A New Approach to the $L(2,1)$-Labeling of Some Products of Graphs

Wai Chee Shiu, Zhendong Shao, Kin Keung Poon, and David Zhang, *Senior Member, IEEE*

***Abstract*—The frequency assignment problem is to assign a frequency which is a nonnegative integer to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. This frequency assignment problem can be modelled with vertex labelings of graphs. An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between x and y in G. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v): v \in V(G)\} = k$. In this paper, we develop a dramatically new approach on the analysis of the adjacency matrices of the graphs to estimate the upper bounds of λ-numbers of the four standard graph products. By the new approach, we can achieve more accurate results and with significant improvement of the previous bounds.**

***Index Terms*—Channel assignment, $L(2,1)$-labeling, Cartesian product, lexicographic product, direct product, strong product.**

I. Introduction

The frequency assignment problem is to assign a frequency which is a nonnegative integer to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. Hale [9] formulated this into a graph vertex coloring problem.

In 1991, Roberts [19] proposed a variation of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if they are of distance 2 in the graph. Based on this problem, Griggs and Yeh [8] considered an $L(2,1)-$*labeling* on a simple graph. An $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between x and y in G. A k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling.

From then on, a large number of articles have been published devoted to the study of the frequency assignment problem and its connections to graph labelings, in particular, to the class of $L(2,1)$-labelings and its generalizations: Over 100 references on the subject are provided in a very comprehensive survey [3]. In addition to graph theory and combinatorial techniques, other interesting approaches in studying these labelings include neural networks [7], [14]; genetic algorithms [17], and simulated annealing [18]. Most of these papers are considering the values of λ on particular classes of graphs.

From the algorithmic point of view, it is not surprising that it is NP-complete to decide whether a given graph G allows an $L(2,1)$-labeling of span at most n [8]. Hence, good lower and upper bounds for λ are clearly welcome. For instance, if G is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$. The upper bound is attainable by Moore graphs (diameter 2 graph with order Δ^2+1), see [8]. Such graphs exist only if $\Delta = 2, 3, 7$, and possibly 57.

The above considerations motivated Griggs and Yeh [8] to conjecture that for any graph G with the maximum degree $\Delta \geq 2$, the best upper bound on $\lambda(G)$ is Δ^2 (Griggs-Yeh conjecture). Noted that this is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$. Griggs and Yeh provided an upper bound $\Delta^2 + 2\Delta$ for general graphs with maximum degree Δ. Chang and Kuo [4] improved the bound to $\Delta^2+\Delta$ and later on Král and Škrekovski [16] further reduced the bound to $\Delta^2+\Delta-1$.

Graph products play an important role in connecting various useful networks and they also serve as natural tools for different concepts in many areas of research. For example, the diagonal mesh with respect to multiprocessor network is representable by the direct product of two odd cycles [22] and one of the central concepts of information theory, the Shannon capacity, is most naturally expressed with the strong product of graphs, cf. [23].

The Cartesian product, the lexicographic product, the direct product and the strong product constitute the four standard graph products [10]. In [21] and [13], Shao *et al.* proved that the L(2,1)-labeling number of the four standard product graphs are bounded by the square of its maximum degree respectively. Hence, the Griggs–Yeh conjecture holds (with some minor exception). Recently, Shao *et al.* [20] improved the upper bounds obtained in [13] with a more refined analysis of neighborhoods in product graphs than the analysis in [13].

The main contribution of this paper is to present a new approach to derive the upper bounds of λ-numbers of the four standard graph products.

Manuscript received November 6, 2007; revised January 7, 2008. First published July 16, 2008; last published August 13, 2008 (projected). This work was supported in part by FRG, Hong Kong Baptist University. The work of D. Zhang was supported in part by the UGC/CRC fund for HKSAR and the NSFC and the 863 funds under Contract 60620160097 and Contract 2006AA01Z193. This paper was recommended by Associate Editor S. Tsukiyama.

W. C. Shiu was with the Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong (e-mail: wcshiu@hkbu.edu.hk).

Z. Shao is with the Department of Computer Science, The University of Western Ontario, London, ON N6A 5B7, Canada (e-mail: zhdshao0026@163.com).

K. K. Poon is with the Department of Mathematics, Science, Social Sciences and Technology, Hong Kong Institute of Education, Hong Kong (e-mail: kkpoon@ied.edu.hk).

D. Zhang is with the Department of Computing, Hong Kong Polytechnic University, Kowloon, Hong Kong (e-mail: csdzhang@comp.polyu.edu.hk).

