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Global dynamics of a two-species clustering model with Lotka–Volterra competition

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Abstract. This paper is concerned with the global dynamics of a two-species Grindrod clustering model with Lotka–Volterra competition. The model takes the advective flux to depend directly upon local population densities without requiring intermediate signals like attractants or repellents to form the aggregation so as to increase the chances of survival of individuals like human populations forming small nucleated settlements. By imposing appropriate boundary conditions, we establish the global boundedness of solutions in two-dimensional bounded domains. Moreover, we prove the global stability of spatially homogeneous steady states under appropriate conditions on system parameters, and show that the rate of convergence to the coexistence steady state is exponential while the rate of convergence to the competitive exclusion steady state is algebraic.

Mathematics Subject Classification. 35M33, 35A01, 35B40, 35Q92.

Keywords. Clustering model, Lotka–Volterra competition, Boundedness, Global stability, Lyapunov functional.

1. Introduction

The reaction-diffusion systems have been advocated to interpret numerous biological phenomena such as wave propagations [13,41], pattern formation [16, 25], ecological invasions [11,19], competition of species [18,23], wound healing [39], and so on (cf. [5,33]). However, in many biological processes involving directed motions, such as chemotaxis and predators seeking prey (prey-taxis), reaction-diffusion models may not be adequate to describe how organisms move and disperse. For instance, Rowell in [37] explains how models with random diffusion fail to explain certain ecological phenomena and do not accurately reflect the non-Brownian motion of individuals. The Lotka–Volterra type predator–prey system with random diffusion only is unable to produce

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spatial patterns (cf. [47]) to interpret the spatiotemporal heterogeneity observed in the field experiment [24,45]. Rational movement along resource gradients has been thought to reduce the diffusive effect and result in clustering and formation of colonies to increase the chances of survival of individuals like human populations forming small nucleated settlements which grow as the population saturates locally. Incorporating both random and rational movements, Grindrod [15] proposed models of individual clustering in single-species and multi-species communities by taking the advective flux to depend directly upon local population densities without requiring intermediate signals like attractants or repellents.

Let us first briefly review the origin of Grindrod clustering models [15]. Classically, models for the spatial dispersion of biological populations have the form

$$u_t = \Delta\phi(u) + f(u,x,t),$$

where $u(x,t)$ denotes the population density at location x and time t, and $f(u,x,t)$ represents population kinetics due to the birth and death; ϕ satisfies $\phi(0) = 0$ and $\phi'(u) > 0$ for $u > 0$. As highlighted in [15], the above model contains no aggregation mechanism such as swarming, herding, and clustering of individuals, which can serve as a balancing factor between death and birth rates and increase survival chances. To incorporate this phenomenon, a modified population balance equation reads

$$\partial_t u = -\nabla\cdot(u\boldsymbol{V}(u,t,x)) + uE(u,t,x),$$

where $\boldsymbol{V}$ and E are the average velocity of individuals and the net rate of reproduction per individual, respectively. E typically has the form

$$E(u) = \begin{cases} 1-u, & \text{monostable case,}\\ (1-u)(u-a) \text{ for some } a\in(0,1), & \text{bistable case.}\end{cases}$$

Considering random diffusion with a probability $\delta\in(0,1)$, and deterministic dispersion with the probability $1-\delta$ in an average velocity $\mathbf{w}$ to increase the expected net rate of reproduction, $\boldsymbol{V}$ responding to u and E is given by $\boldsymbol{V} = -\delta\frac{\nabla u}{u} + (1-\delta)\mathbf{w}$. The former obeys Fickian diffusion $\frac{\nabla u}{u}$, while the latter is supposed to increase the net rate of reproduction per individual, such as $\mathbf{w}\approx\lambda\nabla E$ with $\lambda>0$. This leads to the following model [15]

$$\begin{cases} \partial_t u = \delta\Delta u - (1-\delta)\nabla\cdot(u\mathbf{w}) + uE(u,t,x),\\ -\varepsilon\Delta\mathbf{w} + \mathbf{w} = \lambda\nabla E(u),\end{cases}$$

where $\varepsilon>0$ is a small constant accounting for the small fluctuation to smooth out any sharp local variations in ∇E. After some rescalings, and assuming that the environment is homogeneous, the single-species model proposed in [15] reads

