac{b}{2}\right)\,. \end{aligned}$$

Set the threshold value

$$t = \sum_{i=1}^m \left(x + \frac{x_i}{2}\right)^2 + \frac{1}{2}\sum_{i=1}^m (z + z_i)^2$$

$$
\begin{aligned}
& +2m\left(x+y+\frac{z}{2}+\frac{b}{2}\right)^2 \\
= & \sum_{i=1}^{m}\left(x+\frac{x_i}{2}\right)^2+\frac{1}{2}\sum_{i=1}^{m} z_i^2 \\
& +\sum_{i=1}^{m} z_i z+\left(1+m+\frac{1}{2m}\right) z^2 .
\end{aligned}
$$

We will show that there exist one-to-one functions ϕ and ψ for the N3DM instance such that $x_i + y_{\phi(i)} + z_{\psi(i)} = b$ $(i = 1, 2, \ldots, m)$ if and only if the constructed instance of $P||\sum C_j^2$ has a schedule π such that $Q(\pi) \leq t$.

First, suppose that ϕ and ψ are one-to-one functions such that $x_i + y_{\phi(i)} + z_{\psi(i)} = b$ $(i = 1, 2, \ldots, m)$. We form the schedule π by scheduling jobs $J_{3i-2}, J_{3\phi(i)-1}, J_{3\psi(i)}$ in that order on the ith machine. It holds that

$$
\begin{aligned}
Q(\pi) &= \sum_{i=1}^{m}\left(C_{3i-2}(\pi)^2+C_{3\phi(i)-1}(\pi)^2+C_{3\psi(i)}(\pi)^2\right) \\
&= \sum_{i=1}^{m}\left(\left(x+\frac{x_i}{2}\right)^2+\left(x+\frac{x_i}{2}+y+\frac{y_{\phi(i)}}{2}\right)^2\right. \\
&\qquad \left. +\left(x+\frac{x_i}{2}+y+\frac{y_{\phi(i)}}{2}+z+z_{\psi(i)}\right)^2\right) \\
&= \sum_{i=1}^{m}\left(\left(x+\frac{x_i}{2}\right)^2+\frac{1}{2}\left(z+z_{\psi(i)}\right)^2\right. \\
&\qquad \left. +2\left(x+\frac{x_i}{2}+y+\frac{y_{\phi(i)}}{2}+\frac{z+z_{\psi(i)}}{2}\right)^2\right) \\
&= t .
\end{aligned}
$$

Conversely, suppose that the constructed instance of $P||\sum C_j^2$ has a schedule π such that $Q(\pi) \leq t$. Without loss of generality, we may require that the jobs on the same machine are scheduled according to the *SPT* rule. Then, for any J_{3i-2} , J_{3j-1} , J_{3k} on the same machine, their relative order is J_{3i-2} , J_{3j-1} , J_{3k}. The following three claims further restrict the form of π.

(i) $J_3, J_6, \ldots, J_{3m}$ are assigned to different machines in π.

Suppose to the contrary that there are some two jobs among $J_3, J_6, \ldots, J_{3m}$ to be processed on the same machine. Then,

$$
\begin{aligned}
Q(\pi) &= \sum_{i=1}^{3m} C_i(\pi)^2 \\
&> \sum_{i=1}^{m} C_{3i}(\pi)^2 \\
&> \min_{1\leq k\leq m}\left(\sum_{i=1; i\neq k}^{m}(z+z_i)^2+(2z+z_k)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&> (m+3)z^2 + \sum_{i=1}^{m} \left(2zz_i + z_i^2\right) \\
&> t + \left(2 - \frac{1}{2m}\right) z^2 - \sum_{i=1}^{m} \left(x + \frac{x_i}{2}\right)^2 \\
&> t\,,
\end{aligned}
$$

where the last inequality follows from the definitions of x and z. Then we have a contradiction.

(ii) $J_2, J_5, \ldots, J_{3m-1}$ are assigned to different machines in π.