Digital Object Identifier 10.1109/TCSII.2008.922450

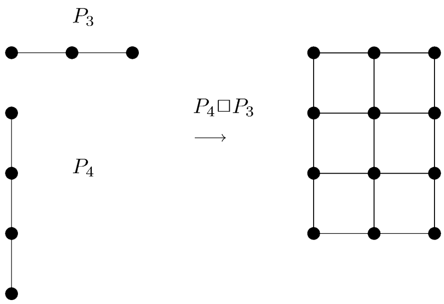

Fig. 1. Cartesian product of graphs.

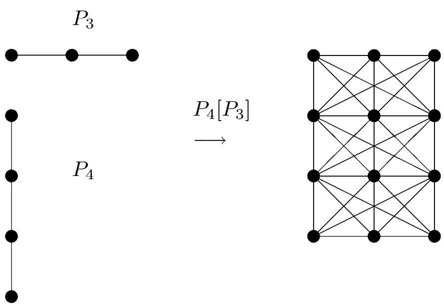

Fig. 2. Composition of graphs.

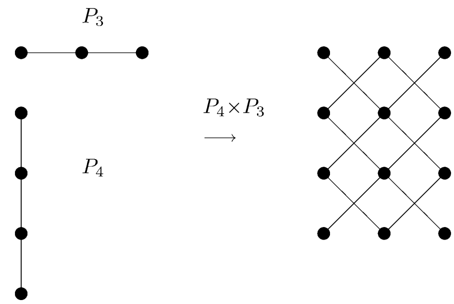

Fig. 3. Direct product of graphs.

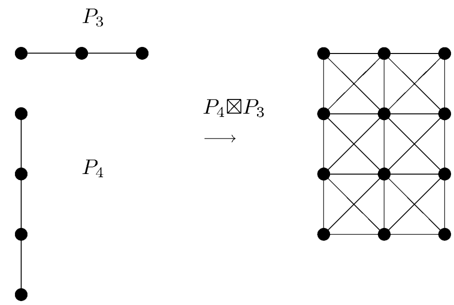

Fig. 4. Strong product of graphs.

A heuristic labeling algorithm is presented that forms the basis for these considerations in Section 3 while the four standard products of graphs are considered respectively in Section 4. Improvements (if any) with respect to the previously known upper bounds are explicitly computed.

Throughout the paper, all graphs are assumed to be simple (i.e., no loop and no parallel edge).

II. Four Standard Products of Graphs

Let G and H be two graphs of orders ν_1 and ν_2, respectively. Let Δ_1 and Δ_2 be the maximum degrees of G and H, respectively.

There are four standard products of graphs, namely, the Cartesian product, composition product (i.e., lexicographic product), direct product and strong product. Let $V(G) = \{u_1, \dots, u_{\nu_1}\}$ and $V(H) = \{v_1, \dots, v_{\nu_2}\}$ be the vertex sets of G and H, respectively. The vertex sets of these four product graphs are the same, which is $V(G) \times V(H)$. In this paper, we shall list the vertex set $V(G) \times V(H)$ in a lexicographic order.

The *Cartesian product* of G and H is denoted by $G \Box H$. In $G \Box H$, the vertex (v, w) is adjacent to the vertex (v', w') if and only if either $v = v'$ and $ww' \in E(H)$, or $w = w'$ and $vv' \in E(G)$. Fig. 1 shows the Cartesian product of P_4 and P_3.

The *composition* (or *lexicographic product*) of G with H is denoted by $G[H]$ or $G \circ H$. In $G[H]$, (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. Fig. 2 shows the composition of P_4 with P_3.

The *direct product* $G \times H$ of G and H is the graph in which the vertex (v, w) is adjacent to the vertex (v', w') if and only if $vv' \in E(G)$ and $ww' \in E(H)$. Fig. 3 shows the direct product of P_4 and P_3.

The *strong product* $G \boxtimes H$ of G and H is the graph, in which the vertex (v, w) is adjacent to the vertex (v', w') if and only if $v = v'$ and $ww' \in E(H)$, or $w = w'$ and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$. Fig. 4 shows the strong product of P_4 and P_3.

TABLE I

product graph	adjacency matrix
$G \Box H$	$A_1 \otimes I_2 + I_1 \otimes A_2$
$G[H]$	$A_1 \otimes J_2 + I_1 \otimes A_2$
$G \times H$	$A_1 \otimes A_2$
$G \boxtimes H$	$A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$

Suppose A_1 and A_2 are the adjacency matrices of G and H, respectively. We can write down the adjacency matrices of these four product graphs. Those matrices involve the Kronecker product $\otimes$ of the matrices (cf. [5]). Namely, we have Table I, where I_1 is the identity matrix of order ν_1, I_2 is the identity matrix of order ν_2, J_2 is the square matrix of order ν_2 with all entries 1.

