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Lattice Boltzmann simulation of viscoplastic fluids on natural convection in inclined enclosure with inner cold circular/elliptical cylinders (Part III: Four cylinders)

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Abstract

In this paper, natural convection in a heated cavity with four inner cold circular/elliptical cylinders in a diamond shape filled with viscoplastic fluids has been simulated by Lattice Boltzmann Method (LBM). In this study, the Bingham model without any regularization has been studied and moreover viscous dissipation effect has been analysed. Fluid flow, heat transfer, and yielded/unyielded parts have been conducted for certain pertinent parameters of Rayleigh number (Ra = 10^4, 10^5 and 10^6), Eckert number, the size of the inner cylinder, various inclined angles of the cavity ($\theta = 0°$, $40°$, $80°$, $120°$), the ratio of the inner cylinder radii (A = 0.5, 1, and 2), and different positions of the inner cylinder. Moreover, the Bingham number (Bn) is studied in a wide range of different studied parameters. Results indicate that the enhancement of the Rayleigh number augments the heat transfer, with a decrease in the size of the unyielded zones. For specific Rayleigh number and the position of the cylinder, the increase in the Bingham number declines the heat transfer and expands the unyielded sections between the inner cylinders and the enclosure. The enhancement of the ratio of the inner cylinder radii augments the heat transfer and declines the unyielded sections. The increase in the equal vertical and horizontal distances between cylinders ($\delta = \Omega$) enhances heat transfer, and moreover, enlarges the unyielded zones. The rise of the distance between horizontal cylinders in fixed distance of the vertical cylinders($\Omega = 0.2$) augments heat transfer while expands the unyielded zones. The enhancement of the distance between vertical cylinders in fixed distance of the horizontal cylinders($\delta = 0.2$) augments heat transfer while alters the unyielded zones. The increase in the inclined angle of the enclosure alters the heat transfer and the yielded/unyielded zones noticeably. The rise of Eckert number even for higher range of practical values (Ec = 0.01, 0.1, and 1) alters the heat transfer and unyielded parts marginally, so the viscous dissipation term can be negligible in this study.

Key words: Viscoplastic fluid, Natural convection, LBM, Circular/elliptical

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cylinders, Viscous dissipation

1 Introduction

Natural convection of viscoplastic fluids in an enclosure due to its wide applications and interest in various chemical, metal, and food industries has been considered recently by researchers. Vola et al. [1] studied the natural convection in a cavity filled with a viscoplastic fluid using the Bingham model without any regularisation of the constitutive law. They applied a numerical method based on the combination of the characteristic/Galerkin method to cope with convection and of the FortinGlowinski decomposition/coordination method to deal with the non-differentiable and nonlinear terms that derive from the constitutive law. However, the streamlines and isotherms for various yield stress values were limited to one value of the Rayleigh number (Ra = 10^4). Turan et al. [2] conducted a study into the simulations of natural convection in square enclosures filled with an incompressible Bingham fluid. The considered flow was laminar and steady. The commercial package FLUENT was utilised to solve the problem. In this study, a second-order central differencing scheme was used for the diffusive terms and a second order up-wind scheme for the convective terms. Coupling of the pressure and velocity fields was achieved using the SIMPLE algorithm. It should be noted that the default Bingham model in FLUENT is a bi viscosity model. The heat transfer and the flow velocities were investigated over a wide range of Rayleigh and Prandtl numbers. They found that the average Nusselt number augments with the rise of the Rayleigh number for both Newtonian and Bingham fluids, whereas the Nusselt numbers of Bingham fluids were smaller than those in Newtonian fluids for a fixed nominal Rayleigh number. They also mentioned that the mean Nusselt number of Bingham fluids decreased with an increase in the Bingham number. Moreover, it was observed that the conduction dominated regime occurs at large values of Bingham numbers. Finally, they reported that for low Bingham numbers, the mean Nusselt number increases with the enhancement of the Prandtl number; by contrast, the opposite behaviour was observed for large values of Bingham numbers. Turan et al. [3] continued their studies with analysing the effect of different aspect ratios (the ratio of the height to the length) of the cavity, adding to their previous results that the average Nusselt number follows a non-monotonic pattern with the aspect ratio for specific val-

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ues of the Rayleigh and Prandtl numbers for both Newtonian and Bingham fluids. At small aspect ratios, the conduction is dominant whereas convection remains predominantly responsible for the heat transfer for large values of aspect ratios. In addition, it was found that the conduction dominated regime occurred at higher values of the Bingham numbers for increasing values of the aspect ratio for a given value of the Rayleigh number. Turan et al. [4] scrutinised the laminar Rayleigh-Bnard convection of yield stress fluids in a square enclosure. The applied method and the achieved results were similar to the two previous studies. Huilgol and Kefayati [5] studied natural convection in a square cavity with differentially heated vertical sides and filled with a Bingham fluid without any regularisation. The finite element method (FEM) based on the operator splitting method was utilised to solve the problem. It was observed that for specific Rayleigh and Prandtl numbers, the increase in the Bingham number decreases the heat transfer. Furthermore, it was found that the growth of the Bingham number expands the unyielded sections in the cavity. Finally, they mentioned that for fixed Rayleigh and Bingham numbers, the unyielded regions grow with the augmentation of the Prandtl number. Karimfazli et al. [6] explored the feasibility of a novel method for the regulation of heat transfer across a cavity. They used computational simulations to resolve the Navier-Stokes and energy equations for different yield stresses.

Many studies have conducted the effect of the presence of the body inside the enclosure on the natural convection of Newtonian fluids and focused on the diverse body shapes such as circular, square and triangular cylinders [7–14]. Just recently, natural convection of viscoplastic fluids is investigated, using regularized models inside cavities with hot and cold bodies. Baranwal and Chhabra [15] studied laminar natural convection heat transfer to Bingham plastic fluids from two differentially heated isothermal cylinders confined in a square enclosure. They utilized regularization approaches of biviscosity and the Bercovier and Engelman models. They used the finite element method-based solver, COMSOL Multiphysics (version 4.3a) to solve the governing equations. Dutta et al. [16] investigated the effects of tilt angle and fluid yield stress on the laminar natural convection from an isothermal square bar cylinder in a Bingham plastic fluid confined in a square duct. They also applied the same regularization approaches of biviscosity and the Bercovier and Engelman models. They also applied the finite element method-based solver, COMSOL Multiphysics (version 4.3a) to solve the governing equations.

Lattice Boltzmann method (LBM) has been demonstrated to be a very effective mesoscopic numerical method to model a broad variety of complex fluid flow phenomena [17–28]. Lattice Boltzmann method (LBM) combined with Finite Difference Method (FDM) has been applied for this problem [29]. It was demonstrated to be a successful mesoscopic method for simulation of Non-Newtonian fluids. Independency of the method to the relaxation time in contrast with common LBM provokes the method to solve different non-

Newtonian fluid energy equations successfully as the method protects the positive points of LBM simultaneously. Huilgol and Kefayati [30] explained and derived the two and three dimensional equations of continuum mechanics for this method and demonstrated that the theoretical development can be applied to all fluids, whether they be Newtonian, or power law fluids, or viscoelastic and viscoplastic fluids. Following the previous study, Huilgol and Kefayati [31] derived the two and three dimensional equations of this method for the cartesian, cylindrical and spherical coordinates. Double-diffusive natural convection of non-Newtonian power-law fluid in a square cavity was studied by the cited method while entropy generations through fluid friction, heat transfer, and mass transfer were analyzed [32]. In the following study, heat and mass transfer as well as entropy generations in natural convection of non-Newtonian power-law fluids in an inclined porous cavity were studied, applying the method [33,34]. Kefayati and Huilgol [35] applied this method to simulate the steady flow in a pipe of square cross-section when the pipe is filled with a Bingham fluid. The problem was solved employing the Bingham model without any regularisation. In the next step, Kefayati and Huilgol [36] utilized the mesoscopic method to conduct a two-dimensional simulation of steady mixed convection in a square enclosure with differentially heated sidewalls when the enclosure is filled with a Bingham fluid. The problem was solved by the Bingham model without any regularisations and also by applying the regularised Papanatasiou model. Double-diffusive natural convection and entropy generations of Bingham fluid in a square cavity and open cavities were simulated by this method [37,38]. Double-diffusive natural convection and entropy generations, studying Soret and Dufour effects and viscous dissipation in a heated enclosure with an inner cold cylinder filled with non-Newtonian Carreau fluid were simulated by the method [39,40].

The main aim of this study is to simulate natural convection of Bingham fluid in a heated cavity with four inner cold cylinders as the yielded/unyielded sections have been displayed. In this study, the Bingham model without any regularization has been studied and moreover viscous dissipation effect also has been analyzed. LBM has been employed to study the problem numerically. Moreover, it is endeavored to express the effects of different parameters on heat transfer as well as yielded/unyielded zones. The obtained results were validated with previous numerical investigations and the effects of the main parameters (Rayleigh number, Bingham number, Eckert number, the size and position of the cold cylinder) on unyielded parts and heat transfer are researched.

2 Theoretical formulation

The geometry of the present problem is shown in Fig.1. The temperatures of the enclosure walls have been considered to be maintained at high temperature of T_H as the inner cylinders are kept at low temperature of T_C. The lengths of the enclosure sidewalls are L where the first, second, third, and fourth cylinders centers are defined by (x_{c1}, y_{c1}),(x_{c2}, y_{c2}),(x_{c3}, y_{c3}),(x_{c4}, y_{c4}); respectively. The cylinders have been set in a diamond shape with same area. The first cylinder is located on the vertical bottom side of the cavity $y_{c1} < 0.5L$, the second cylinder is located on the horizontal top side of the cavity $x_{c2} > 0.5L$, the third cylinder is located on the top vertical side of the cavity $y_{c3} > 0.5L$, the fourth cylinder is located on the bottom horizontal side of the cavity $x_{c4} < 0.5L$. The centers of the first and third cylinders are fixed in the horizontal direction ($x_{c1} = x_{c3} = 0.5L$) where their vertical distances from the center of the cavity are the same. In addition, the centers of the second and fourth cylinders are fixed in the vertical direction ($y_{c2} = y_{c4} = 0.5L$) where their horizontal distances from the center of the cavity are the same. The angle between the x direction and the bottom side of the cavity (θ) is altered counterclockwise. The inner circular/elliptical cylinders have the horizontal radius (a), the vertical radius (b), the vertical distance between the centers of the first and third cylinders is defined by the parameter of $\Omega = \frac{|y_{c3}-y_{c1}|}{L}$, the horizontal distance between the centers of the second and fourth cylinders is defined by the parameter of $\delta = \frac{|x_{c2}-x_{c4}|}{L}$, and the ratio of radii (A $= b/a$). The cavity is filled with a viscoplastic fluid. The prandtl number is fixed at Pr=0.1. There is no heat generation and thermal radiation. The flow is incompressible, and laminar. The density variation is approximated by the standard Boussinesq model for temperature. The viscous dissipation in the energy equation has been analyzed in this study.

2.1 Dimensional equations

Based on the above assumptions, and applying the Boussinesq approximation, the mass, momentum, and energy equations are [35,36]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + g\rho sin\theta\left[1 + \beta_T(T - T_C)\right], \tag{2.2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right) + g\rho cos\theta\left[1 + \beta_T(T - T_C)\right], \tag{2.3}$$

In the above equations, ($\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$), t, $\boldsymbol{\tau}$ are the Cartesian coordinates, time, and shear stress; respectively. In addition, ($\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$), T, ρ, and g are the dimensional velocities, temperature, density, and gravity acceleration; respectively. β_T is the coefficient of thermal expansion and ρ is density. Now, let the pressure p be written as the sum $p = p_s + p_d$, where the static part p_s accounts for gravity alone, and p_d is the dynamic part. Thus,

$$-\frac{\partial p_s}{\partial \mathbf{x}} = \rho g \left(sin\theta\,\mathbf{i} + cos\theta\,\mathbf{j} \right)\cdot \tag{2.4}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{1}{\rho c_p}\left[\tau_{xx}\left(\frac{\partial u}{\partial x}\right)+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{yy}\left(\frac{\partial v}{\partial y}\right)\right] \tag{2.5}$$

α and c_p are the thermal diffusivity and the specific heat at constant pressure, respectively [41–45].