Let t_i denote the start time of job J_{3i} in π. Then,

$$
\begin{aligned}
Q(\pi) &> \sum_{i=1}^{m} t_i^2 + \sum_{i=1}^{m} (t_i + z + z_i)^2 \\
&> 2\sum_{i=1}^{m} t_i^2 + mz^2 + \sum_{i=1}^{m} \left(z_i^2 + 2zt_i + 2zz_i\right)\,.
\end{aligned}
$$

Since J_{3i} should be the last job on its machine, it holds that

$$
\begin{aligned}
\sum_{i=1}^{m} (2zt_i + zz_i) &= 2z \sum_{i=1}^{m} \left(x + y + \frac{x_i}{2} + \frac{y_i}{2} + \frac{z_i}{2}\right) \\
&= 2zm \left(x + y + \frac{b}{2}\right) \\
&= z^2\,. \qquad (1)
\end{aligned}
$$

Then,

$$
Q(\pi) > t + 2\sum_{i=1}^{m} t_i^2 - \sum_{i=1}^{m} \left(x + \frac{x_i}{2}\right)^2 - \frac{z^2}{2m}\,.
$$

If some two jobs among $J_2, J_5, \ldots, J_{3m-1}$ are processed on the same machine, then

$$
\sum_{i=1}^{m} t_i^2 > (m+3)y^2 + \sum_{i=1}^{m} \left(x + \frac{x_i}{2}\right)^2\,.
$$

Noticing that $2(m+3)y^2 \geq \frac{z^2}{2m}$, we have that $Q(\pi) > t$, a contradiction.

(iii) $J_1, J_4, \ldots, J_{3m-2}$ are assigned to different machines in π.

Let s_i denote the start time of job J_{3i-1} in π. Then,

$$
\begin{aligned}
Q(\pi) &\geq \sum_{i=1}^{m} s_i^2 + \sum_{i=1}^{m} t_i^2 + \sum_{i=1}^{m} (t_i + z + z_i)^2 \\
&= \sum_{i=1}^{m} s_i^2 + 2\sum_{i=1}^{m} t_i^2 + mz^2 \\
&\quad + \sum_{i=1}^{m} \left(z_i^2 + 2zt_i + 2zz_i + 2t_i z_i\right)\,.
\end{aligned}
$$

Noticing (1) and $\sum_{i=1}^m s_i^2 \geq \sum_{i=1}^m \left(x + \frac{x_i}{2}\right)^2$, we have that

$$Q(\pi) > t + 2\sum_{i=1}^m t_i^2 + 2\sum_{i=1}^m t_i z_i - \frac{z^2}{2m}.$$

Since $t_i > y$ and $\sum_{i=1}^m t_i^2 = \sum_{i=1}^m \left(s_i + y + \frac{y_i}{2}\right)^2$, we further obtain that

$$\begin{aligned} Q(\pi) &> t + 2\sum_{i=1}^m s_i^2 + 2my^2 + 4y\sum_{i=1}^m s_i \\ &\quad +2y\sum_{i=1}^m y_i + 2y\sum_{i=1}^m z_i - \frac{z^2}{2m} \\ &= t + 2\sum_{i=1}^m s_i^2 + 2my^2 + 4y\sum_{i=1}^m \left(x + \frac{x_i}{2}\right) \\ &\quad +2y\sum_{i=1}^m (y_i + z_i) - \frac{z^2}{2m} \\ &= t + 2\sum_{i=1}^m s_i^2 + 2my(y + 2x + b) - \frac{z^2}{2m} \\ &= t + 2\sum_{i=1}^m s_i^2 - 2m\left(x + \frac{b}{2}\right)^2. \end{aligned}$$

If some two jobs among $J_1, J_4, \ldots, J_{3m-2}$ are processed on the same machine, then $\sum_{i=1}^m s_i^2 > (m+3)x^2$, and hence $Q(\pi) > t$, a contradiction.

By claims (i) $\sim$ (iii), we may define one-to-one functions $\phi(i)$ and $\psi(i)$ such that $J_{3i-2}, J_{3\phi(i)-1}, J_{3\psi(i)}$ are the jobs processed on the same machine in π. Since

$$\begin{aligned} Q(\pi) &= \sum_{i=1}^m \left(C_{3i-2}(\pi)^2 + C_{3\phi(i)-1}(\pi)^2 + C_{3\psi(i)}(\pi)^2\right) \\ &= \sum_{i=1}^m \left(\left(x + \frac{x_i}{2}\right)^2 + \frac{1}{2}\left(z + z_{\psi(i)}\right)^2 \right. \\ &\quad \left. + 2\left(x + \frac{x_i}{2} + y + \frac{y_{\phi(i)}}{2} + \frac{z + z_{\psi(i)}}{2}\right)^2\right), \end{aligned}$$

it follows from $Q(\pi) \leq t$ that $x_i + y_{\phi(i)} + z_{\psi(i)} = b$ $(i = 1, 2, \ldots, m)$. This completes the proof of the strong NP-hardness of $P||\sum C_j^2$. □

3 Analysis of the SPT rule

In this section, we show that the SPT rule is asymptotically optimal for $P||\sum C_j^2$ under certain assumptions on the probability distribution of job processing times.