III. Labeling Algorithm

A subset X of $V(G)$ is called an i-*stable set* (or i-*independent set*), if the distance between any two vertices in X is greater than i. A 1-stable (independent) set is a usual independent set. A *maximal* 2-stable subset X of a set Y is a 2-stable subset of Y such that X is not a proper subset of any 2-stable subset of Y.

Chang and Kuo [4] proposed the following algorithm to obtain an L(2,1)-labeling and the maximum value of that labeling on a given graph.

Algorithm 2.1.

Input: A graph $G = (V, E)$.

Output: The value k is the maximum label.

Idea: In each step, find a maximal 2-stable set from unlabeled vertices that are of distance at least two away from those vertices labeled in the previous step. Then label all vertices in that 2-stable set with the index i in the current stage. The index i starts from 0 and then increases by 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \phi$;
$V = V(G); i = 0$.

Iteration:

1) Determine Y_i and X_i.
   - $Y_i = \{x \in V : x$ is unlabeled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$.
   - X_i a maximal 2-stable subset of Y_i.
   - If $Y_i = \phi$, then set $X_i = \phi$.
2) Label vertices in X_i (if any) by i.
3) $V \leftarrow V \setminus X_i$.
4) If $V \neq \phi$, then $i \leftarrow i + 1$ and go to Step 1.
5) Record the current i as k (which is the maximum label). Stop.

Thus k is an upper bound on $\lambda(G)$. In addidtion, we would like to obtain a bound in terms of the maximum degree $\Delta(G)$ of G instead of in terms of the chromatic number $\chi(G)$.

Let $f : V \to \{0, \ldots, k\}$ be a labeling obtained in the Algorithm 2.1 and x be a vertex with the largest label k. Denote

$$\begin{aligned} I_1 =&\{i : 0 \leq i \leq k-1 \text{ and } d(x, y) = 1, \\ &\text{for some } y \in X_i\} = \{f(y) : d(x, y) = 1, \\ &\text{for some } y \in X_i, 1 \leq i \leq k-1\} \\ I_2 =&\{i : 0 \leq i \leq k-1 \text{ and } d(x, y) \leq 2, \\ &\text{for some } y \in X_i\} = \{f(y) : d(x, y) \leq 2, \\ &\text{for some } y \in X_i, 1 \leq i \leq k-1\} \\ I_3 =&\{i : 0 \leq i \leq k-1 \text{ and } d(x, y) \geq 3, \\ &\text{for all } y \in X_i\} = \{0, 1, \ldots, k-1\} \setminus I_2. \end{aligned}$$

It is clear that $|I_2| + |I_3| = k$. For any $i \in I_3$, $x \notin Y_i$; otherwise $X_i \cup \{x\}$ is a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x, y) = 1$ for some vertices y in X_{i-1}; i.e., $i - 1 \in I_1$. So, $|I_3| \leq |I_1|$. Hence, $k = |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to find the upper bound of k, it suffices to estimate $|I_2| + |I_1|$ in terms of $\Delta(G)$.

Before eliminating the upper bound of k, we introduce a notation first. Let M be a matrix with n rows. For $1 \leq i \leq n, r_i(M)$ denote the number of nonzero entries in the ith row of M excluding the diagonal entry.

Let A be the adjacency matrix of G with respect to the list of vertices $\{v_1, \ldots, v_n\}$. Then it is well-known that the (i, j)th entry of A^k is the number of different (v_i, v_j)-walks in G of length k, for $k \geq 0$.

Thus, $r_i(A) = \deg(v_i), r_i(A^2)$ is the number of vertices joining by a walk of length 2 from v_i excluding v_i itself and $r_i(A^2 + A)$ is the number of vertices of distance 1 or 2 from v_i. So that

$$r_i(A^2) \leq \deg(v_i)(\Delta(G) - 1) \quad (1)$$

$$r_i(A^2 + A) \leq \deg(v_i)\Delta(G). \quad (2)$$

For convenience, the notations which have been introduced in this section will also be used in the following section.

IV. Main Results

The upper bounds of those four standard product graphs were studied in [13], [20], [21]. In this section, we will reconsider those cases by our new approach. Most of the upper bounds are improved.