2.2 Constitutive model

Bingham [46] constituted the viscoplastic fluids as follows:

$$\begin{cases} \mathbf{A}(\mathbf{u}) = \mathbf{0}, & K(\boldsymbol{\tau}) \leq \tau_y, \\ \boldsymbol{\tau} = \left(\eta + \frac{\tau_y}{K(\mathbf{u})}\right)\mathbf{A}(\mathbf{u}), & K(\boldsymbol{\tau}) > \tau_y, \end{cases} \tag{2.6}$$

where the viscosity η and the yield stress τ_y are constant, and the two invariants $K(\mathbf{u})$ and $K(\boldsymbol{\tau})$ are defined below:

$$2K^2(\mathbf{u}) = \mathbf{A}(\mathbf{u}) : \mathbf{A}(\mathbf{u}), \quad 2K^2(\boldsymbol{\tau}) = \boldsymbol{\tau} : \boldsymbol{\tau}. \tag{2.7}$$

where

$$\mathbf{A}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T. \tag{2.8}$$

Due to the discontinuity in the Bingham model, approximate models such as the Papanastasiou [47], Bercovier and Engelman [48], and the bi-viscosity [49] models are used by researchers and different software packages. However, a constitutive equation for a Bingham fluid fully equivalent to the original form can be used. This method was proposed and developed by Duvaut and Lions [50] and Glowinski [51] and the constitutive equation takes the form

$$\boldsymbol{\tau} = \eta\mathbf{A}(\mathbf{u}) + \sqrt{2}\,\tau_y\,\boldsymbol{\Lambda}, \quad \mathbf{1} : \boldsymbol{\Lambda} = 0, \tag{2.9}$$

where one may call the second order, symmetric, tensor $\mathbf{\Lambda}$ the *viscoplasticity constraint tensor*. Note that the traceless condition $\mathbf{1} : \mathbf{\Lambda} = 0$ has been imposed on this tensor so that the stress tensor $\boldsymbol{\tau}$ satisfies the condition tr $\boldsymbol{\tau} = 0$. In order to demarcate the flow field into unyielded/yielded zones, one requires that the tensor $\mathbf{\Lambda}$ meet the following conditions:

$$\mathbf{\Lambda} : \mathbf{\Lambda} = \begin{cases} < 1, & \mathbf{A}(\mathbf{u}) = \mathbf{0}, \\ 1, & \mathbf{A}(\mathbf{u}) \neq \mathbf{0}. \end{cases} \tag{2.10}$$

These conditions satisfy those imposed on the stress tensor, viz., $K(\boldsymbol{\tau}) \leq \tau_y$ when $\mathbf{A}(\mathbf{u}) = \mathbf{0}$, and $\tau_y < K(\boldsymbol{\tau})$ when $\mathbf{A}(\mathbf{u}) \neq \mathbf{0}$. The problem of determining where the flow is rigid and where it is liquid-like has been shifted to finding the tensor $\mathbf{\Lambda}$ in the flow field such that is satisfies Eq.(2.10). What has been proposed is important for the following reasons:

(1) The constitutive equations Eqs. (2.9) - (2.10) are defined over the entire flow domain, not just where the fluid has yielded.
(2) One searches for the solution velocity field $\mathbf{u}$ and the viscoplasticity constraint tensor $\mathbf{\Lambda}$ to determine the yielded/unyielded regions. There are no singularities because one is not trying to find the location of the yield surface(s) through the limit of $\mathbf{A}(\mathbf{u})/K(\mathbf{u})$ as $\mathbf{A}(\mathbf{u}) \to \mathbf{0}$.
(3) However, the equations of motion now involve two unknown fields: a vector field $\mathbf{u}$, and a symmetric tensor field $\mathbf{\Lambda}$. The latter requires that there should exist a connection between the velocity field $\mathbf{u}$ and $\mathbf{\Lambda}$. Under Dirichlet boundary conditions, it is possible to prove such a relation. Here, we provide a summary of the results.
$\mathbf{\Lambda}$ can be obtained from a simple projection operation as follows [5,35,36]:

$$\mathbf{\Lambda} = P_{\mathcal{M}}\Big(\mathbf{\Lambda} + r\tau_y \mathbf{A}(\mathbf{v})\Big), \quad \forall r > 0, \tag{2.11}$$

where $\mathcal{M} = \{\boldsymbol{\mu} | \boldsymbol{\mu} = (\mu_{ij})_{1\leq i,j\leq 2} \in (L^2(\Omega))^4, \|\boldsymbol{\mu}\| \leq 1 \text{ a.e. on } \Omega\}$ and

$$P_{\mathcal{M}} : (L^2(\Omega))^4 \to \mathcal{M} \tag{2.12}$$

is the projection operator defined so that $P_{\mathcal{M}}(\boldsymbol{\mu}) = \boldsymbol{\mu}$, if $\|\boldsymbol{\mu}\| \leq 1$, and $P_{\mathcal{M}}(\boldsymbol{\mu}) = \boldsymbol{\mu}/\,\|\boldsymbol{\mu}\|$ otherwise. Note that in the context of Eq. (2.11), the tensor $\boldsymbol{\mu} = \mathbf{\Lambda} + r\tau_y\mathbf{A}(\mathbf{v})$ and it is symmetric. Further, the tensor $\boldsymbol{\mu}$ must be dimensionless for $\mathbf{\Lambda}$ is also dimensionless.

where $r > 0$ is a real number to be specified. Successive iterations are performed till convergence is achieved to the desired level of accuracy. Note that the yield surface is the boundary between $||\mathbf{\Lambda}|| < 1$ and $||\mathbf{\Lambda}|| = 1$. Hence, the solution of the boundary value problem delivers in the limit both the velocity field as well as the shape and location of the yield surface.

2.3 Boundary conditions

The flow domain is given by $\omega = (0, L) \times (0, L)$, and the boundary $\Gamma = \partial\omega$. It is the union of eight disjoint subsets:

$$\Gamma_1 = \{(x, y), x = 0, 0 \le y \le L\}, \tag{2.13a}$$
$$\Gamma_2 = \{(x, y), x = L, 0 \le y \le L\}, \tag{2.13b}$$
$$\Gamma_3 = \{(x, y), 0 \le x \le L, y = 0\}, \tag{2.13c}$$
$$\Gamma_4 = \{(x, y), 0 \le x \le L, y = L\}, \tag{2.13d}$$
$$\Gamma_5 = \left\{(x, y), \frac{(x - x_{c1})^2}{a^2} + \frac{(y - y_{c1})^2}{b^2} = 1\right\}, \tag{2.13e}$$
$$\Gamma_6 = \left\{(x, y), \frac{(x - x_{c2})^2}{a^2} + \frac{(y - y_{c2})^2}{b^2} = 1\right\}, \tag{2.13f}$$
$$\Gamma_7 = \left\{(x, y), \frac{(x - x_{c3})^2}{a^2} + \frac{(y - y_{c3})^2}{b^2} = 1\right\}, \tag{2.13g}$$
$$\Gamma_8 = \left\{(x, y), \frac{(x - x_{c4})^2}{a^2} + \frac{(y - y_{c4})^2}{b^2} = 1\right\}. \tag{2.13h}$$

x_{c1} and y_{c1} are the horizontal and vertical center positions of the first cylinder. x_{c2} and y_{c2} are the horizontal and vertical center positions of the inner second cylinder. x_{c3} and y_{c3} are the horizontal and vertical center positions of the inner third cylinder. x_{c4} and y_{c4} are the horizontal and vertical center positions of the inner fourth cylinder. The parameters of a and b are the horizontal and vertical radii of the inner cylinders.

The boundary condition for the velocity is straightforward:

$$\mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_2} = \mathbf{u}|_{\Gamma_3} = \mathbf{u}|_{\Gamma_4} = \mathbf{u}|_{\Gamma_5} = \mathbf{u}|_{\Gamma_6} = \mathbf{u}|_{\Gamma_7} = \mathbf{u}|_{\Gamma_8} = \mathbf{0}. \tag{2.14}$$

The boundary conditions for the temperature are:

$$T|_{\Gamma_1} = T|_{\Gamma_2} = T|_{\Gamma_3} = T|_{\Gamma_4} = T_H, \quad T|_{\Gamma_5} = T|_{\Gamma_6} = T|_{\Gamma_7} = T|_{\Gamma_8} = T_C \tag{2.15}$$

2.4 Non-dimensional equations

We define the buoyancy velocity scale $U = \left(\frac{\alpha}{L}\right) Ra^{0.5}$ where Ra is the Rayleigh number. In order to proceed to the numerical solution of the system, the following non dimensional variables are introduced [52].

$$t^* = \frac{t\,U}{L}, \quad x^* = x/L, \quad y^* = y/L, \quad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad {p_d}^* = \frac{p_d}{\rho\, U^2}, \tag{2.16}$$

$$T^* = (T - T_C)/\Delta T, \quad \Delta T = T_H - T_C, \quad \boldsymbol{\tau}^* = \frac{\sqrt{2}\,\boldsymbol{\tau}\, L}{\eta\, U} \tag{2.17}$$

By substitution of Eqs. (2.16) - (2.17) into Eqs. (2.1) - (2.5) and dropping asterisks, the following system of non-dimensional equations are derived:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.18}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p_d}{\partial x} + \frac{Pr}{\sqrt{\mathrm{Ra}}}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + \mathrm{Pr}\ T sin\theta \tag{2.19}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p_d}{\partial y} + \frac{Pr}{\sqrt{\mathrm{Ra}}}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) + \mathrm{Pr}\ T cos\theta \tag{2.20}$$

$$\begin{aligned}\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \frac{1}{\sqrt{Ra}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\\ +\frac{Pr\, Ec}{\sqrt{Ra}}\ &\left[\tau_{xx}\left(\frac{\partial u}{\partial x}\right) + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{yy}\left(\frac{\partial v}{\partial y}\right)\right]\end{aligned} \tag{2.21}$$

In the case of the exact Bingham model [35,36], the non-dimensional stresses are given by

$$\tau_{xx} = \left[2\left(\frac{\partial u}{\partial x}\right) + \ \mathrm{Bn}\ \Lambda_{xx}\right]\ , \tag{2.22a}$$

$$\tau_{yy} = \left[2\left(\frac{\partial v}{\partial y}\right) + \ \mathrm{Bn}\ \Lambda_{yy}\right]\ , \tag{2.22b}$$

$$\tau_{xy} = \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \ \mathrm{Bn}\ \Lambda_{xy}\right]\ , \tag{2.22c}$$

$$\mathbf{\Lambda}^{n+1} = P_{\mathcal{M}}\Big(\mathbf{\Lambda}^n + \mathrm{Pr}\ \mathrm{Bn}\ \mathbf{A}(\mathbf{u})^n\Big), \tag{2.22d}$$

The mentioned non-dimensional parameters in the above equations are as follows:

Rayleigh number:

$$Ra = \frac{\rho\, \beta_T\, g L^3 \Delta T}{\eta\, \alpha} \tag{2.23}$$

Prandtl number:

$$\mathrm{Pr} = \frac{\eta}{\rho\, \alpha}\ , \tag{2.24}$$

Bingham number:

$$Bn = \frac{\sqrt{2}\, \tau_y\, L}{\eta\, U}\ , \tag{2.25}$$

Eckert number:

$$Ec = \frac{U^2}{c_p\,\Delta T}\,, \tag{2.26}$$

Since the buoyancy velocity scale is $U = \left(\frac{\alpha}{L}\right)\,Ra^{0.5}$, ΔT and L are equal to unity, the Eckert number has the following relation to the specific heat at constant pressure, Rayleigh number, and thermal diffusivity

$$Ec \propto \frac{\alpha^2\,Ra}{c_p}\,, \tag{2.27}$$

The α and c_p for different viscoplastic fluids in various temperatures can be ranged over (10^{-3} – 10^{-6} m^2/s) and (10^2 – 10^4 J/kg K). So, in the highest studied Rayleigh numbers ($Ra = 10^6$), and the thermal diffusivity ($\alpha = 10^{-3}$) as well as the lowest value of the specific heat at constant pressure, $c_p = 10^3$, the highest amount of the Eckert number would be Ec = 0.01. Therefore, the Eckert number in practical view would be in the range of Ec $\ll$ 0.01. From the Eq.(2.21), it can be observed the viscous dissipation term (The last term in the right side of the equation) is too small and negligible. However, some simulations are conducted to prove the cited theorem which the viscous dissipation is negligible. We have analysed the Eckert number in a higher range at Ec = 0.01, 0.1 and 1 to observe the viscous dissipation term effect on heat transfer, and yielded/unyielded parts.

The local and the average Nusselt numbers at the cavity sides are as

$$Nu = \left(-\frac{\partial T}{\partial r}\right)_{r=0}, \tag{2.28a}$$

$$Nu_{avg} = \int_0^1 Nu\,ds,\ . \tag{2.28b}$$

where r denotes the unit normal direction on a specific side wall s.
The total average Nusselt numbers are as

$$Nu_{tot_{avg}} = Nu_{L_{avg}} + Nu_{R_{avg}} + Nu_{B_{avg}} + Nu_{T_{avg}} \tag{2.29}$$

In the above equation, the subscribes of tot, L, R, B, T, avg means total, the left wall of the cavity, the right wall of the cavity, the bottom wall of the cavity, the top wall of the cavity, and average; respectively.

3 The numerical method

The LBM equations and their relationships with continuum equations have been explained in details in Huilgol and Kefayati [30]. Here, just a brief description about the main equations would be cited. In addition, the applied algorithm has been described and the studied problem equations in the LBM are mentioned.

3.1 The Continuity and Momentum equations

To have the continuity and momentum equations, a discrete particle distribution function f_α is defined over a D2Q9 lattice where it should satisfy an evolution equation:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla_{\mathbf{x}} f_\alpha - F_\alpha = -\frac{1}{\varepsilon\,\phi}(f_\alpha - f_\alpha^{eq}), \tag{3.1}$$

where ε is a small parameter to be prescribed when numerical simulations are considered. ϕ is the relaxation time.

Associated to each node is a lattice velocity vector $\boldsymbol{\xi}_\alpha$. It is defined as follows:

$$\boldsymbol{\xi}_\alpha = \begin{cases} (0,0), & \alpha = 0, \\ \sigma(\cos\Theta_\alpha, \sin\Theta_\alpha) & \alpha = 1,3,5,7, \\ \sigma\sqrt{2}(\cos\Theta_\alpha, \sin\Theta_\alpha), & \alpha = 2,4,6,8. \end{cases} \tag{3.2}$$

Here, the angles Θ_α are defined through $\Theta_\alpha = (\alpha-1)\pi/4,\ \alpha = 1,\cdots,8$. The constant σ has to be chosen with care for it affects numerical stability; its choice depends on the problem. The method for finding the parameter σ which satisfies the Courant-Friedrichs-Lewy (CFL) condition is described in [30].