$$\begin{cases} u_t = d\Delta u - \chi\nabla\cdot(u\mathbf{w}) + \gamma uE(u), & x\in\Omega,\ t>0,\\ -\varepsilon\Delta\mathbf{w} + \mathbf{w} = \nabla E(u), & x\in\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) is a bounded domain with a smooth boundary, Variables $u(x,t)$ and $E(u)$, and the parameter ε have the same meaning as above, $\mathbf{w}(x,t)$

denotes the average velocity increasing the expected rate of reproduction of individuals up to a rescaling. The parameters d, χ, γ are all positive.

In the case of multi-species communities, the interspecific interactions (like competition or cooperation) between different species are indispensable. In particular the m-species Grindrod clustering model with competitive interactions reads as (cf. [15])

$$\begin{cases} \partial_t u_i = d_i \Delta u_i - \chi_i \nabla \cdot (u_i \mathbf{w}_i) + u_i E_i, & x \in \Omega,\ t > 0, \\ -\varepsilon_i \Delta \mathbf{w}_i + \mathbf{w}_i - \nabla E_i = 0, & x \in \Omega,\ t > 0, \\ u_i(x,0) = u_{i0}(x), & x \in \Omega, \end{cases} \tag{1.2}$$

with

$$E_i := E_i(u_1, u_2, \cdots, u_m) = a_i - \sum_{j=1}^{m} b_{ij} u_j, \quad i = 1, 2, \cdots, m,$$

where all parameters $d_i, \chi_i, \varepsilon_i, a_i, b_{ij}$ are positive. The original no-flux boundary condition (i.e. no individuals can cross the boundary) proposed in [15, formula (2.4)] for the two-species Grindrod clustering model (i.e., $m = 2$) is

$$\nabla u_i \cdot \mathbf{n} = \mathbf{w}_i \cdot \mathbf{n} = 0, \text{ on } \partial\Omega, \quad i = 1, 2,$$

where $\mathbf{n}$ denotes the unit outer normal vector to the boundary $\partial\Omega$. However, the above boundary condition $\mathbf{w}_i \cdot \mathbf{n} \mid_{\partial\Omega} = 0$ for $\mathbf{w}_i$ is inadequate to warrant the global well-posedness of the model (1.2) in multi-dimensions ($n \geq 2$). This limitation was identified by Nasreddine [34–36] for the single-species Grindroid clustering model, where the additional boundary condition $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} \times \mathbf{n} \mid_{\partial\Omega} = 0$ is suggested for the velocity $\mathbf{w}$. Such a kind of boundary condition is not peculiar, see e.g. [12,38] for other models incorporating this kind of boundary condition. Accordingly, for the m-species Grindroid clustering model (1.2), we incorporate the boundary condition $\frac{\partial \mathbf{w}_i}{\partial \mathbf{n}} \times \mathbf{n} \mid_{\partial\Omega} = 0$ for $\mathbf{w}_i$ ($i = 1, 2, \cdots, m$). Therefore, the boundary conditions of (1.2) to be considered are

$$\nabla u_i \cdot \mathbf{n} = \mathbf{w}_i \cdot \mathbf{n} = 0,\ \partial_{\mathbf{n}} \mathbf{w}_i \times \mathbf{n} = \mathbf{0}, \text{ on } \partial\Omega, \quad i = 1, 2, \cdots, m. \tag{1.3}$$