Theorem 4.1: Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Then

$$\lambda(G \square H) \leq \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2.$$

Proof: Note from Table I that, the adjacency matrix of $G \square H$ is $A = A_1 \otimes I_2 + I_1 \otimes A_2$. Then

$$\begin{aligned} A^2 + A &= A_1^2 \otimes I_2^2 + 2A_1 \otimes A_2 \\ &\quad + I_1^2 \otimes A_2^2 + A_1 \otimes I_2 + I_1 \otimes A_2 \\ &= \left(A_1^2 + A_1\right) \otimes I_2 + I_1 \otimes \left(A_2^2 + A_2\right) \\ &\quad + 2A_1 \otimes A_2. \end{aligned}$$

Note that the rules of algebra of Kronecker product matrices can be found in [5].

Let k be the maximum label obtained by the Algorithm 2.1. Let $(u_i, v_j) \in V(G) \times V(H)$ be the vertex with the label k. We look at the (u_i, v_j)th row of the matrix $A^2 + A$. We have

$$\begin{aligned} r_{(u_i,v_j)}(A^2 + A) \leq& r_{(u_i,v_j)}\left(\left(A_1^2 + A_1\right) \otimes I_2\right) \\ &+ r_{(u_i,v_j)}(I_1 \otimes (A_2^2 + A_2)) \\ &+ r_{(u_i,v_j)}(2(A_1 \otimes A_2)) \\ =& r_i\left(A_1^2 + A_1\right) r_j(I_2) \\ &+ r_i(I_1) r_j\left(A_2^2 + A_2\right) \\ &+ r_i(A_1) r_j(A_2) \\ =& \deg_G(u_i)\Delta_1 + \deg_H(v_j)\Delta_2 \\ &+ \deg_G(u_i)\deg_H(v_j). \end{aligned}$$

Note that the last equality is obtained by applying (2).

Also we have known that $|I_1| \leq \Delta(G \square H) = \Delta_1 + \Delta_2$. Thus,

$$\begin{aligned} \lambda(G \square H) &\leq |I_2| + |I_1| \\ &\leq \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2. \end{aligned}$$ ■

The above result agrees with Shao and Yeh's result in [21].

Theorem 4.2: Let Δ_1 and Δ_2 be the maximum degree of G and H, respectively and let ν_2 be the order of H. Then

$$\lambda(G[H]) \leq \Delta_1^2\nu_2 + \Delta_2^2 + \Delta_1\nu_2 + \Delta_2.$$

Proof: From Table I, we get that the adjacency matrix of $G[H]$ is $A = A_1 \otimes J_2 + I_1 \otimes A_2$. Then

$$\begin{aligned} A^2 + A =& \nu_2 A_1^2 \otimes J_2 + A_1 \otimes J_2A_2 \\ &+ A_1 \otimes A_2J_2 + I_1 \otimes A_2^2 \\ &+ A_1 \otimes J_2 + I_1 \otimes A_2 \\ =& \nu_2 A_1^2 \otimes J_2 + A_1 \otimes (J_2A_2 + A_2J_2 + J_2) \\ &+ I_1 \otimes \left(A_2^2 + A_2\right). \end{aligned}$$

Since all entries of the involved matrices are nonnegative, the number of nonzero entries in the (u_i, v_j)th entry of $\nu_2 A_1^2 \otimes J_2 + A_1 \otimes (J_2A_2 + A_2J_2 + J_2)$ is the same as that of $A_1^2 \otimes J_2 + A_1 \otimes J_2 = (A_1^2 + A_1) \otimes J_2$. Thus, the number of nonzero entries in the (u_i, v_j)th entry of $A^2 + A$ excluding the diagonal entry is at most $\nu_2 \deg_G(u_i)\Delta_1 + \deg_H(v_j)\Delta_2$. Note that $|I_1| \leq \Delta(G[H]) = \Delta_1\nu_2 + \Delta_2$. Thus,

$$|I_2| + |I_1| \leq \Delta_1^2\nu_2 + \Delta_2^2 + \Delta_1\nu_2 + \Delta_2.$$

This completes the proof. ■

Hence if $H = K_1$, then $G[H] \cong G$ and $|I_2|+|I_1| \le \Delta_1^2+\Delta_1$. It agrees with Chang and Kuo's result [4].

In [21], it was proved that $\lambda(G[H]) \le \Delta^2 + \Delta - 2\nu_2\Delta_1$, where the maximum degree of $G[H]$ is $\Delta = \nu_2\Delta_1 + \Delta_2$.