The equilibrium distribution function, f_α^{eq}, is different from the conventional ones adopted by previous researchers, who normally expand the Maxwellian distribution function. In the present approach, we expand f_α^{eq} as a quadratic in terms of $\boldsymbol{\xi}_\alpha$, using the notation of linear algebra [30,31]:

$$f_\alpha^{eq} = A_\alpha + \boldsymbol{\xi}_\alpha \cdot \mathbf{B}_\alpha + (\boldsymbol{\xi}_\alpha \otimes \boldsymbol{\xi}_\alpha) : \mathbf{C}_\alpha,\ \alpha = 0,1,2,\cdots,8. \tag{3.3}$$

The relation between the above parameters and non-dimensional macroscopic values are as follows:

$$A_0 = \rho - \frac{2p}{\sigma^2} - \frac{\rho|\mathbf{u}|^2}{\sigma^2} + \frac{Pr}{\sqrt{Ra}}\left(\frac{\tau_{xx}+\tau_{yy}}{2}\right), \quad A_\alpha = 0,\ \alpha = 1,2,\cdots,8. \tag{3.4}$$

$$\mathbf{B}_1 = \frac{\rho\mathbf{u}}{2\sigma^2} = \mathbf{B}_\alpha,\ \alpha = 1,3,5,7; \quad \mathbf{B}_\alpha = \mathbf{0},\ \alpha = 0,2,4,6,8. \tag{3.5}$$

Next, the matrices $\mathbf{C}_\alpha$ are such that $\mathbf{C}_0 = 0;\ \mathbf{C}_1 = \mathbf{C}_\alpha,\ \alpha = 1,3,5,7;\ \mathbf{C}_2 = \mathbf{C}_\alpha,\ \alpha = 2,4,6,8,$ where

$$\mathbf{C}_1 = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}, \quad C_{11} = \frac{1}{2\sigma^4}\left(p + \rho u^2 - \frac{Pr}{\sqrt{Ra}}\tau_{xx}\right), \quad C_{22} = \frac{1}{2\sigma^4}\left(p + \rho v^2 - \frac{Pr}{\sqrt{Ra}}\tau_{yy}\right), \tag{3.6}$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & C_{12} \\ C_{21} & 0 \end{bmatrix}, \quad C_{12} = C_{21} = \frac{1}{8\sigma^4}\left(\rho uv - \frac{Pr}{\sqrt{Ra}}\tau_{xy}\right). \tag{3.7}$$

The body force term F_α in (3.1) can be defined as

$$F_\alpha = 0, \quad \alpha = 0, 2, 4, 6, 8, \tag{3.8a}$$

$$F_\alpha = \frac{1}{2\sigma^2}\mathbf{N}\cdot\boldsymbol{\xi}_\alpha, \quad \alpha = 1, 3, 5, 7 \tag{3.8b}$$

where

$$\mathbf{N} = \mathrm{Pr}\ T\,(sin\theta\,\mathbf{i} + cos\theta\,\mathbf{j}) \tag{3.9}$$

The main equations of the discrete particle distribution function (3.1) is solved by the splitting method of Toro [53]. Hence, the equations can be separated into two parts. The first one is the streaming section which is written as

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla_{\mathbf{x}} f_\alpha - F_\alpha = 0. \tag{3.10}$$

Eqs.(3.10) has been solved with the method of Lax and Wendroff [54] and the following equations are used.

$$\begin{aligned} f_\alpha^{n+1}(i,j) = f_\alpha^{n}(i,j) - \frac{\Delta t}{2\Delta x}\xi_\alpha(i)\left[f_\alpha^{n}(i+1,j) - f_\alpha^{n}(i-1,j)\right] \\ - \frac{\Delta t}{2\Delta y}\xi_\alpha(j)\left[f_\alpha^{n}(i,j+1) - f_\alpha^{n}(i,j-1)\right] + \\ \frac{\Delta t^2}{2\Delta x^2}{\xi_\alpha}^2(i)\left[f_\alpha^{n}(i+1,j) - 2f_\alpha^{n}(i,j) + f_\alpha^{n}(i-1,j)\right] + F_\alpha(i)\Delta t + \\ \frac{\Delta t^2}{2\Delta y^2}{\xi_\alpha}^2(j)\left[f_\alpha^{n}(i,j+1) - 2f_\alpha^{n}(i,j) + f_\alpha^{n}(i,j-1)\right] + F_\alpha(j)\Delta t\ , \end{aligned} \tag{3.11}$$

In Eqs.(3.11), we have put

$$\xi_\alpha(i) = \boldsymbol{\xi}_\alpha\cdot\mathbf{i}, \quad \xi_\alpha(j) = \boldsymbol{\xi}_\alpha\cdot\mathbf{j}, \quad F_\alpha(i) = \mathbf{F}_\alpha\cdot\mathbf{i}, \quad F_\alpha(j) = \mathbf{F}_\alpha\cdot\mathbf{j}. \tag{3.12}$$

The second part is the collision section which is as follows:

$$\frac{\partial f_\alpha}{\partial t} = -\frac{1}{\varepsilon\phi}(f_\alpha(\mathbf{x},t) - f_\alpha^{eq}(\mathbf{x},t)), \tag{3.13}$$

Eqs.(3.13) can be solved by using the Euler method and the choice of $\varepsilon\phi$ is taken as the time step (Δt). That is

$$\frac{f_\alpha(\mathbf{x},t+\Delta t) - f_\alpha(\mathbf{x},t)}{\Delta t} = -\frac{1}{\varepsilon\phi}(f_\alpha(\mathbf{x},t) - f_\alpha^{eq}(\mathbf{x},t)), \tag{3.14}$$

from which one obtains

$$f_\alpha(\mathbf{x},t+\Delta t) = f_\alpha^{eq}(\mathbf{x},t), \tag{3.15}$$

3.2 The Energy Equation

In order to obtain the energy equation, an internal energy distribution function g_α is introduced and it is assumed to satisfy an evolution equation similar to that for f_α. Thus,

$$\frac{\partial g_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla_{\mathbf{x}} g_\alpha - G_\alpha = -\frac{1}{\varepsilon\phi}(g_\alpha - g_\alpha^{eq}). \tag{3.16}$$

G_α refers to the external supply e.g. radiation in the energy equation. Here, g_α^{eq} has a monomial expansion:

$$g_\alpha^{eq} = D_\alpha + \boldsymbol{\xi}_\alpha \cdot \mathbf{E}_\alpha, \tag{3.17}$$

One way of satisfying the above is to assume, as before, that the scalars are given by $D_\alpha = D_1,\ \alpha = 1,3,5,7$, and $D_\alpha = D_2,\ \alpha = 2,4,6,8,$. In this problem, the non-dimensional parameters are obtained as follows (See Eqs. 3.39,3.42 in Huilgol and Kefayati [30]):

$$D_0 = T, \quad D_1 = 0, \quad D_2 = 0. \tag{3.18}$$

Regarding the vectors, it is assumed that $\mathbf{E}_0 = \mathbf{0}$, $\mathbf{E}_\alpha = \mathbf{E}_1$, $\alpha = 1,3,5,7; \mathbf{E}_\alpha = \mathbf{E}_2$, $\alpha = 2,4,6,8$, where (See Eqs. 3.40,3.43 in Huilgol and Kefayati [30])

$$\mathbf{E}_1 = \frac{\left[\mathbf{u}\,T - \frac{1}{\sqrt{Ra}}\frac{\partial T}{\partial \mathbf{x}} - \frac{Pr\,Ec}{\sqrt{Ra}}\ \left[(u\,\tau_{xx} + v\,\tau_{xy}) + (u\,\tau_{yx} + v\,\tau_{yy})\right]\right]}{2\,\sigma^2}. \tag{3.19}$$

Finally, $G_\alpha = 0$.

The main equations of the internal energy distribution function are solved by the splitting method of Toro [53]. Hence, the equations can be separated into two parts. The first one is the streaming section which is written as

$$\frac{\partial g_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla_{\mathbf{x}} g_\alpha = 0. \tag{3.20}$$

Eqs.(3.20) have been solved with the method of Lax and Wendroff [54] and the following equations are used.

$$\begin{aligned} g_\alpha^{n+1}(i,j) = g_\alpha^n(i,j) - \frac{\Delta t}{2\Delta x}\xi_\alpha(i)\left[g_\alpha^n(i+1,j) - g_\alpha^n(i-1,j)\right] \\ - \frac{\Delta t}{2\Delta y}\xi_\alpha(j)\left[g_\alpha^n(i,j+1) - g_\alpha^n(i,j-1)\right] + \\ \frac{\Delta t^2}{2\Delta x^2}{\xi_\alpha}^2(i)\left[g_\alpha^n(i+1,j) - 2g_\alpha^n(i,j) + g_\alpha^n(i-1,j)\right] + \\ \frac{\Delta t^2}{2\Delta y^2}{\xi_\alpha}^2(j)\left[g_\alpha^n(i,j+1) - 2g_\alpha^n(i,j) + g_\alpha^n(i,j-1)\right] \end{aligned} \tag{3.21}$$

The second part is the collision section which is as follows:

$$\frac{\partial g_\alpha}{\partial t} = -\frac{1}{\varepsilon\phi}(g_\alpha(\mathbf{x},t) - g_\alpha^{eq}(\mathbf{x},t)). \tag{3.22}$$

Eqs.(3.22) can be solved by using the Euler method and the choice of $\varepsilon\phi$ is taken as the time step (Δt). That is

$$\frac{g_\alpha(\mathbf{x},t+\Delta t) - g_\alpha(\mathbf{x},t)}{\Delta t} = -\frac{1}{\varepsilon\phi}(g_\alpha(\mathbf{x},t) - g_\alpha^{eq}(\mathbf{x},t)), \tag{3.23}$$

from which one obtains

$$g_\alpha(\mathbf{x},t+\Delta t) = g_\alpha^{eq}(\mathbf{x},t). \tag{3.24}$$

3.3 Boundary conditions

One of the main advantages of the current approach is that boundary conditions can be incorporated in a manner similar to macroscopic methods, in contrast with other methods utilised for solving LBM equations. The latter employ complicated special relationships for the discrete particle distribution function (f_α) and the internal energy distribution function (g_α) for each kind of boundary conditions and problems [55,56]. For example, methods such as on-grid and mid-grid bounce back are used when the velocity is zero on the boundary; when the boundary is in motion, bounce-back is used along with a set of linear equations to determine the boundary values f_α. In the method used here, the boundary conditions of f_α and g_α can be obtained directly from the macroscopic values on the boundaries due to the relationships of the macroscopic values with f_α and g_α. As a result, in this method, boundary conditions, especially the Dirichlet conditions, can be included in various problems similar to macroscopic methods and no special equations for f_α and g_α are needed to incorporate the boundary conditions. Therefore, we apply the cited macroscopic values (velocities and temperatures) on the boundary conditions in the Eqs. (2.13a)–(2.13h) on the cited boundaries Eqs. (2.14),(2.15) directly. In addition, for the curved boundaries, just we need to employ the curved equations Eqs. (2.13e)–(2.13h) in the Cartesian coordinates.

4 Applied parameters, code validation and grid independence

Lattice Boltzmann Method (LBM) scheme is utilized to simulate laminar natural convection in a heated enclosure with four inner cold cylinders in a diamond shape that is filled with a viscoplastic fluid in the presence of the viscous dissipation in the energy equation. The Prandtl number is fixed at Pr=0.1. This problem is investigated at different parameters of Rayleigh number (Ra = 10^4, 10^5 and 10^6), Eckert number, the size of the inner cylinder, various inclined angles of the cavity ($\theta = 0°$, 40°, 80°, 120°), the ratios of the inner cylinders radii (A = 0.5, 1, and 2), and different positions of the inner cylinder. Moreover, the Bingham number (Bn) is studied in a wide range of different studied parameters. The applied code for the fluid flow and heat transfer is validated by the study of Mun et al. [13] in the Fig.2 at Ra = 10^5, $\Omega = \delta = 0.4$, $\theta = 0°$, a = b = 0.1 L, and Pr = 0.71 for the case of Newtonian fluids in a cooled enclosure with a heated cylinder in the center of the cavity. In addition, the average Nusselt number on the one hot circular cylinder is validated between present results and the studies of Zhang et al. [14], Kim et al. [8], and Park et al. [10] in the Table.2 in different Rayleigh numbers. LBM is applied for isothermal and non-isothermal problems of Bingham fluid recently [35,36] which demonstrates the accuracy of the utilized code for non-Newtonian Bingham fluids properly. An extensive mesh testing procedure was conducted to guarantee a grid independent solution. Six different mesh combinations were explored for the case of Ra= 10^6, Bn=1, Ec = 0, $\Omega = \delta = 0.4$, $\theta = 0°$, and a = b = 0.1 L. The average Nusselt numbers on the hot wall have been studied. It was confirmed that the grid size (300*300) ensures a grid independent solution as portrayed by Table 1. In addition, we set the time step $\Delta t = 0.0001$ for this calculation and based on on the validations, the final (developed) stage was defined at the non-dimensional time $t^* = 40$. To see the process of convection and yielding before the final stage, Fig.3 presents the isotherms, streamlines, and yielded/unyielded zone at Ra = 10^6, Bn = 8, Ec = 0, a = 0.05 L, b = 0.1 L, $\theta = 0°$, and $\Omega = \delta = 0.4$ in different non-dimensional time (t^* = 10, 20, 30, and 40). The running time for the grid size (300 * 300) is 5791 seconds. The value of σ was varied in each iteration according to the Appendix.

5 Results and discussion

Fig.4 shows the isotherms, streamlines and yielded/unyielded parts in different Rayleigh numbers at $\delta = \Omega = 0.4$, Bn = 1, Ec = 0, $\theta = 0°$, and a = b = 0.1 L. At Ra=10^4, the temperature contours are circular around the cylinder which demonstrate the conduction is dominated in the enclosure. As the Rayleigh number increases, the movements of the isotherms between the cold cylinder and hot walls ameliorate significantly and they become progressively curved. Moreover, the gradient of temperature on the hot wall augments with the rise of Rayleigh number. In fact, it occurs while the thermal boundary layer thickness on the side walls decreases with increasing Rayleigh number. The streamlines exhibit that the convection process has been enhanced by the growth of Rayleigh numbers as the core of the streamline changes and the streamlines traverse more distance in the cavity. The last column displays the yielded (White) and unyielded (Black) regions for the studied Rayleigh numbers at Bn = 1. It is clear that the proportion of the unyielded sections in the enclosure has enhanced with the drop of Rayleigh numbers markedly. Therefore, for a constant Bingham number, the increase in Rayleigh number causes the unyielded zones to decline.