3.1 Preliminaries

A sequence of random variables X_k $(k = 1, 2, \ldots)$ is said to converge in probability to the constant c if for every $\epsilon > 0$, $\lim_{k\to\infty} Pr(|X_k - c| < \epsilon) = 1$, where $Pr(A)$ is the probability of the event A.

Lemma 1 (Lemma 4 in Ng et al. [14]) *If X_k and Y_k $(k = 1, 2, \ldots)$ converge in probability to c and d respectively, where $d \neq 0$, then $\frac{X_k}{Y_k}$ converges in probability to $\frac{c}{d}$.*

Let $E(X)$ and $D(X)$ represent the expectation and variance of the random variable X, respectively.

Lemma 2 (Lemma 5 in [14]) *If $\lim_{k\to\infty} E(X_k) = c$ and $\lim_{k\to\infty} D(X_k) = 0$, then X_k converges in probability to c.*

Let $X_1, X_2, \ldots, X_n$ be n random variables. Reordering the random variables increasingly, we obtain the order statistics $X_{[1]}, X_{[2]}, \ldots, X_{[n]}$.

Lemma 3 (David [7]) *Let $X_1, X_2, \ldots, X_n$ be n independent random variables uniformly distributed in $(0, 1)$. Then, for any nonnegative integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_l \leq n$ and $\alpha_1, \alpha_2, \ldots, \alpha_l$, it holds that*

$$E\left(\prod_{i=1}^{l} X_{[r_i]}^{\alpha_i}\right) = \frac{n!}{\left(n + \sum_{i=1}^{l} \alpha_i\right)!} \prod_{i=1}^{l} \frac{\left(r_i - 1 + \sum_{j=1}^{i} \alpha_j\right)!}{\left(r_i - 1 + \sum_{j=1}^{i-1} \alpha_j\right)!}.$$

The next lemma gives an estimation of the sum of powers of positive integers.

Lemma 4 (Spiegel and Liu [16]) $1^\lambda + 2^\lambda + \cdots + n^\lambda = \frac{n^{\lambda+1}}{\lambda+1} + O(n^\lambda)$.

In the last part of this subsection, we give a lower bound for $P||\sum C_j^2$. Let $n = km + v$ $(0 \leq v < m)$ and $J_{[i]}$ $(i = 1, 2, \ldots, n)$ denote the job with the ith shortest processing time. Let

$$L = \frac{1}{m} \sum_{i=0}^{k} \left(\sum_{j=1}^{im+v} p_{[j]}\right)^2.$$

Lemma 5 *L is a lower bound on the optimal value of $P||\sum C_j^2$.*

Proof The proof is done by induction on n. If $n \leq m$, then $L = \frac{1}{m}(\sum_{j=1}^{n} p_{[j]})^2$ and the conclusion holds. Consider the case of $n \geq m + 1$. Let π_* be an optimal schedule and J_{n_i} $(i = 1, 2, \ldots, m)$ be the last job on the ith machine in π_*. Let $\mathcal{I}$ be the set consisting of the $(k-1)m+v$ jobs preceding J_{n_i} $(i=1,2,\ldots,m)$ in π_* and Q_0 be the sum of their quadratic completion times. Then, it holds that

$$Q(\pi_*) = Q_0 + \sum_{i=1}^{m} C_{n_i}(\pi_*)^2. \tag{2}$$

By reducing the processing times of some jobs in $\mathcal{I}$, we can obtain a job set with processing times $p_{[1]}, p_{[2]}, \ldots, p_{[(k-1)m+v]}$. Then, by the induction hypothesis, we have that

$$Q_0 \geq \frac{1}{m} \sum_{i=0}^{k-1} \left(\sum_{j=1}^{im+v} p_{[j]} \right)^2 . \tag{3}$$

Since $\sum_{i=1}^{m} C_{n_i}(\pi_*) = \sum_{j=1}^{km+v} p_{[j]}$, it holds that

$$\sum_{i=1}^{m} C_{n_i}(\pi_*)^2 \geq \frac{1}{m} \left(\sum_{j=1}^{km+v} p_{[j]} \right)^2 . \tag{4}$$

Combining (2), (3) and (4), we obtain $Q(\pi_*) \geq L$. □

3.2 Asymptotic optimality

We say that a schedule is asymptotically optimal if its relative error converges in probability to zero. Let π_{spt} denote the *SPT* schedule. In this subsection, we show the asymptotic optimality of π_{spt}. For this purpose, we assume that $p_1, p_2, \ldots, p_n$ are independent random variables uniformly distributed in $(0,1)$.