As usual, for vectors $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n)$, the cross product $\mathbf{a} \times \mathbf{b}$ is the number $a_1 b_2 - a_2 b_1$ if $n = 2$ and the vector $(a_2 b_3 - a_3 b_2, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1)$ if $n = 3$. In one dimension ($n = 1$), the condition $\partial_{\mathbf{n}} \mathbf{w}_i \times \mathbf{n} = \mathbf{0}$ is not needed. The Dirchlet boundary condition $\mathbf{w}|_{\partial\Omega} = 0$ satisfies the condition for $\mathbf{w}$ in (1.3). Though the Grindrod models were proposed three decades ago, there is no mathematical result available, except some preliminary results obtained for the single-species Grindrod model (1.1) with boundary conditions in (1.3). Nasreddine [34] proved the local-in-time existence of strong solutions of (1.1) with (1.3) in multi-dimensions for $(u, \mathbf{w}) \in W^{1,p}(\Omega)$ with $p > n$ and global existence of strong solutions to (1.1) in one dimension with (1.3) for both monostable and bistable functions $E(u)$ as well as L^2 convergence of solutions to constant steady states in the monostable case. The global existence of strong solutions of (1.1) with (1.3) in two dimensions was later established in [36], where the solution bound in $W^{1,p}(\Omega)$ depends on time and the possibility of blow-up at infinite time was not precluded. The diffusion vanishing problem of (1.1) with (1.3) as $\varepsilon \to 0$ in one dimension was investigated in [35], and the existence of traveling wave solutions of (1.1) in $\mathbb{R}$ was established in [21] for $E(u) = 1 - u$. The planar and radial traveling wave solutions of single-species and two-species Grindrod models with monostable function $E(u)$ in $\mathbb{R}^n$ was investigated in [26] alongside numerical simulations and found that directed motion can have substantial impacts not only on wave speed but also on the existence and structure of emergent patterns. The pattern formation of the two-species Grindrod model (i.e. (1.2) with $m = 2$) with (1.3) in a two-dimensional convex domain was studied in [27], by assuming $\mathbf{w}_i = \nabla\phi_i$ for some potential functions ϕ_i $(i = 1, 2)$, for three interspecific interactions: competition, generalist predator–prey and predator–prey. In particular, how the advective dispersal of species in heterogeneous resources and hazards leads to asymptotic steady states that retain spatial aggregation or clustering in regions of resource abundance and away from hazards was examined. By investigating pattern formation of the multi-species Grindrod model (1.2) approximated by a non-local cross-diffusion model, the authors of [40] proved that the Turing patterns, which were impossible for the two-species models, may arise for m-species Grindrod models with $m \geq 3$.

From the above literature review, we see that the qualitative understanding of the Grindrod clustering models remains poorly understood, especially whether the solution blows up in infinite time in multi-dimensions is inconclusive and the large-time behavior of solutions is also unclear. This paper is devoted to exploring these basic questions. Without loss of generality, we consider the two-species Grindrod clustering model with competitions

$$\begin{cases} \partial_t u_1 = d_1\Delta u_1 - \chi_1\nabla\cdot(u_1\mathbf{w}_1) + u_1E_1(u_1,u_2), & x\in\Omega,\ t>0,\\ \partial_t u_2 = d_2\Delta u_2 - \chi_2\nabla\cdot(u_2\mathbf{w}_2) + u_2E_2(u_1,u_2), & x\in\Omega,\ t>0,\\ -\varepsilon_1\Delta\mathbf{w}_1 + \mathbf{w}_1 = \nabla E_1(u_1,u_2), & x\in\Omega,\ t>0,\\ -\varepsilon_2\Delta\mathbf{w}_2 + \mathbf{w}_2 = \nabla E_2(u_1,u_2), & x\in\Omega,\ t>0,\\ \nabla u_i\cdot\mathbf{n} = \mathbf{w}_i\cdot\mathbf{n} = 0,\ \partial_{\mathbf{n}}\mathbf{w}_i\times\mathbf{n} = 0, \quad i=1,2, & x\in\partial\Omega,\ t>0,\\ (u_1,u_2)(x,0) = (u_{10},u_{20})(x), & x\in\Omega, \end{cases} \tag{1.4}$$

with

$$E_1(u_1,u_2) := \gamma_1 - u_1 - cu_2 \quad \text{and} \quad E_2(u_1,u_2) := \gamma_2 - bu_1 - u_2, \tag{1.5}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with a smooth boundary and $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$, and all parameters $d_i, \chi_i, \varepsilon_i, \gamma_i, b, c$, $i\in\{1,2\}$, are positive. We underline that the boundary conditions in (1.4) (see also (1.3)) means that $\partial_{\mathbf{n}}\mathbf{w}_i$ is parallel to $\mathbf{n}$ on $\partial\Omega$, which along with the boundary condition $\mathbf{w}_i\cdot\mathbf{n}\mid_{\partial\Omega}=0$ implies

$$\mathbf{w}_i\cdot\partial_{\mathbf{n}}\mathbf{w}_j\mid_{\partial\Omega}=0, \quad i,j\in\{1,2\}. \tag{1.6}$$