Table 3 indicates that the average Nusselt number increases as Rayleigh number rises at Ec = 0, Bn = 1, a = b = 0.1 L, $\theta = 0°$, and $\delta = \Omega = 0.4$. The unyielded sections occupy more spaces in the cavity as the Bingham number augments and the rise of the Bingham number causes the yielded regions to disappear gradually. In other words, there is a critical Bingham number Bn_c above which the fluid is completely unyielded. These numbers are listed in Table 4 for different Rayleigh numbers.

Fig.5 depicts the isotherms in different radii and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and $\theta = 0°$. In this figure, the isotherms for different ratios of the radii (A = 0.5, 1, and 2) have been studied. The figure demonstrates that the convection process of isotherms (The convection process means the movement of the isotherms between the cold cylinder and the hot walls) augments clearly as the ratio of the radius (A) increases. When Bingham number increase, the convection process diminishes gradually. At high Bingham numbers, the isotherms shape into the form of the circular/elliptical. This pattern confirms the conduction process is replaced with convection. As the ratio of radius increases, the pattern to reach the conduction part happens in higher Bingham numbers.

Fig.6 exhibits the streamlines in different radii and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and $\theta = 0°$. Two symmetric circulation with two vortices are shown in the streamlines in different parameters. The cores of the streamlines become strong as the ratio of the radius increases. It demonstrates that the rise of the ration of radius enhances the convection process. Since the Bingham number increases, the curved shape of the streamline decline and the core of the circulation move from the bottom to center. Finally, the form of the streamline shows that the conduction dominates the flow field and a relatively weak convective flow can be observed at the higher Bingham number.

Fig.7 displays the yielded/unyielded in different radii and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and $\theta = 0°$. Since, the cylinders are located in the middle of the cavity, the unyielded zones are symmetric for different radii and Bingham numbers. At Bn = 1, the main unyielded regions appear on the the top left and right corners and around the cylinders. As the ratio of the radii (A) increases the unyielded zones around cylinders and the top corners sections alter. In addition, curved shapes of unyielded parts around the cylinder at A = 1 decreases gradually with the rise of the ratio of radii and nearly disappear at A = 2. At Bn = 4, the unyielded zone is created on the bottom of the bottom cylinder as the top corners also enlarge. It demonstrate the increase in the ratio of radii from A = 0.5 to 1 enlarges the unyielded zone, but the unyielded part diminishes considerably with the rise of A = 1 to 2. At Bn = 8, for A = 0.5 and 2, the unyielded zone; interestingly, under the bottom cylinder declines while other section, especially, close to side walls and the top side enhance significantly. At A = 1, the format of the unyielded zone is similar to the pattern a Bn = 4 while the unyielded parts extent. As Bingham number rises from Bn = 8 to 12, the unyielded zone increases significantly at A = 0.5 and 1 as the only two symmetric yielded parts were observed in the middle of the cavity. At A = 2, the rise of the unyielded part is less than A = 0.5 and 1 where the unyielded zone develops steadily. The increase in Bingham number from Bn = 12 to 16, causes the space between the inner cylinder and the cavity fills utterly with the unyielded materials for A = 0.5 and 1 except small spaces around the cylinder. For A = 2, there are just three narrow yielded parts next to the cylinder while other parts are unyielded. The figure demonstrates that for A = 1, 2 and 4, just small yielded spots are observed at Bn = 20.

Fig.8 shows the local Nusselt numbers on the left wall (Nu_L) and the bottom wall (Nu_b) in different Bingham numbers at $\Omega = 0.4$, Ra = 10^6, Ec = 0, $\theta = 0°$, and a = b = 0.1 L. The local Nusselt numbers on the left wall (Nu_L) depicts that there is a high peak in different Bingham numbers where the position of this peak value on the left wall alters in various Bingham numbers. Moreover, the sinusoidal manner of the local Nusselt number at Bn = 1 and 4 changes to curved behaviour in higher Bingham numbers. At higher Bingham numbers (Bn = 8, 12 , and 16), the peak value is observed nearly at Y = 0.7, 0.5 and 0.5 for the Nu_L; respectively as it has a curved shape and in a symmetric form. The local Nusselt numbers on the bottom wall (Nu_b) at Bn = 1 and 4 has a sinusoidal manner which the pick values are seen at Y = 0.3 and 0.8. As Bingham number increases from Bn = 4 to 8, the sinusoidal behavior alters to a curved shape and the values of local Nusselt number declines considerably. The rise of the Bingham number from Bn = 8 to 12 causes the local Nusselt number to decline slightly, but the local Nusselt numbers for Bingham numbers of Bn = 12, and 16 are nearly the same.

Fig.9 demonstrates that the average Nusselt numbers on the top wall (Nu_{Tavg}), the left wall (Nu_{Lavg}) and the bottom wall (Nu_{Bavg}) in different Bingham numbers and aspect ratios of the radii at $\delta = \Omega = 0.4$, Ra = 10^6, Ec = 0, and $\theta = 0°$. It should be noted that the average Nusselt numbers on the right and left walls are the same because of the symmetric shape, so the average Nusselt number of the right wall is not depicted here. The figure shows that Nu_{Bavg} in different ratios of radii drop gradually as the Bingham number increases. It is clear that at $Bn \leq 12$, the increase in the ratio of the radii for various Bingham numbers augments Nu_{Bavg}. In addition, Nu_{Lavg} drops substantially from Bn = 1 to 4 while it enhances against the rise of Bingham number at $Bn \geq 8$, but, the effect of the ratios of radii is irregular. It was demonstrated the highest values of the Nu_{Lavg} are obtained at A = 1 in different Bingham numbers. Nu_{Tavg} increases gradually as the Bingham number rises in different studied ratios of radii. Further, the maximum values of the Nu_{Tavg} was observed at A = 1 among the studied ratio of the radii.

Table 5 states Average Nusselt numbers in different Bingham numbers and the ratio of radii (A) at Ec = 0, Ra = 10^6, $\theta = 0°$, and $\delta = \Omega = 0.4$. It demonstrates that the total average Nusselt number decreases gradually as Bingham number rises for different ratio of radii (A). It shows that the least total average Nusselt numbers are observed at A = 0.5 for different Bingham numbers. But, the highest value among different ratio of radii (A) and Bingham number is observed at A = 1 except Bn 4 where the the average Nusselt number is higher at A = 2 marginally.

Fig.10 depicts the isotherms, streamlines and the yielded/unyielded zones in different vertical (Ω) and horizontal (δ) positions at Ra = 10^6, $\theta = 0°$, Bn = 8, Ec = 0, and a = b = 0.1 L. In this figure, the distances between the centres of the horizontal and vertical cylinders have been selected the same values. The isotherms demonstrate that the curved shapes of the isotherms augment considerably with the rise of vertical (Ω) and horizontal (δ) positions and strengthen the convection process significantly. The streamlines also prove this pattern as the core of the streamline becomes strong and finally at $\Omega = \delta = 0.7$ six vortex are replaced with the two symmetric circulations in the cavity. It demonstrates that the yielded/unyielded parts alter significantly. At $\Omega = \delta = 0.3$, the unyielded zone fills the space between the four cylinders and around them, and moreover two curved unyielded parts are present close to side walls. The uniyielded parts diminish generally and their shapes alter as the vertical and horizontal positions enhances from $\Omega = \delta = 0.3$ to 0.4. It shows that an unyielded part generates under the bottom cylinder and the unyielded parts close to side walls decrease substantially. In addition, the unyielded part on the top of the top cylinder drops. The increase in the vertical and horizontal positions from $\Omega = \delta = 0.4$ to 0.5 creates an unyielded part in the top of the top cylinder and some unyielded parts in the middle of the cylinders; however, the ratio of yielded/unyielded parts do not change considerably. The rise of the distances $\Omega = \delta = 0.5$ to 0.6 enhance the unyielded part considerably while at $\Omega = \delta = 0.7$, the cavity is filled utterly with the unyielded material.

Table 6 cites total average Nusselt numbers in different vertical (Ω) and horizontal (δ) positions at Ra = 10^6, $\theta = 0°$, Bn = 8, Ec = 0, and a = b = 0.1 L. It demonstrates that the total average Nusselt number enhances gradually with the increase in the vertical (Ω) and horizontal (δ) positions.

Fig.11 illustrates the isotherms, streamlines and the yielded/unyielded zones in different horizontal (δ) positions at Ra = 10^6, $\theta = 0°$, $\Omega = 0.4$, Ec = 0, Bn = 8, and a = b = 0.1 L. In this figure, the distance between centres of the cylinders which are located vertically is fixed at $\Omega = 0.4$ where the distance between centres of the horizontal cylinders (δ) is changing. It depicts that the movements of the isotherms enhance considerably and result in the augmentation of convection process. The streamline confirms the trend as the movement of circulation enhances where two cores are appeared at $\delta = 0.7$. It demonstrates that the unyielded parts have nearly the same pattern at $\delta = 0.3$, 0.4 and 0.5 although the sizes of the unyielded parts are different. But at $\delta = 0.6$, the unyielded parts enhance significantly around the cylinders although the unyielded part under the bottom cylinder diminishes. At $\delta = 0.7$, there is just two small yielded parts around the horizontal cylinders while other parts are unyielded.

Table 7 mentions total average Nusselt numbers in different horizontal (δ) positions at Ra = 10^6, $\theta = 0°$, $\Omega = 0.4$, Ec = 0, Bn = 8, and a = b = 0.1 L. It shows that the total average Nusselt number augments steadily by the rise of horizontal (δ) positions.

Fig.12 exposes the isotherms, streamlines and the yielded/unyielded zones in different vertical positions (Ω) at Ra = 10^6, $\theta = 0°$, $\delta = 0.4$, Ec = 0, Bn = 8, and a = b = 0.1 L. In this figure, the distance between centres of the cylinders which are located horizontally is fixed at $\delta = 0.4$ where the distance between centres of the vertical cylinders (Ω) is changing. It depicts that the movements of the isotherms enhance considerably and result in the augmentation of convection process. The streamline confirms the trend as the movement of circulation enhances. It demonstrates that as the vertical position (Ω) increases from $\Omega = 0.3$ to 0.4, the unyielded parts diminish significantly and the positions of the unyielded parts alter in the cavity. In fact, the massive unyielded parts close to the side walls and top of the top cylinder dwindle considerably. However, an unyielded part is generated in the bottom of the cylinder. At $\Omega = 0.5$, 0.6, and 0.7, the patterns of the unyielded parts are similar where two unyielded sections close to the top and bottom vertical cylinders are generated as well as unyielded sections upside the horizontal cylinders and close to side walls.

Table 8 specifies total average Nusselt numbers in different vertical positions (Ω) at Ra = 10^6, $\theta = 0°$, $\delta = 0.4$, Ec = 0, Bn = 8, and a = b = 0.1 L. It demonstrates that the total average Nusselt number rises gradually with the increase in the vertical positions (Ω).

Fig.13 portrays the isotherms in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L. When the inclined angles increase, the buoyancy force divide into vertical and horizontal forces throughout the cavity. Therefore, the shapes of the isotherms change and the alteration provokes the heat transfer process in the cavity to ameliorate or reduce. The rise of Bingham number enhances the gradient of isotherms on the cylinder and the movements of the isotherms between the cylinder and the hot walls diminish substantially. Therefore, heat transfer declines considerably as Bingham number increases in various inclined angles.

Fig.14 demonstrates the streamlines in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L. The inclined angle causes the buoyancy forces to apply to different directions and not only in the vertical direction. So, the circulation and values of the two symmetric streamlines around the cylinder alter. This alteration confirms that the convection process changes due to the inclined angle. As Bingham number increases, the streamlines weaken evidently and therefore convection process drops.

Fig.15 depicts the yielded/unyielded sections in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L. At Bn = 1 and 4, the unyielded part expands on the top corner of the cavity at $\theta = 40°$ and 120° while the sizes of unyielded zones do not alter significantly at $\theta = 80°$ compared to $\theta = 0°$. For Bn = 8, the increase in the inclined angles from $\theta = 0°$ to $\theta = 40°$ and 120° causes two unyielded parts on the bottom and top corners of the cavity are generated. But, at $\theta = 80°$, the unyielded zone diminishes slightly compared to $\theta = 0°$. For Bn = 12, interestingly, the most unyielded parts are observed at $\theta = 0°$ in contrast with the previous trends of smaller Bingham numbers. At Bn = 16 and 20, there are only yielded parts around the cylinder and other sections are unyielded. Finally, it demonstrates Bn = 20 is the close value to the critical Bingham number for different inclined angles.

Table 9 states total average Nusselt numbers in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L. For Bn = 1, the total average Nusselt number is nearly the same for the inclined angles of $\theta = 0° = 40°$. The total average Nusselt number increases considerably as the inclined angle increases from $\theta = 40°$ to 80°, but it drops from $\theta = 80°$ to 120°. At Bn = 4, the maximum and minimum average Nusselt numbers are obtained at inclined angles of $\theta = 0°$ and

80°; respectively. Similar with the Bingham number of Bn = 4, it shows that for Bn = 8 and 12, the highest and least values of average Nusselt numbers are observed at inclined angles of $\theta = 0°$ and 80°; respectively. At Bn = 16 and 120, the total average Nusselt numbers are nearly the same.

Fig.16 exhibits the isotherms, streamlines and yielded/unyielded parts in different Eckert numbers at at $\Omega = 0.4$, Bn = 8, and a = b = 0.1 L, and $\theta = 0°$. It demonstrates that the increase in Eckert number provokes the isotherms, streamlines and the unielded/yielded sections to change slightly.

Table 10 mentions average Nusselt numbers in different Eckert numbers at Bn = 8, Ec=0, Ra = 10^6, a= b = 0.1 L, $\theta = 0°$, and $\Omega = 0.4$. It indicates that the total average Nusselt number increases slightly as the Eckert number increases.