Since $\Delta^2 + \Delta - 2\nu_2\Delta_1 - (\Delta_1^2\nu_2 + \Delta_2^2 + \Delta_1\nu_2 + \Delta_2) = \Delta_1^2\nu_2(\nu_2 - 1) + 2\Delta_1(\Delta_2 - 1)\nu_2$, we have reduced the bound by $\Delta_1^2\nu_2(\nu_2 - 1) + 2\Delta_1(\Delta_2 - 1)\nu_2$.

In [15] and [20], they obtained an upper bound for the $L(2,1)$-labeling number of the direct product of two graphs in terms of the maximum degrees of the graphs involved. We shall improve this bound.

Theorem 4.3: Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Then

$$\lambda(G \times H) \le \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2.$$

Proof: From Table I, we get that the adjacency matrix of $G \times H$ is $A = A_1 \otimes A_2$. Then

$$A^2 + A = A_1^2 \otimes A_2^2 + A_1 \otimes A_2.$$

Similar to the proof of the previous theorem, by (1) we have

$$\begin{aligned} |I_2| + |I_1| &\le [\Delta_1(\Delta_1 - 1)\Delta_2(\Delta_2 - 1) + \Delta_1\Delta_2] + \Delta_1\Delta_2 \\ &= \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2. \end{aligned}$$

This completes the proof. ■

In [20], it was proved that $\lambda(G \times H) \le \Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1)$, where the maximum degree of $G \times H$ is $\Delta = \Delta_1\Delta_2$. Theorem 4.3 is an improvement of this result. Since $\Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) - (\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2) = \Delta_1^2 + \Delta_2^2 - \Delta_1 - \Delta_2$, we have thus reduced the bound by $\Delta_1^2 + \Delta_2^2 - \Delta_1 - \Delta_2$.

In [12] the λ-numbers of the strong product of cycles are considered. In [15] and [20], they obtained a general upper bound for the λ-number of strong products in terms of maximum degrees of the factor graphs (and the product).

Theorem 4.4: Let Δ_1, and Δ_2 be the maximum degree of G and H, respectively. Then

$$\lambda(G \boxtimes H) \le \Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2.$$

Proof: From Table I, we get that the adjacency matrix of $G \boxtimes H$ is $A = A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$. Then

$$\begin{aligned} A^2 + A =&(A_1 \otimes A_2)^2 + (A_1 \otimes I_2 + I_1 \otimes A_2)^2 \\ &+ (A_1 \otimes A_2)(A_1 \otimes I_2 + I_1 \otimes A_2) \\ &+ (A_1 \otimes I_2 + I_1 \otimes A_2)(A_1 \otimes A_2) \\ =&(A_1^2 + A_1) \otimes \left(A_2^2 + A_2\right) + A_1^2 \otimes I_2 \\ &+ I_1 \otimes A_2^2. \end{aligned}$$

Similar to the proof of the previous theorem, by (1) and (2) we have

$$\begin{aligned} |I_2| + |I_1| \le&[\Delta_1^2\Delta_2^2 + \Delta_1(\Delta_1 - 1) \\ &+ \Delta_2(\Delta_2 - 1)] + \Delta_1 + \Delta_2 + \Delta_1\Delta_2 \\ =&\Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2. \end{aligned}$$

This completes the proof. ■

In [20], it was proved that $\lambda(G \boxtimes H) \le \Delta^2 + \Delta - (\Delta_1 + \Delta_2 + 4)\Delta_1\Delta_2\Delta^2 + \Delta_1 + \Delta_2 - 5\Delta_1\Delta_2$, where the maximum degree of $G \boxtimes H$ is $\Delta = \Delta_1\Delta_2 + \Delta_1 + \Delta_2$. Since $\Delta^2 + \Delta - (\Delta_1 + \Delta_2 + 4)\Delta_1\Delta_2 - (\Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2) = (\Delta_1 + \Delta_2 - 2)\Delta_1\Delta_2 + \Delta_1 + \Delta_2$, we have reduced the bound by $(\Delta_1 + \Delta_2 - 2)\Delta_1\Delta_2 + \Delta_1 + \Delta_2$.

V. Conclusion

By our new developed approach, most of the previous results about the upper bounds of λ-numbers of the four standard graph products have been improved significantly. In addition, the new approach is easy to follow and will reduce many unnecessary counting procedures that occurred in many previous papers. In other words, we believe that our method is a new direction for researchers and engineers to derive the upper bounds of λ-numbers more efficiently.

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