We remark that without advection (i.e. $\chi_1 = \chi_2 = 0$), the first two equations of (1.4)–(1.5) have no components $\mathbf{w}_i (i = 1, 2)$ and become the well-known competition-diffusion Lotka–Volterra model which has been well studied (cf. [4,30]). The competition models with advection have also been widely studied in literatures (cf. [1,2,6–9,14,50]). All these works have assumed that the advection is biased to the concentration gradient of given resources. The competition dynamics in advective environments like the river or stream were also studied (cf. [29,31]). We refer to [48,49] for the study of global dynamics of diffusion-advection competition models with more general diffusive and/or advective coefficients. When the advection is modeled by the prey-taxis in a predator–prey system, the competition dynamics were investigated in [44]. Evidently, the advection considered in the Grindrod model (1.4)–(1.5) are different from those considered in the existing studies mentioned above.

In this paper, we shall establish the global boundedness and time-asymptotic dynamics of solutions to (1.4). Our first result concerning the global existence and boundedness of classical solutions is stated in the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Assume $u_{10}, u_{20} \in W^{1,p}(\Omega)$ with $p > 2$ and $u_{10}, u_{20} \gneqq 0$. Then the system (1.4) admits a unique classical solution $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ satisfying*

$$\begin{cases} u_i \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ \mathbf{w}_i \in \left[C^{2,1}(\bar{\Omega} \times (0,\infty))\right]^2, \end{cases} \quad i = 1, 2,$$

and $u_1, u_2 > 0$ in $\Omega \times (0,\infty)$. Moreover, there exists a positive constant C independent of t such that

$$\sum_{i=1}^{2} \left(\|u_i(\cdot,t)\|_{W^{1,p}(\Omega)} + \|\mathbf{w}_i(\cdot,t)\|_{W^{1,p}(\Omega)} \right) \le C \quad \text{for all } t > 0. \tag{1.7}$$

Next, we explore asymptotic dynamics of (1.4). Except the extinction steady state $(0, 0, \mathbf{0}, \mathbf{0})$, the system (1.4) has three possible homogeneous steady states $(u_{1s}, u_{2s}, \mathbf{0}, \mathbf{0})$ depending on the value of parameters b, c, γ_1, γ_2. They can be classified into the following three categories similar to the classical Lotka–Volterra competition system (cf. [30]):

- **Case 1.** Weak competition: $c < \frac{\gamma_1}{\gamma_2} < \frac{1}{b}$.
- **Case 2.** Competitive exclusion: $\frac{\gamma_1}{\gamma_2} < \min\{\frac{1}{b}, c\}$ (resp. $\frac{\gamma_1}{\gamma_2} > \max\{\frac{1}{b}, c\}$).
- **Case 3.** Strong competition: $\frac{1}{b} < \frac{\gamma_1}{\gamma_2} < c$.

Then the corresponding homogeneous steady state $(u_{1\,s}, u_{2\,s}, \mathbf{0}, \mathbf{0})$ can be solved as follows:

$$(u_{1s}, u_{2s}) = \begin{cases} (0, \gamma_2) \text{ or } (\gamma_1, 0) \text{ or } (u_1^*, u_2^*), & \text{in } \textbf{Case 1}, \\ (0, \gamma_2) \text{ or } (\gamma_1, 0), & \text{in } \textbf{Case 2}, \end{cases}$$

where

$$(u_1^*, u_2^*) := \left(\frac{\gamma_1 - \gamma_2 c}{1 - bc}, \frac{\gamma_2 - \gamma_1 b}{1 - bc} \right). \tag{1.8}$$

To state our main results on the large time behavior of solutions, we introduce some notations. Denote the function

$$f(x, y) := \frac{1 + x^2 y^2}{1 - xy} > 1 \quad \text{for } x, y > 0 \text{ and } xy < 1, \tag{1.9}$$

and define two positive constants

$$K_1 := \frac{16 d_1 \varepsilon_1}{\chi_1^2} \quad \text{and} \quad K_2 := \frac{16 d_2 \varepsilon_2}{\chi_2^2}. \tag{1.10}$$