According to the *SPT* rule, for $l = 1, 2, \ldots, v$, the lth machine processes $k+1$ jobs: $J_{[jm+l]}$ $(j = 0, 1, \ldots, k)$, and for $l = v+1, v+2, \ldots, m$, the lth machine processes k jobs: $J_{[(j-1)m+l]}$ $(j = 1, 2, \ldots, k)$. Then,

$$\begin{aligned} Q(\pi_{spt}) &= \sum_{l=1}^{v} \sum_{i=0}^{k} \left(\sum_{j=0}^{i} p_{[jm+l]} \right)^2 \\ &\quad + \sum_{l=v+1}^{m} \sum_{i=1}^{k} \left(\sum_{j=1}^{i} p_{[(j-1)m+l]} \right)^2 \\ &\leq m \sum_{i=0}^{k} \left(\sum_{j=0}^{i} p_{[jm+v]} \right)^2 \stackrel{\text{def}}{=} X, \end{aligned}$$

where $p_{[0]} = 0$ is assumed if $v = 0$. On the other hand, we have

$$\begin{aligned} Q(\pi_*) \geq L &\geq \frac{1}{m} \sum_{i=1}^{k} \left(m \sum_{j=0}^{i-1} p_{[jm+v]} \right)^2 \\ &= m \sum_{i=0}^{k-1} \left(\sum_{j=0}^{i} p_{[jm+v]} \right)^2 \stackrel{\text{def}}{=} Y . \end{aligned}$$

Then,

$$Q(\pi_{spt}) - Q(\pi_*) \le X - Y = m\left(\sum_{j=0}^{k} p_{[jm+v]}\right)^2 .$$

Noticing $\sum_{i=0}^{k-1}\sum_{j=0}^{i} p_{[jm+v]} = \sum_{j=0}^{k-1}(k-j)p_{[jm+v]}$, we have

$$Q(\pi_*) \ge Y \ge \frac{m}{k}\left(\sum_{j=0}^{k-1}(k-j)p_{[jm+v]}\right)^2 \stackrel{\text{def}}{=} Z .$$

Lemma 6 $\frac{X-Y}{k^2}$ $(k = 1, 2, \ldots)$ *converges in probability to* $\frac{m}{4}$.

Proof It follows from Lemma 3 that

$$E(p_{[jm+v]}p_{[lm+v]}) = \frac{(lm+v)(jm+v+1)}{(km+v+1)(km+v+2)}$$

if $l \le j$. Thus,

$$\begin{aligned} E(X-Y) &= m\sum_{j=0}^{k} E\left(p_{[jm+v]}^2\right) \\ &\quad +2m\sum_{j=1}^{k}\sum_{l=0}^{j-1} E\left(p_{[jm+v]}p_{[lm+v]}\right) \\ &= 2m\sum_{j=1}^{k}\sum_{l=0}^{j-1}\frac{lj}{k^2} + O(k) \\ &= \frac{mk^2}{4} + O(k) . \end{aligned}$$

Since $D(X-Y) = E(X-Y)^2 - (E(X-Y))^2$ and

$$\begin{aligned} E(X-Y)^2 &= m^2 E\left(\sum_{j=0}^{k} p_{[jm+v]}\right)^4 \\ &= m^2 \sum_{\alpha_0+\cdots+\alpha_k=4} \frac{4!}{\alpha_0!\cdots\alpha_k!} E\left(\prod_{j=0}^{k} p_{[jm+v]}^{\alpha_j}\right) \\ &= m^2 \sum_{\alpha_0+\cdots+\alpha_k=4} \frac{4!n!}{\alpha_0!\cdots\alpha_k!(n+4)!} \prod_{j=0}^{k} \frac{\left(jm+v-1+\sum_{i=0}^{j}\alpha_i\right)!}{\left(jm+v-1+\sum_{i=0}^{j-1}\alpha_i\right)!} \\ &= m^2 \sum_{\alpha_0+\cdots+\alpha_k=4} \frac{4!n!}{\alpha_0!\cdots\alpha_k!(n+4)!} \left(\prod_{j=0}^{k}(jm)^{\alpha_j} + O(k^3)\right) \\ &= \frac{m^2 n!}{(n+4)!}\left(\left(\sum_{j=0}^{k} jm\right)^4 + O(k^7)\right) \\ &= \frac{m^2k^4}{16} + O(k^3) , \end{aligned}$$

$D(X-Y)=O(k^3)$ holds. By Lemma 2, $\frac{X-Y}{k^2}$ $(k=1,2,\ldots)$ converges in probability to $\frac{m}{4}$. □

Lemma 7 $\frac{Z}{k^3}$ $(k=1,2,\ldots)$ *converges in probability to* $\frac{m}{36}$.