6 Concluding Remarks

Natural convection of viscoplastic fluids in an inclined heated enclosure with four inner cold cylinders in a diamond shape in the presence of viscous dissipation has been analyzed by Lattice Boltzmann method (LBM). In this study, the Bingham model without any regularization has been studied for the simulation of viscoplastic fluids. Fluid flow, heat transfer, and yielded/unyielded have been conducted for certain pertinent parameters of Rayleigh number (Ra = 10^4, 10^5 and 10^6), Bingham number (Bn = 1 - Bn_c), Eckert number, the size of the inner cylinder, various inclined angles of the cavity ($\theta = 0°$, 40°, 80°, 120°), the ratio of radii of the inner cylinders (A = 0.5, 1, and 2), and different positions of the inner cylinder. The main conclusions of the present investigation can be summarized as follows:

- Heat transfer enhances with augmentation of Rayleigh number in different studied parameters.
- The average Nusselt numbers demonstrate that the heat transfer declines with the rise of the Bingham number in various studied parameters.
- It was found that the rise of Bingham number augments the unyielded part, but provokes the heat transfer to drop gradually.
- The enhancement of Rayleigh number increases the critical Bingham number significantly.
- The increase in the ratio of the cylinder radii (A = b/a) enhance heat transfer.
- The unyielded zones diminish gradually as the ratio of the cylinder radii (A = b/a) increases.
- The increase in the distances between horizontal and vertical cylinders equally ($\delta = \Omega$) enhances heat transfer considerably and causes the

unyielded zones to develop significantly.

- The rise of the distance between two horizontal cylinders for a fixed distance between vertical cylinders ($\Omega = 0.2$) augments heat transfer substantially and provokes the unyielded zones to expand considerably.
- The enhancement of the distance between two vertical cylinders for a fixed distance between horizontal cylinders ($\delta = 0.2$) grows heat transfer steadily and changes unyielded zones.
- The alteration of inclined angle changes heat transfer and the unyielded zones significantly.
- The rise of Eckert number enhances the average Nusselt number gradually, but the unyielded regions alter marginally.

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Appendix

Here, we shall discuss the stability of the numerical scheme. Finding the parameter σ, we multiply f_α^{eq} with $|\boldsymbol{\xi}_\alpha|^2/2$ and take the sum, which leads to

$$\sum_{\alpha=0}^{8} \frac{1}{2} f_\alpha^{eq} |\boldsymbol{\xi}_\alpha|^2 = p + \frac{1}{2}\rho\,|\mathbf{u}|^2 - \frac{\tau_{xx} + \tau_{yy}}{2}. \tag{A1}$$

Next, it is easy to verify that

$$\sum_{\alpha=0}^{8} F_\alpha |\boldsymbol{\xi}_\alpha|^2 = 0. \tag{A2}$$

Hence,

$$\frac{\partial}{\partial t}\left[p + \frac{1}{2}\rho|\mathbf{u}|^2 - \frac{\tau_{xx} + \tau_{yy}}{2}\right] + \frac{\sigma^2}{2}\rho(\nabla \cdot \mathbf{u}) = O\left(\varepsilon\right). \tag{A3}$$

The Courant-Friedrichs-Lewy (CFL) condition states that [57,58]

$$K = \frac{u\Delta t}{\Delta x} + \frac{v\Delta t}{\Delta y} \leq 1. \tag{A4}$$

This can be used in (A3) and we obtain

$$\left[|\mathbf{u}|^2 + \frac{2p - \tau_{xx} - \tau_{yy}}{\rho}\right] + \sigma^2 K = O\left(\varepsilon\right). \tag{A5}$$

Thus, the lattice speed σ must satisfy

$$\sigma = K_c\sqrt{\left|\frac{\tau_{xx} + \tau_{yy} - 2p}{\rho} - |\mathbf{u}|^2\right|}, \quad K_c = \frac{1}{\sqrt{K}} \geq 1. \tag{A6}$$

Since the pressure p has to be uniquely defined in a Bingham fluid, one requires that $\tau_{xx} + \tau_{yy} = 0$;. Thus, reduces to

$$\sigma = K_c\sqrt{\left|\frac{-2p}{\rho} - |\mathbf{u}|^2\right|}, \quad K_c = \frac{1}{\sqrt{K}} \geq 1 \tag{A7}$$

As a result, the value σ is modified and changes in each iteration as defined through (A7).

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Nomenclature

A The first Rivlin-Ericksen tensor
A The ratio of the inner cylinder radii
a Horizontal radius
Bn Bingham number
b Vertical radius
c Lattice speed
c_p Specific heat capacity at constant pressure
Ec Eckert number
F External forces
f_α Density distribution functions for the specific node of α
f_α^{eq} Equilibrium density distribution functions for the specific node of α
g_α Internal energy distribution functions for the specific node of α
g_α^{eq} Equilibrium internal energy distribution functions for the specific node of α
g Gravity
k Thermal conductivity
L Length of the cavity
N Body force
Nu Nusselt number
p Pressure
Pr Prandtl number
Ra Rayleigh number
T Temperature
t Time
x, y Cartesian coordinates
x_{c1}, y_{c1} The horizontal and vertical positions of the first cylinder center
x_{c2}, y_{c2} The horizontal and vertical positions of the second cylinder center
x_{c3}, y_{c3} The horizontal and vertical positions of the third cylinder center
x_{c4}, y_{c4} The horizontal and vertical positions of the fourth cylinder center
u Velocity in x direction
U The buoyancy velocity scale
v Velocity in y direction

Greek letters

β_T Thermal expansion coefficient
ϕ Relaxation time
τ Shear stress
τ_y Yield stress
ξ Discrete particle speeds
Δx Lattice spacing in x direction

Δy Lattice spacing in y direction
Δt Time increment
α Thermal diffusivity
ρ Density of fluid
ψ Stream function value
$\boldsymbol{\Lambda}$ The viscoplasticity constraint
θ The inclined angle of the cavity
Ω The vertical distance between the centres of the first and third cylinders
δ The horizontal distance between the centres of the second and fourth cylinders

Subscripts
avg Average
B Bottom
C Cold
$c1$ Center of the first cylinder
$c2$ Center of the second cylinder
d Dynamic
H Hot
L Left
x, y Cartesian coordinates
α Specific node
R Right
s Static
T Top
tot Total

Table 1
Grid independence study at Ra = 10^6, Bn = 1, Ec = 0, a = b = 0.1 L, $\theta = 0°$, Ω = 0.4

Mesh size	Nu_{avg}
100*100	10.229
150*150	10.791
200*200	11.285
250*250	11.392
300*300	11.558
350*350	11.558

Table 2
Comparison of the average Nusselt number on the one hot circular cylinder (a = b = 0.2 L) between the present result and previous studies for Newtonian fluids (Bn = 0) in different Rayleigh numbers at $\theta = 0°$, and ($x_c = y_c = 0.5$ L)

	Present study	Zhang et al. [14]	Kim et al. [8]	Park et al. [10]
Ra = 10^3	5.105	5.103	5.093	5.107
Ra = 10^4	5.116	5.087	5.108	5.128
Ra = 10^5	7.791	7.651	7.767	7.836
Ra = 10^6	14.201	14.024	14.110	14.462

Table 3
Average Nusselt numbers in different Rayleigh numbers at Ec = 0, Bn = 1, a = b = 0.1 L, $\theta = 0°$, and $\delta = \Omega = 0.4$

	Ra = 10^4	Ra = 10^5	Ra = 10^6
Nu_{avg}	8.844	9.199	11.558

Table 4
The critical Bingham number in different Rayleigh numbers at Ec = 0, a = b = 0.1 L, $\theta = 0°$, and $\Omega = 0.4$

	Ra = 10^4	Ra = 10^5	Ra = 10^6
Bn_c	3	8	22

Table 5
Average Nusselt numbers in different Bingham numbers and the ratio of radii (A) at Ec = 0, Ra = 10^6, $\theta = 0°$, and $\delta = \Omega = 0.4$

	Bn=1	Bn=4	Bn=8	Bn=12	Bn=16	Bn=20
A = 0.5	7.070	3.668	2.577	2.576	2.578	2.577
A = 1	11.558	7.781	4.516	4.047	4.039	4.040
A = 2	11.226	7.885	3.989	2.720	2.585	2.578

Table 6
Average Nusselt numbers in different positions of the inner cylinders at a = b = 0.1 L, Ec = 0, $\theta = 0°$, Bn = 8, and Ra = 10^6

	$\Omega = \delta = 0.3$	$\Omega = \delta = 0.4$	$\Omega = \delta = 0.5$	$\Omega = \delta = 0.6$	$\Omega = \delta = 0.7$
Nu_{avg}	2.175	4.516	7.092	10.857	18.288

Table 7
Average Nusselt numbers in different distances between the horizontal cylinders (δ) at $\theta = 0°$, Ec = 0, Ra = 10^6, $\Omega = 0.4$, Bn = 8, and a= b = 0.1 L

	$\delta = 0.3$	$\delta = 0.4$	$\delta = 0.5$	$\delta = 0.6$	$\delta = 0.7$
Nu_{avg}	3.431	4.516	5.573	7.440	11.164

Table 8
Average Nusselt numbers in different distances between the vertical cylinders (Ω) at $\theta = 0°$, Ec = 0, Ra = 10^6, $\delta = 0.4$, Bn = 8, and a= b = 0.1 L

	$\Omega = 0.3$	$\Omega = 0.4$	$\Omega = 0.5$	$\Omega = 0.6$	$\Omega = 0.7$
Nu_{avg}	3.140	4.516	6.431	8.365	11.819

Table 9
Average Nusselt numbers in different Bingham numbers and inclined angles (θ) at Ec = 0, Ra = 10^6, a= b = 0.1 L, and $\delta = \Omega = 0.4$

	Bn=1	Bn=4	Bn=8	Bn=12	Bn=16	Bn=20
$\theta = 0°$						
Nu_{avg}	7.070	3.668	2.557	2.576	2.578	2.577
$\theta = 40°$						
Nu_{avg}	7.062	4.471	2.863	2.594	2.581	2.580
$\theta = 80°$						
Nu_{avg}	10.907	7.628	3.864	2.705	2.591	2.580
$\theta = 120°$						
Nu_{avg}	8.901	5.902	3.250	2.644	2.583	2.577

Table 10
Average Nusselt numbers in different Eckert numbers at Bn = 8, Ec=0, Ra = 10^6, a= b = 0.1 L, $\theta = 0°$, and $\delta = \Omega = 0.4$

	Ec = 0	Ec = 0.01	Ec = 0.1	Ec = 1
$Nu_{tot_{avg}}$	2.557	2.598	2.644	2.685

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Fig.13 The comparison of the isotherms in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

Fig.14 The comparison of the streamlines in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

Fig.15 The comparison of the yielded/unyielded zones in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

Fig.16 The comparison of the isotherms, streamlines and yielded/unyielded parts in different Eckert numbers (Ec = 0 (red line) and Ec = 1 (dashed green line)) at $\delta = \Omega = 0.4$, Bn = 8, and a = b = 0.1 L, and $\theta = 0°$

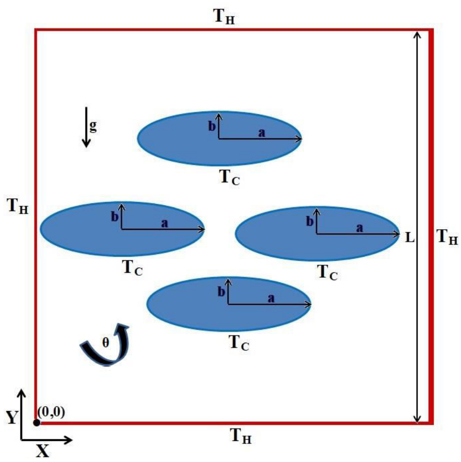


Fig. 1. The geometry of the present problem

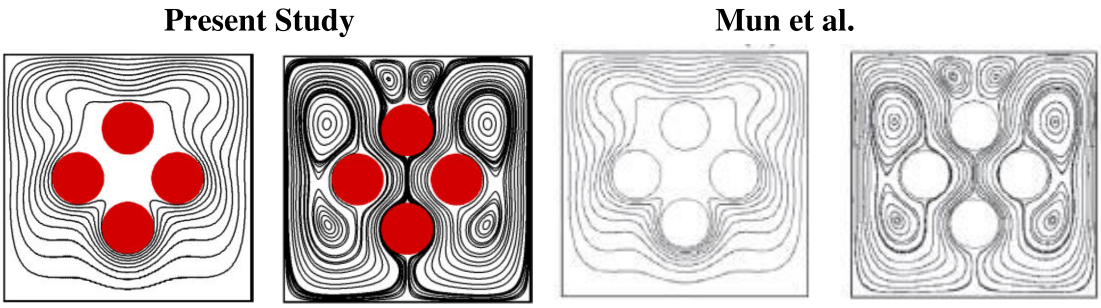


Fig. 2. The comparison of the isotherms and streamlines between present study and Mun et al. [13] at Ra = 10^5, Pr=0.71, $\theta = 0°$, $\delta = \Omega = 0.4$, and a = b = 0.1 L