Moreover, when $c < \frac{\gamma_1}{\gamma_2} < \frac{1}{b}$ (weak competition), define two positive constants

$$K_1^* := \frac{2u_1^*\left(\frac{K_2}{1-bc} - u_2^*(1+bc)\right)}{K_2 - f(b,c)u_2^*} \quad \text{if } K_2 > f(b,c)u_2^* \tag{1.11}$$

and

$$K_2^* := \frac{2u_2^*\left(\frac{K_1}{1-bc} - u_1^*(1+bc)\right)}{K_1 - f(b,c)u_1^*} \quad \text{if } K_1 > f(b,c)u_1^*. \tag{1.12}$$

Our second result is stated in the following.

Theorem 1.2. *Suppose that the conditions in Theorem 1.1 hold. Then the solution $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ of the system (1.4) obtained in Theorem 1.1 has the following convergence properties.*

(i) Assume $c < \frac{\gamma_1}{\gamma_2} < \frac{1}{b}$ and (u_1^*, u_2^*) is given by (1.8). If (K_1, K_2) defined in (1.10) satisfies

$$K_1 > f(b,c)u_1^* \quad \text{and} \quad K_2 > K_2^*, \tag{1.13}$$

or

$$K_2 > f(b,c)u_2^* \quad \text{and} \quad K_1 > K_1^*, \tag{1.14}$$

then there exist positive constants C and λ independent of t such that

$$\sum_{i=1}^{2}\left(\|u_i(\cdot,t) - u_i^*\|_{W^{1,\infty}(\Omega)} + \|\mathbf{w}_i\|_{W^{1,\infty}(\Omega)}\right) \le Ce^{-\lambda t} \quad \text{as } t \to \infty.$$

(ii) Assume $\frac{\gamma_1}{\gamma_2} < \min\{\frac{1}{b}, c\}$. If

$$K_2 > f\left(b, \frac{\gamma_1}{\gamma_2}\right)\gamma_2 = \frac{\gamma_2^2 + b^2\gamma_1^2}{\gamma_2 - b\gamma_1}, \tag{1.15}$$

then there exists a positive constant C independent of t such that

$$\|u_1(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_2(\cdot,t) - \gamma_2\|_{W^{1,\infty}(\Omega)} + \sum_{i=1}^{2}\|\mathbf{w}_i\|_{W^{1,\infty}(\Omega)} \le \frac{C}{1+t} \quad \text{as } t \to \infty.$$

(iii) Assume $\frac{\gamma_1}{\gamma_2} > \max\{\frac{1}{b}, c\}$. If

$$K_1 > f\left(\frac{\gamma_2}{\gamma_1}, c\right)\gamma_1 = \frac{\gamma_1^2 + c^2\gamma_2^2}{\gamma_1 - c\gamma_2},$$

then there exists a positive constant C independent of t such that

$$\|u_1(\cdot,t) - \gamma_1\|_{W^{1,\infty}(\Omega)} + \|u_2(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \sum_{i=1}^{2}\|\mathbf{w}_i\|_{W^{1,\infty}(\Omega)} \le \frac{C}{1+t} \quad \text{as } t \to \infty.$$

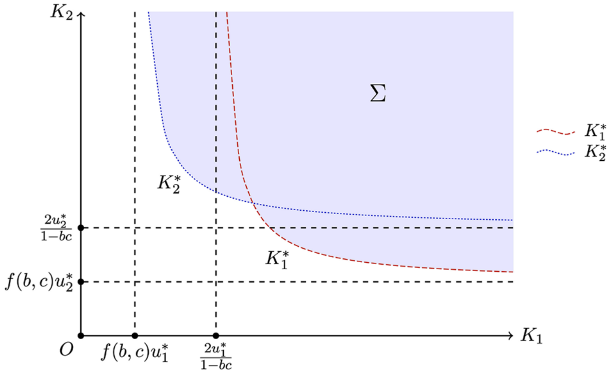

FIGURE 1. Illustration of Σ defined by (1.16)