Proof Using Lemmas 3 and 4, we have

$$\begin{aligned} E(Z) &= \frac{m}{k}\sum_{j=0}^{k-1}(k-j)^2 E\left(p^2_{[jm+v]}\right) \\ &\quad + \frac{2m}{k}\sum_{j=1}^{k-1}\sum_{l=0}^{j-1}(k-j)(k-l)E\left(p_{[jm+v]}p_{[lm+v]}\right) \\ &= \frac{2m}{k^3}\sum_{j=1}^{k-1}\sum_{l=0}^{j-1}(k-j)(k-l)lj + O(k^2) \\ &= \frac{mk^3}{36} + O(k^2)\,. \end{aligned}$$

Since $D(Z)=E(Z^2)-(E(Z))^2$ and

$$\begin{aligned} E(Z^2) &= \frac{m^2}{k^2}E\left(\sum_{j=0}^{k-1}(k-j)p_{[jm+v]}\right)^4 \\ &= \frac{m^2}{k^2}\sum_{\alpha_0+\cdots+\alpha_{k-1}=4}\frac{4!\prod_{j=0}^{k-1}(k-j)^{\alpha_j}}{\alpha_0!\cdots\alpha_{k-1}!}E\left(\prod_{j=0}^{k-1}p^{\alpha_j}_{[jm+v]}\right) \\ &= \frac{m^2}{k^2}\sum_{\alpha_0+\cdots+\alpha_{k-1}=4}\frac{4!n!\prod_{j=0}^{k-1}(k-j)^{\alpha_j}}{\alpha_0!\cdots\alpha_{k-1}!(n+4)!}\prod_{j=0}^{k-1}\frac{\left(jm+v-1+\sum_{i=0}^{j}\alpha_i\right)!}{\left(jm+v-1+\sum_{i=0}^{j-1}\alpha_i\right)!} \\ &= \frac{m^2}{k^2}\sum_{\alpha_0+\cdots+\alpha_{k-1}=4}\frac{4!n!\prod_{j=0}^{k-1}(k-j)^{\alpha_j}}{\alpha_0!\cdots\alpha_{k-1}!(n+4)!}\left(\prod_{j=0}^{k-1}(jm)^{\alpha_j}+O(k^3)\right) \\ &= \frac{m^2n!}{k^2(n+4)!}\left(\left(\sum_{j=0}^{k-1}(k-j)jm\right)^4 + O(k^3)\left(\sum_{j=0}^{k-1}(k-j)\right)^4\right) \\ &= \frac{m^2k^6}{6^4}+O(k^5)\,, \end{aligned}$$

$D(Z)=O(k^5)$ holds. By Lemma 2, $\frac{Z}{k^3}$ $(k=1,2,\ldots)$ converges in probability to $\frac{m}{36}$. □

Now we can prove the asymptotic optimality of π_{spt}.

Theorem 2 *Suppose that* $p_1,p_2,\ldots,p_n$ *are independent random variables uniformly distributed in* $(0,1)$. *Then, the relative error of* π_{spt} *converges in probability to zero.*

Proof By Lemmas 1, 6 and 7, we know that $\frac{k(X-Y)}{Z}$ $(k=1,2,\ldots)$ converges in probability to constant 9. Then, $\frac{X-Y}{Z}$ converges in probability to zero, which implies the desired conclusion. □

3.3 Computational experience and improvement

To evaluate the empirical performance of the *SPT* rule, we performed a series of computational experiments, where n varied from 20 to 1000 and m varied from 2 to 100. For each combination of n and m, 500 instances were generated with processing times drawn from integers uniformly distributed in $(0, 1000)$. The effectiveness of π_{spt} was measured by $(Q(\pi_{spt}) - L)/L$, i.e., the relative error between $Q(\pi_{spt})$ and the lower bound L defined in Lemma 5. We kept the records of the average relative error and maximum relative error among the 500 instances. The results are given in Tables 1 and 2. Note that we did not test the instances with $n \leq m$ since π_{spt} is optimal for them.