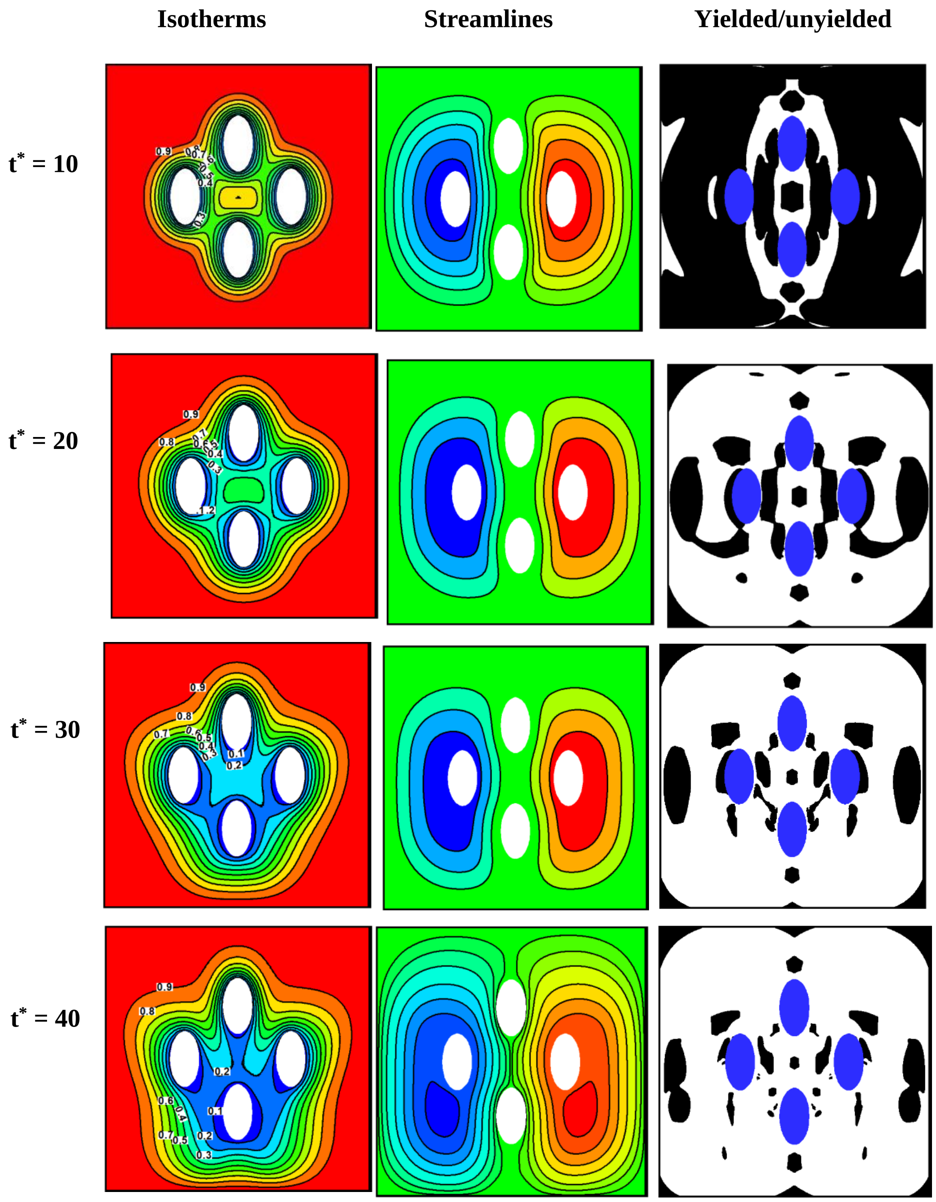


Fig. 3. The comparison of the isotherms, streamlines and yielded/unyielded parts in different non-dimensional time (t^*) at $\delta = \Omega = 0.4$, Bn = 8, Ec = 0, $\theta = 0°$, and a = 0.05 L, b = 0.1 L

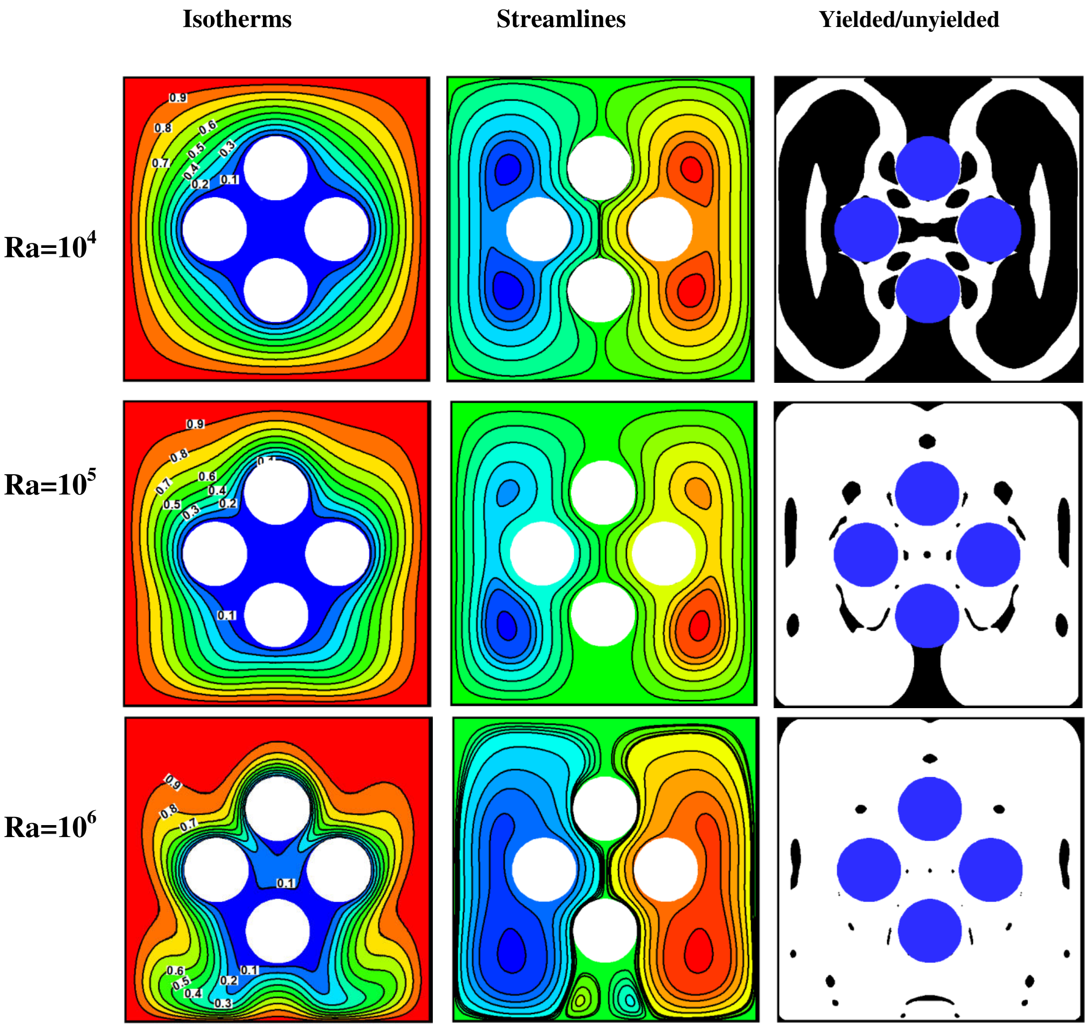


Fig. 4. The comparison of the isotherms, streamlines and yielded/unyielded parts in different Rayleigh numbers at $\delta = \Omega = 0.4$, Bn = 1, Ec = 0, $\theta = 0°$, and a = b = 0.1 L

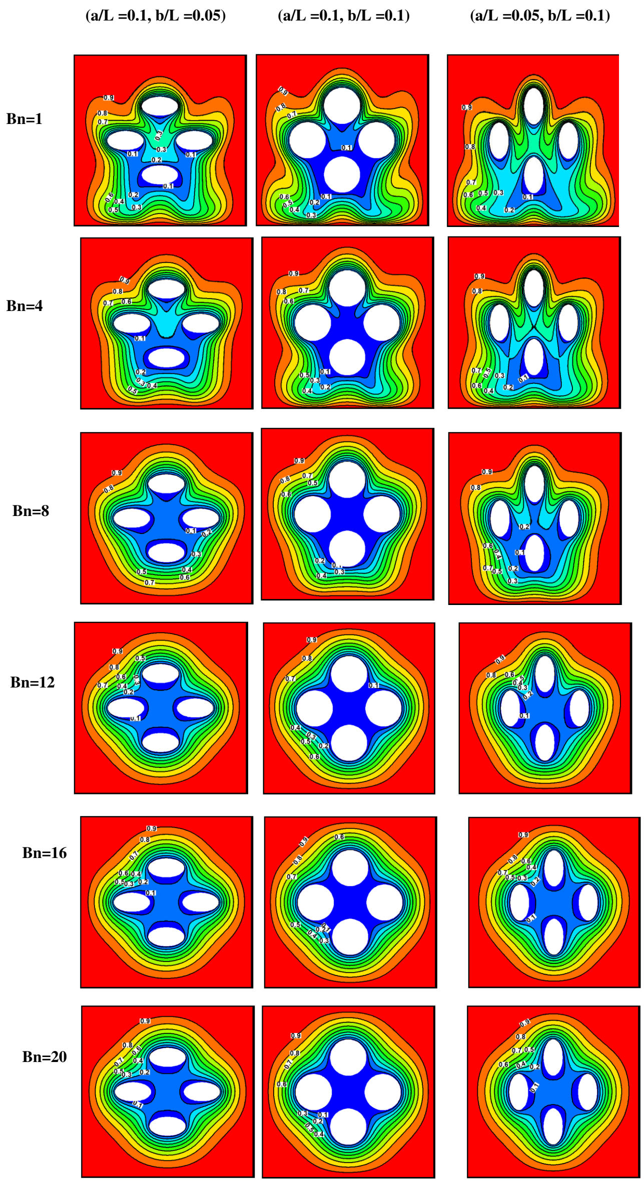


Fig. 5. The comparison of the isotherms in different radii and Bingham numbers at $\delta = \Omega = 0.4$, $\mathrm{Ec} = 0$, and $\theta = 0°$

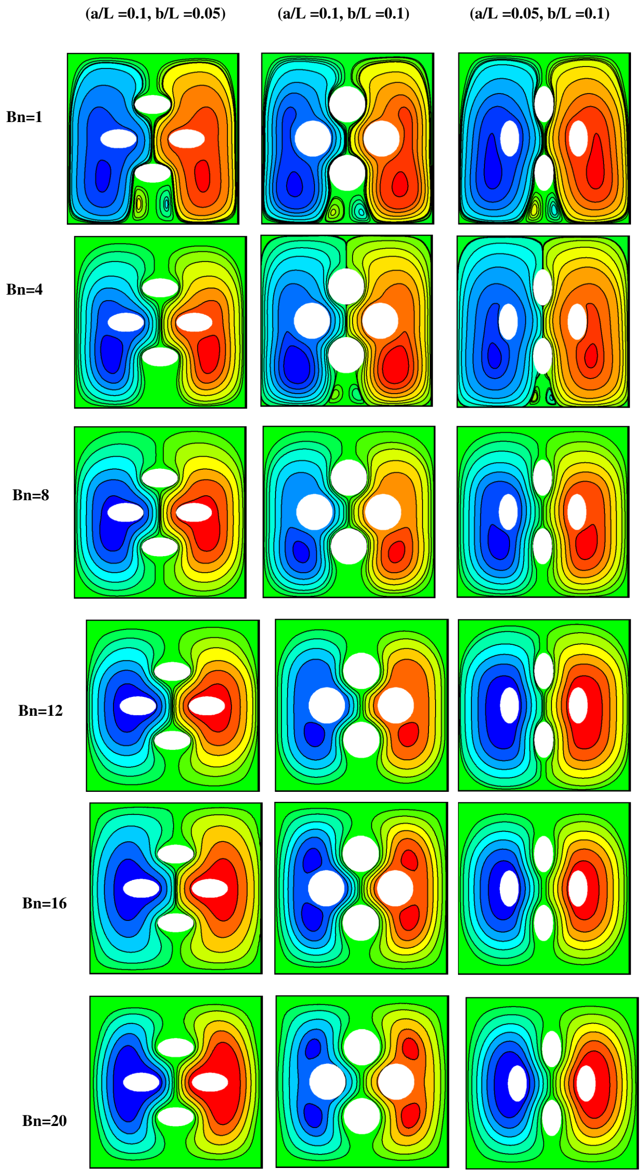


Fig. 6. The comparison of the streamlines in different radii and Bingham numbers at $\delta = \Omega = 0.4$, Ec $= 0$, and $\theta = 0°$

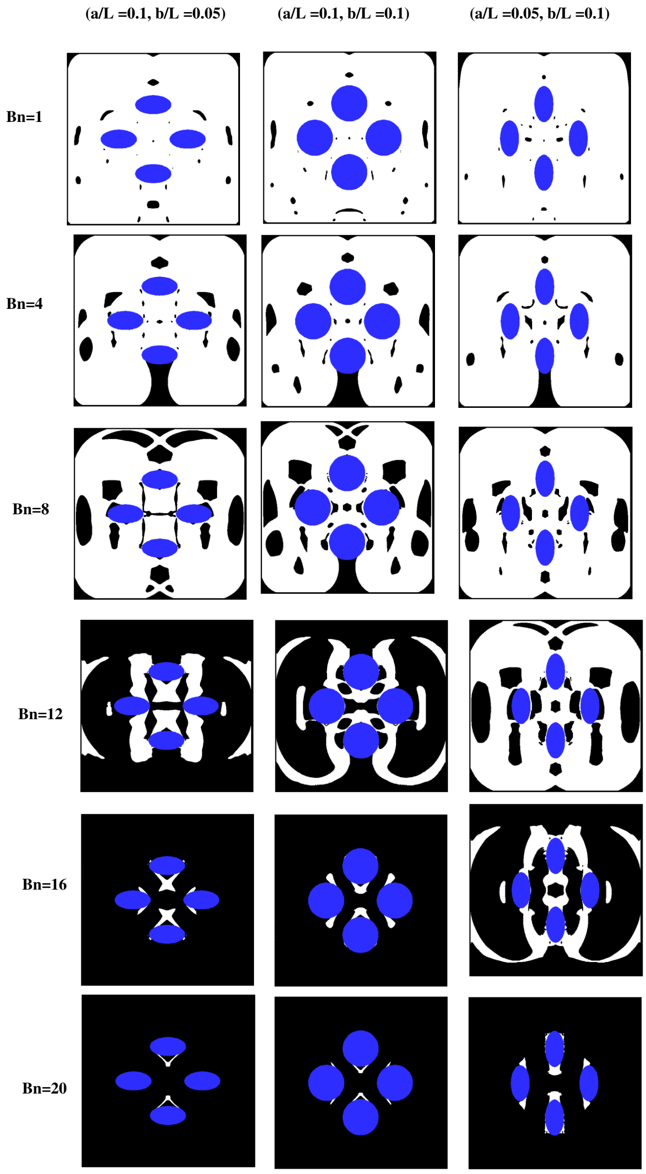


Fig. 7. The comparison of the yielded/unyielded zones in different radii and Bingham numbers at $\delta = \Omega = 0.4$, Ec $= 0$, and $\theta = 0°$