Remark 1.1. It is unclear whether the parameter regime of (K_1, K_2) satisfying (1.13) or (1.14) in Theorem 1.2(i) is admissible. Below we shall confirm this and further show what the admissible regime looks like. First, one can check that (K_1^*, K_2^*) defined by (1.11) and (1.12) satisfies

$$K_1^* > f(b,c)u_1^* > u_1^* \quad \text{and} \quad K_2^* > f(b,c)u_2^* > u_2^*.$$

Viewing K_1^* and K_2^* as functions of K_2 and K_1 according to (1.11) and (1.12), respectively, we get

$$K_1^* = \frac{2u_1^*}{1-bc} + \frac{I_K}{K_2 - f(b,c)u_2^*} \quad \text{and} \quad K_2^* = \frac{2u_2^*}{1-bc} + \frac{I_K}{K_1 - f(b,c)u_1^*},$$

where

$$I_K = \frac{2u_1^* u_2^* (f(b,c) + b^2c^2 - 1)}{(1-bc)} > 0$$

due to $f(b,c) > 1$. Therefore, K_1^* (resp. K_2^*) decreases monotonically with respect to $K_2 \in (f(b,c)u_2^*, +\infty)$ (resp. $K_1 \in (f(b,c)u_1^*, +\infty)$). Let

$$\Sigma := \{(K_1, K_2) \mid (K_1, K_2) \text{ satisfies (1.13) or (1.14)}\}, \tag{1.16}$$

then the region of Σ is showed in Fig. 1.

The rest of this paper is organized as follows. In Sect. 2, we shall address the local existence of solutions to (1.4), and then we will use an extension criterion to prove that the local solution is actually uniformly bounded and exists globally in time in Sect. 3. In Sect. 4, we shall prove the global stabilities stated in Theorem 1.2 by constructing Lyapunov functionals along with compactness arguments.

2. Preliminaries: local existence and some inequalities

Before proceeding, we introduce some notations used throughout the paper.

Notations:

- For brevity, we abbreviate $\int_0^t \int_\Omega f(\cdot,s)dxds$ and $\int_\Omega f(\cdot,t)dx$ as $\int_0^t \int_\Omega f$ and $\int_\Omega f$, respectively. In addition, C and C_i $(i = 1, 2, 3, \cdots)$ stand for generic positive constants which may vary from line to line.
- $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le k\}$ denotes the usual Sobolev space, where $D^\alpha u$ is the weak partial derivative. If $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$.

In this section, we establish the local existence of solutions to the system (1.4) under appropriate initial conditions. Moreover, we shall collect and prove some useful inequalities which will be used in the subsequent sections. To begin with, we consider the regularity of the solution $\mathbf{w}$ to the following system:

$$\begin{cases} -\Delta \mathbf{w} + \mu \mathbf{w} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} \mid_{\partial\Omega} = 0, \ \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n} \mid_{\partial\Omega} = \mathbf{0}, & \text{if } n = 2, 3, \end{cases} \tag{2.1}$$

where $\mu \in \{0, 1\}$ and $\mathbf{f} \in (L^p(\Omega))^n$ for some $1 < p < +\infty$. The system (2.1) has the following properties.

Lemma 2.1. (cf. [38, Theorem 3, Theorem 4, Remark 5]) *Let $\mu = 0$ and $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded domain with a smooth boundary.*

(i) *If $\mathbf{f} \in (L^p(\Omega))^n$ with some $1 < p < +\infty$, then the system (2.1) has a unique solution $\mathbf{w} \in (W^{2,p}(\Omega))^n$ satisfying*

$$\|\mathbf{w}\|_{W^{2,p}(\Omega)} \le C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}, \tag{2.2}$$

where $C(p, n, \Omega)$ is a positive constant depending only on p, n and Ω.

(ii) *If $\mathbf{f} \in (C^{k,\alpha}(\bar{\Omega}))^n$ with some $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, then the system (2.1) has a unique solution $\mathbf{w} \in C^{k+2,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ and*

$$\|\mathbf{w}\|_{C^{k+2,\beta}(\bar{\Omega})} \le C(k, \alpha, \beta, n, \Omega)\|\mathbf{f}\|_{C^{k,\alpha}(\bar{\Omega})}. \tag{2.3}$$

where $C(k, \alpha, \beta, n, \Omega)$ is a positive constant depending only on k, α, β, n and Ω.