From the computational results, we see that both the average and maximum relative errors tend to zero as n increases. The speed of convergence correlates with n/m since each combination of n and m on the same diagonal has a similar value of n/m and the errors on the same diagonal are similar. The convergence rate is fast. For example, when $n/m = 2$, 5, 10, the average relative error is about 10%, 2%, 0.5%, respectively.

The computational results reported in Tables 1 and 2 are very good in many cases, but the error is not negligible when n/m is fairly small. For example, the average relative error comes to about 10% when $n/m = 2$. So we see a need to improve the *SPT* rule. Indeed, the *SPT* rule divides all $n = km + v$ jobs into $k + 1$ groups, where the first group has v jobs and every other group has m jobs, and schedules the ith $(1 \leq i \leq m)$ longest job of each group on the same machine. Hence, the burden on each machine is unbalanced. We modified the rule so that the ith longest job of each group is assigned to the machine that has the ith lightest burden after the preceding groups have been assigned. Let π'_{spt} denote the resulting schedule. It is easy to see that for each group, the total quadratic completion time under π'_{spt} is no more than the total quadratic completion time under π_{spt}. Thus, it holds that $Q(\pi'_{spt}) \leq Q(\pi_{spt})$. For π'_{spt}, our computational results are given in Tables 3 and 4. The results suggest that the modification effectively reduces the relative error.

4 Concluding remarks

In this paper we considered the parallel machine scheduling problem of minimizing the sum of quadratic job completion times. The problem was proved to be strongly NP-hard and the performance of the *SPT* rule was evaluated by probabilistic analysis and computational experiments. Although in our probabilistic analysis, the job processing times were assumed to be uniformly distributed in $(0, 1)$, the analysis with minor adjustments is applicable to job processing times uniformly distributed in any fixed

Table 1: Average relative error of π_{spt} in percentage

	$m=2$	5	10	20	50	100
$n=20$	0.3850	2.7003	9.9016	–	–	–
50	0.0638	0.4862	1.8740	7.0699	–	–
100	0.0164	0.1270	0.5030	1.8851	9.8983	–
200	0.0041	0.0326	0.1310	0.5033	2.8328	9.8734
500	0.0007	0.0053	0.0215	0.0855	0.5075	1.8726
1000	0.0002	0.0013	0.0054	0.0217	0.1320	0.5068

Table 2: Maximum relative error of π_{spt} in percentage

	$m=2$	5	10	20	50	100
$n=20$	1.2969	6.2572	25.4787	–	–	–
50	0.1535	0.8826	3.0029	12.5565	–	–
100	0.0321	0.1994	0.7198	2.6925	15.7039	–
200	0.0068	0.0428	0.1741	0.6550	3.7935	13.1489
500	0.0009	0.0063	0.0260	0.1032	0.5887	2.2670
1000	0.0004	0.0016	0.0062	0.0247	0.1479	0.5673

Table 3: Average relative error of π'_{spt} in percentage

	$m=2$	5	10	20	50	100
$n=20$	0.0178	0.2953	2.6612	–	–	–
50	0.0006	0.0104	0.1012	0.8842	–	–
100	–	0.0007	0.0073	0.0864	2.0964	–
200	–	–	0.0005	0.0060	0.1764	2.0211
500	–	–	–	0.0002	0.0053	0.0841
1000	–	–	–	–	0.0003	0.0052

Table 4: Maximum relative error of π'_{spt} in percentage

	$m=2$	5	10	20	50	100
$n=20$	0.2279	1.5934	10.2781	–	–	–
50	0.0070	0.0580	0.2833	3.5575	–	–
100	–	0.0022	0.0168	0.2527	3.6636	–
200	–	–	0.0010	0.0105	0.2889	2.9041
500	–	–	–	0.0003	0.0084	0.1206
1000	–	–	–	–	0.0006	0.0074

interval.

Noticing that L defined in Lemma 5 is also a lower bound for the preemptive scheduling problem $P|pmtn|\sum C_j^2$, where the processing of any job may be interrupted and resumed at a later time on the same or a different machine, we may conclude that the SPT rule is asymptotically optimal for $P|pmtn|\sum C_j^2$. We note that the computational complexity of $P|pmtn|\sum C_j^2$ remains open and preemption sometimes is necessary. However, preemption is unnecessary for the linear cost problem $P|pmtn|\sum C_j$.

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