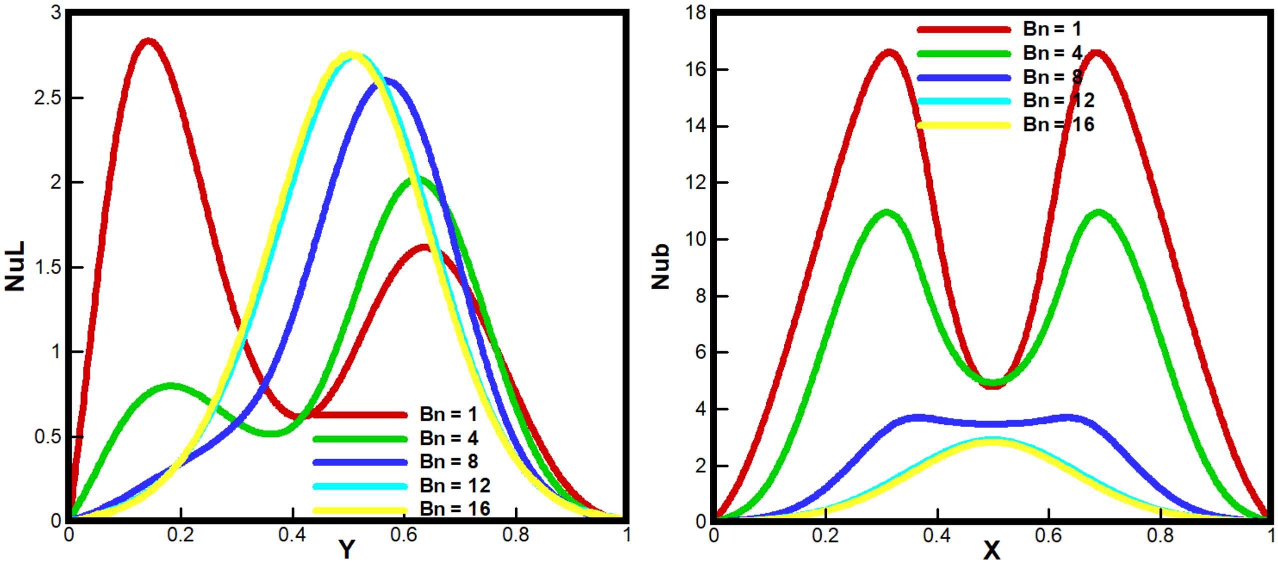
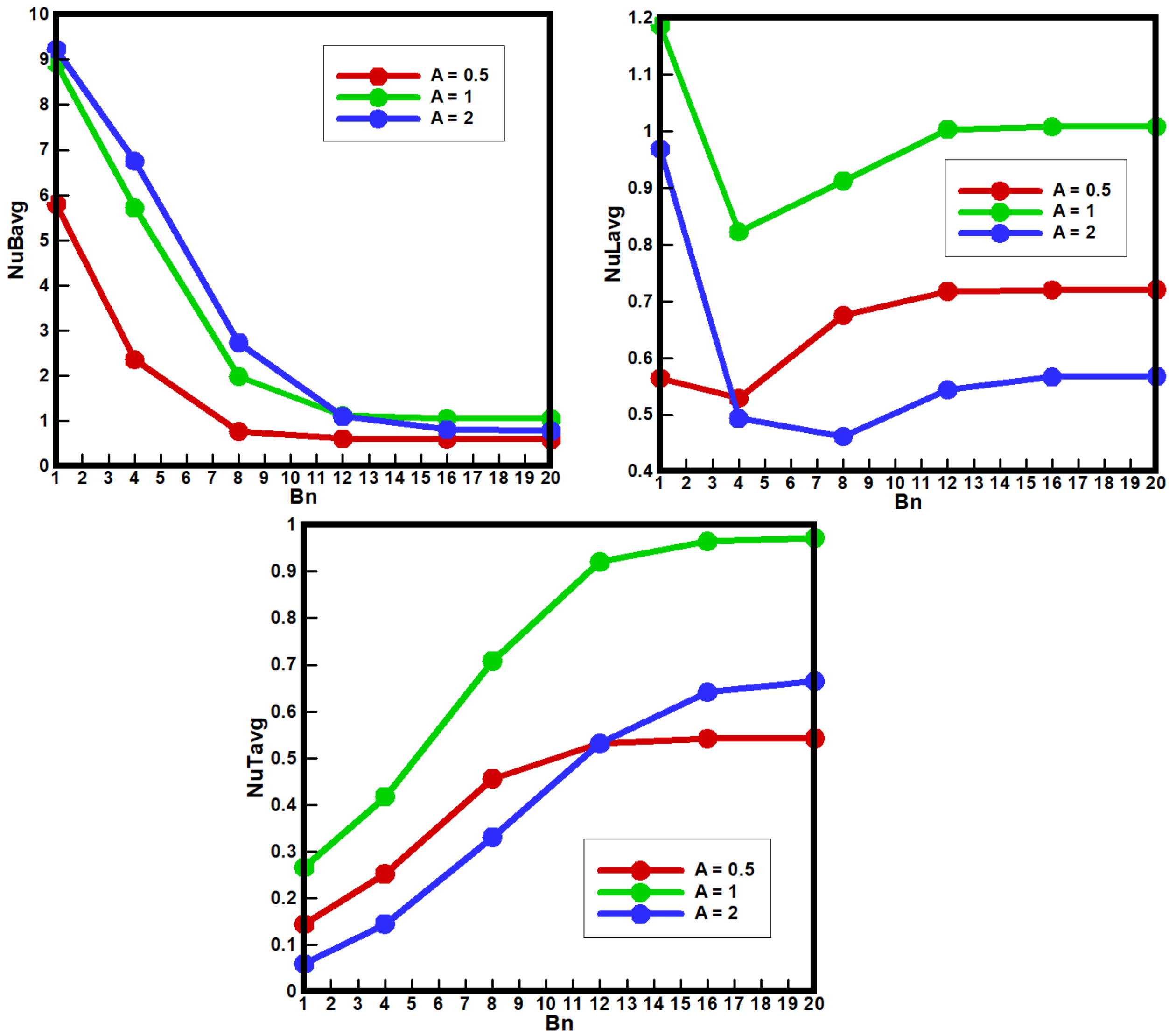


Fig. 8. The local Nusselt numbers on the left wall (Nu_L) and the bottom wall (Nu_b) in different Bingham numbers at $\delta = \Omega = 0.4$, Ra $= 10^6$, Ec $= 0$, $\theta = 0°$, and a $=$ b $= 0.1$ L



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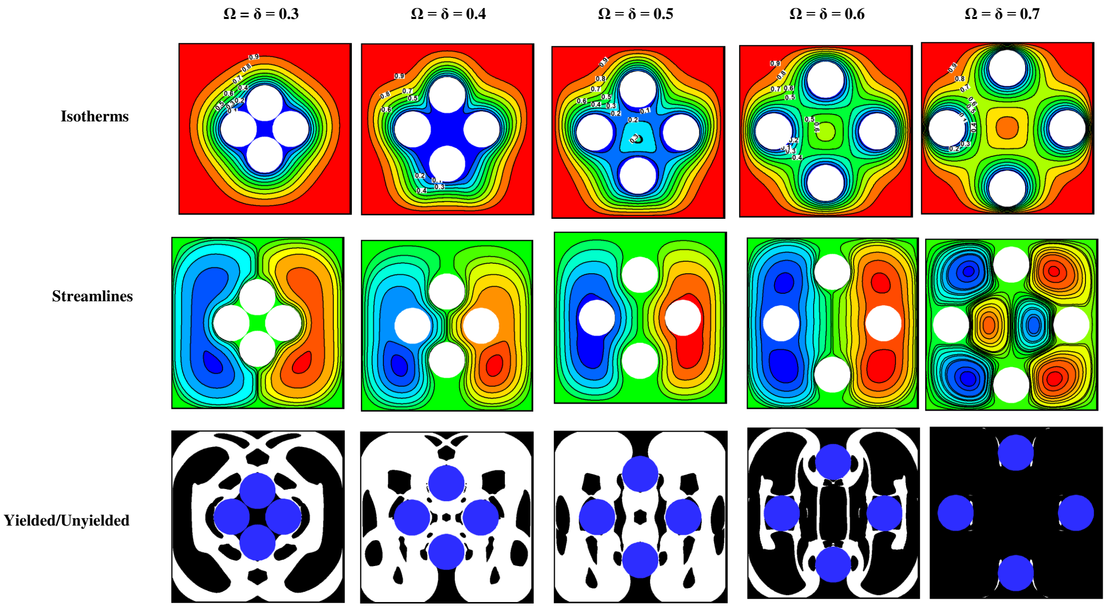

Fig. 10. The comparison of the isotherms, streamlines and the yielded/unyielded zones in different vertical (Ω) and horizontal (δ) positions at Ra $= 10^6$, $\theta = 0°$, Bn $= 8$, Ec $= 0$, and a $=$ b $= 0.1$ L

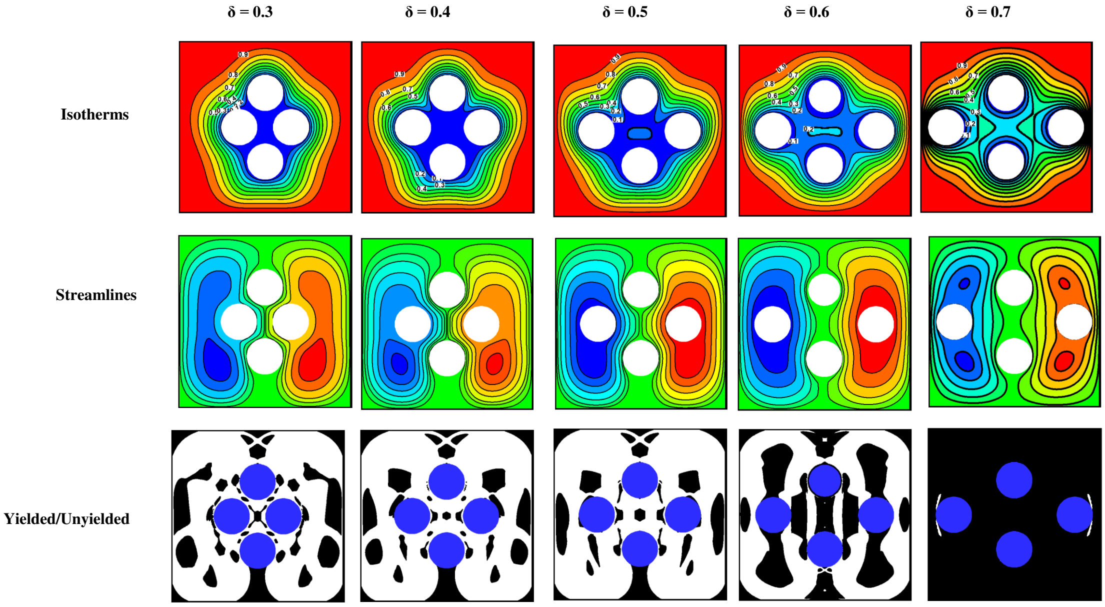

Fig. 11. The comparison of the isotherms, streamlines and the yielded/unyielded zones in different horizontal (δ) positions at Ra $= 10^6$, $\theta = 0°$, $\Omega = 0.4$, Ec $= 0$, Bn $= 8$, and a $=$ b $= 0.1$ L

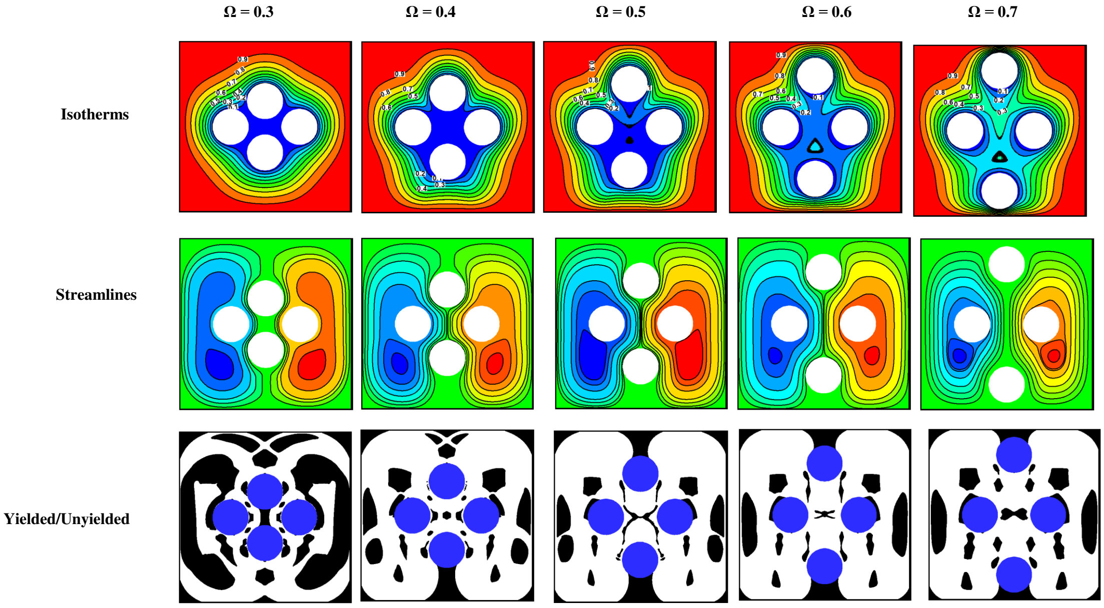

Fig. 12. The comparison of the isotherms, streamlines and the yielded/unyielded zones in different vertical positions (Ω) at Ra $= 10^6$, $\theta = 0°$, $\delta = 0.4$, Ec $= 0$, Bn $=$ 8, and a $=$ b $=$ 0.1 L