Now we prove the following results for the case $\mu = 1$.

Lemma 2.2. *Let $\mu = 1$ and $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded domain with a smooth boundary.*

(i) *If $\mathbf{f} \in \left(H^k(\Omega)\right)^n$ with $k \in \{0, 1, 2\}$, then the system (2.1) has a unique solution $\mathbf{w} \in H^{k+2}(\Omega)$ satisfying*

$$\|\mathbf{w}\|_{H^{k+2}(\Omega)} \le C(k, n, \Omega)\|\mathbf{f}\|_{H^k(\Omega)}, \tag{2.4}$$

where $C(k, n, \Omega)$ is a positive constant depending only on k, n and Ω.

(ii) *If $\mathbf{f} \in (L^p(\Omega))^n$ with some $1 < p < +\infty$, then the system (2.1) has a unique solution $\mathbf{w} \in (W^{2,p}(\Omega))^n$ satisfying*

$$\|\mathbf{w}\|_{W^{2,p}(\Omega)} \le C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}, \tag{2.5}$$

where $C(p, n, \Omega)$ is a positive constant depending only on p, n and Ω.

(iii) *If $\mathbf{f} \in (C^{k,\alpha}(\bar{\Omega}))^n$ with some $\alpha \in (0, 1)$ and $k \in \{0, 1, 2\}$, then the system (2.1) has a unique solution $\mathbf{w} \in C^{k+2,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ and*

$$\|\mathbf{w}\|_{C^{k+2,\beta}(\bar{\Omega})} \le C(k, \alpha, \beta, n, \Omega)\|\mathbf{f}\|_{C^{k,\alpha}(\bar{\Omega})}. \tag{2.6}$$

where $C(k, \alpha, \beta, n, \Omega)$ is a positive constant depending only on k, α, β, n and Ω.

Proof. The existence and uniqueness of the solution to (2.1) for $\mu = 1$ can be established by similar arguments for (2.1) in the case of $\mu = 0$ (cf. [38]). Below we only prove the regularity properties given in (2.4)–(2.6).

(i) Since the proof of (2.4) involves tedious calculations, we place it in Appendix A (see Lemma A1 and Lemma A2 in Appendix A).

(ii) Next we prove (2.5). The first equation of (2.1) can be rewritten as

$$-\Delta \mathbf{w} = \mathbf{f} - \mathbf{w}. \tag{2.7}$$

Therefore, in view of (2.2) and (2.7), it is sufficient to prove that for $p \in (1, \infty)$, there exists a positive constant $C(p, n, \Omega)$ such that

$$\|\mathbf{f} - \mathbf{w}\|_{L^p(\Omega)} \leq C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.8}$$

We consider three cases: $p \geq 2$, $\frac{6}{5} \leq p < 2$ and $1 < p < \frac{6}{5}$. First, if $p \geq 2$, it follows from $\mathbf{f} \in (L^p(\Omega))^n \hookrightarrow (L^2(\Omega))^n$ for $p \geq 2$ and (2.4) that $\mathbf{w} \in (H^2(\Omega))^n$ with $\|\mathbf{w}\|_{H^2(\Omega)} \leq C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}$, which alongside the Sobolev embedding $(H^2(\Omega))^n \hookrightarrow (L^p(\Omega))^n$ yields

$$\|\mathbf{w}\|_{L^p(\Omega)} \leq C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}$$

and hence (2.8) holds for $p \geq 2$. Secondly, if $\frac{6}{5} \leq p < 2$, we claim that

$$\text{the solution } \mathbf{w} \text{ to (2.1) satisfies } \|\mathbf{w}\|_{H^1(\Omega)} \leq C(p, n, \Omega)\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.9}$$

To this end, we define the real Hilbert space (cf. [10])

$$X := \left\{\mathbf{u} \in (H^1(\Omega))^n \mid \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\} \quad \text{with the norm}$$
$$\|\mathbf{u}\|_X := \left(\int_\Omega |\nabla \mathbf{u}|^2\right)^{\frac{1}{2}}.$$