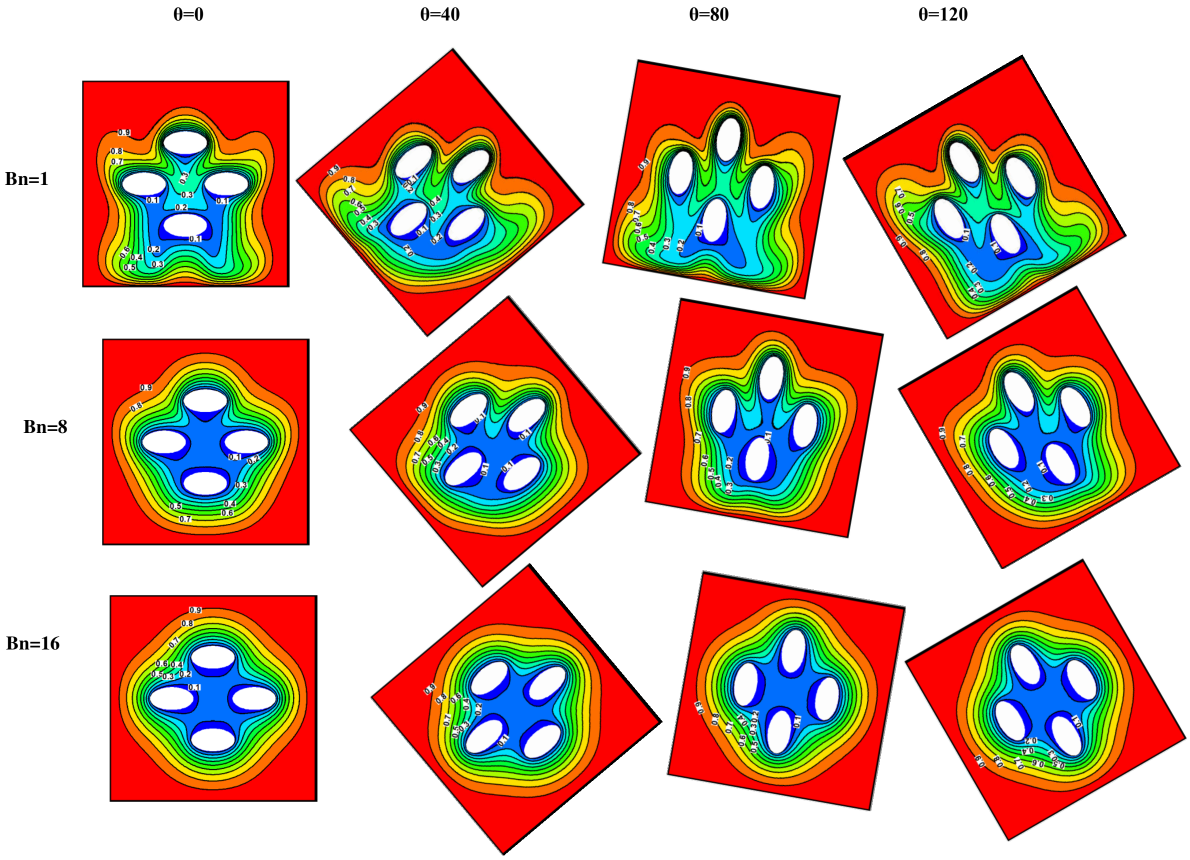

Fig. 13. The comparison of the isotherms in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

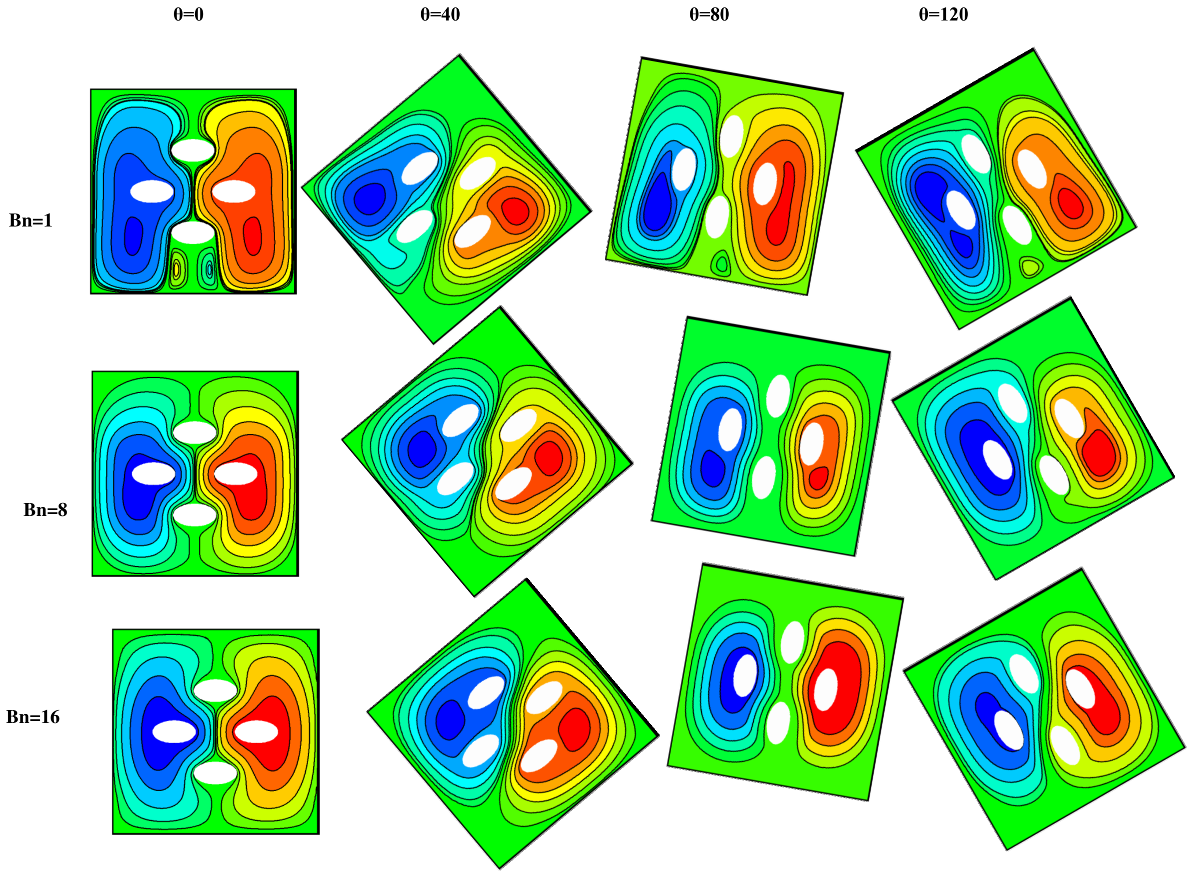

Fig. 14. The comparison of the streamlines in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

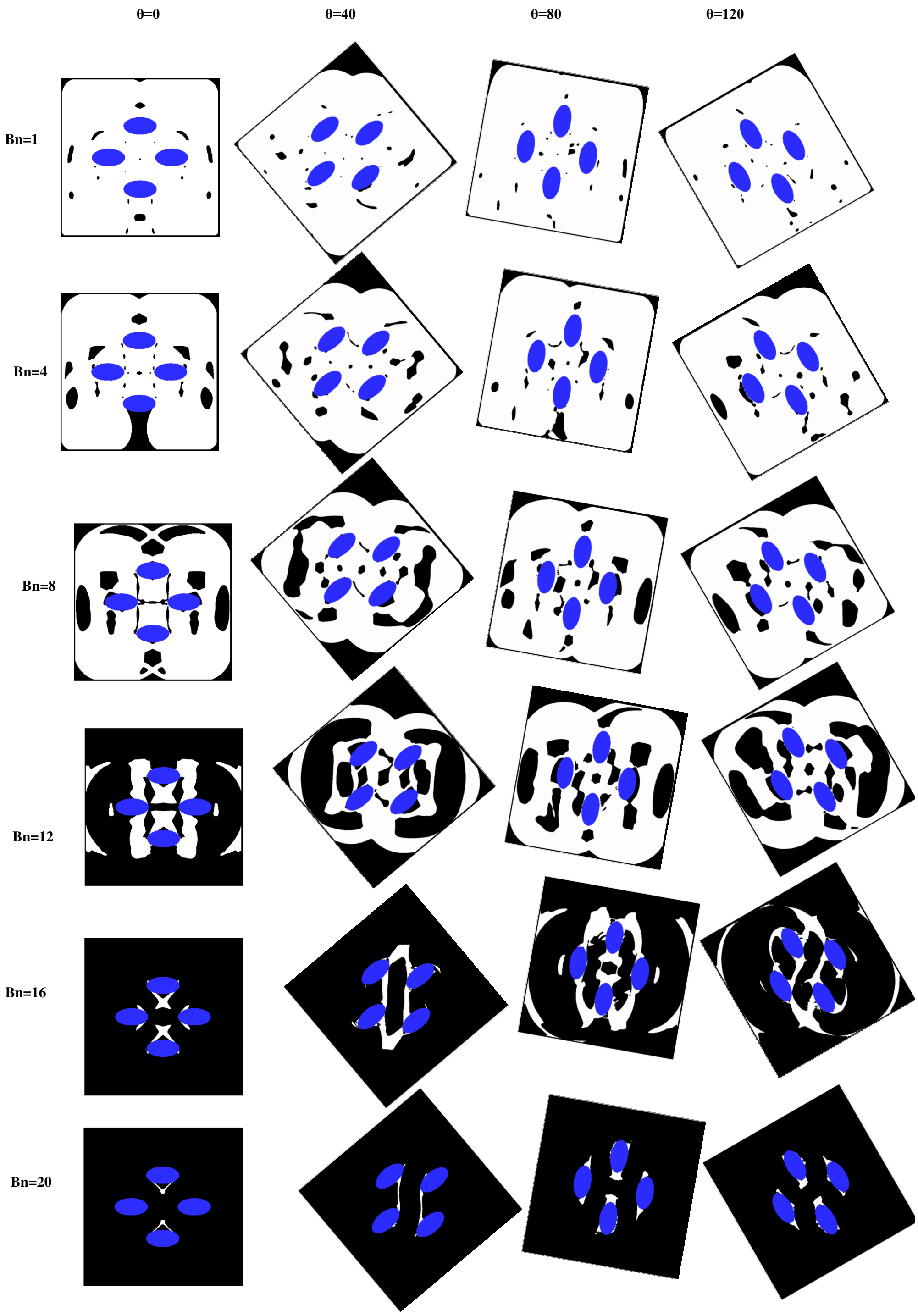

Fig. 15. The comparison of the yielded/unyielded zones in different inclined angles and Bingham numbers at $\delta = \Omega = 0.4$, Ec = 0, and a = 0.1 L, b = 0.05 L

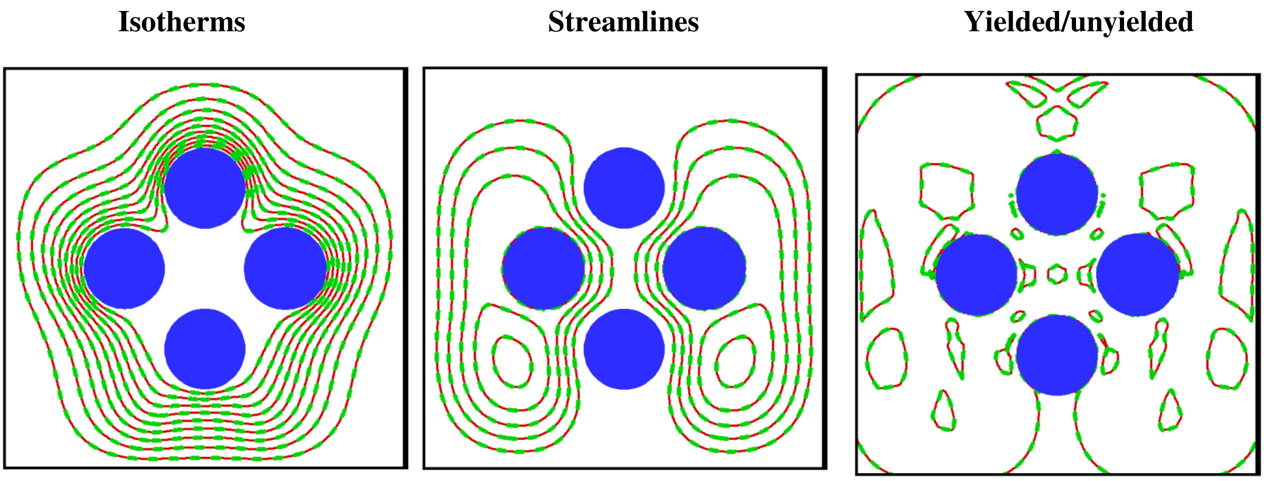

Fig. 16. The comparison of the isotherms, streamlines and yielded/unyielded parts in different Eckert numbers (Ec = 0 (red line) and Ec = 1 (dashed green line)) at $\delta = \Omega = 0.4$, Bn = 8, and a = b = 0.1 L, and $\theta = 0°$