Since $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^{p^*}(\Omega) = (L^p(\Omega))^*$ with $p^* := \frac{p}{p-1}$, we have $\mathbf{f} \in (L^p(\Omega))^n \hookrightarrow (H^1(\Omega))^*$. Define the bilinear form $B[\cdot, \cdot]$ on X by

$$B[\mathbf{u}, \mathbf{v}] := \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \int_\Omega \mathbf{u} \cdot \mathbf{v}, \quad \forall\, \mathbf{u}, \mathbf{v} \in X.$$

Then we have

$$\begin{aligned} |B[\mathbf{u}, \mathbf{v}]| &\leq \|\nabla \mathbf{u}\|_{L^2(\Omega)}\|\nabla \mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)} \\ &\leq \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \\ &\leq C \|\mathbf{u}\|_X \|\mathbf{v}\|_X \quad \forall\, \mathbf{u}, \mathbf{v} \in X, \end{aligned}$$

where we have used the fact that the X norm is equivalent to the usual $H^1(\Omega)$ norm (cf. [12]). Moreover,

$$B[\mathbf{u}, \mathbf{u}] = \|\mathbf{u}\|_{H^1(\Omega)}^2 \geq C \|\mathbf{u}\|_X^2, \quad \forall\, \mathbf{u} \in X. \tag{2.10}$$

Therefore, by the Lax–Milgram theorem, we obtain that for each $\mathbf{f} \in (L^p(\Omega))^n$ with $\frac{6}{5} \le p < 2$, there exists a unique $\mathbf{u}_f \in X$ such that

$$B[\mathbf{u}_f, \mathbf{v}] = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \quad \forall\, \mathbf{v} \in X.$$

Therefore, by (2.10), Hölder's inequality, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and the equivalence of X norm and $H^1(\Omega)$ norm, we have

$$\begin{aligned}\|\mathbf{u}\|_{H^1(\Omega)}^2 \le C\,\|\mathbf{u}\|_X^2 \le CB[\mathbf{u},\mathbf{u}] \le C\int_\Omega \mathbf{f}\cdot\mathbf{u} \le C\|\mathbf{f}\|_{L^p(\Omega)}\|\mathbf{u}\|_{L^{p^*}(\Omega)} \\ \le C\|\mathbf{f}\|_{L^p(\Omega)}\,\|\mathbf{u}\|_{H^1(\Omega)}\end{aligned}$$

for all $\mathbf{u} \in X$, that is

$$\|\mathbf{u}\|_{H^1(\Omega)} \le C\|\mathbf{f}\|_{L^p(\Omega)}, \quad \forall\, \mathbf{u} \in X.$$

Therefore, the claim (2.9) is proved since the solution $\mathbf{w}$ to (2.1) satisfies $\mathbf{w} \in X$, and hence (2.8) holds for $\frac{6}{5} \le p < 2$ due to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$. It remains to consider the case $1 < p < \frac{6}{5}$. We let $\alpha := \frac{p}{3-2p} \in (1,2)$ which satisfies

$$3\alpha = (\alpha - 1)p^*. \tag{2.11}$$

Then $|\mathbf{w}|^\alpha = \left(\sum_{i=1}^n w_i^2\right)^{\frac{\alpha}{2}}$ satisfies

$$\begin{cases} \Delta|\mathbf{w}|^\alpha = \alpha(\alpha-2)|\mathbf{w}|^{\alpha-4}|\mathbf{w}\cdot\nabla\mathbf{w}|^2 + \alpha|\mathbf{w}|^{\alpha-2}\left(|\nabla\mathbf{w}|^2 + \mathbf{w}\cdot\Delta\mathbf{w}\right) & \text{in } \Omega, \\ \partial_{\mathbf{n}}|\mathbf{w}|^\alpha = \alpha\sum_{i,j=1}^n |\mathbf{w}|^{\alpha-2} w_i \partial_j w_i n_j = \alpha|\mathbf{w}|^{\alpha-2}\mathbf{w}\cdot\partial_{\mathbf{n}}\mathbf{w} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

where we have used $\mathbf{w}\cdot\partial_{\mathbf{n}}\mathbf{w} = 0$ on $\partial\Omega$ due to the boundary conditions of $\mathbf{w}$ in (2.1). Integrating the first equation of (2.12) on Ω by parts with $\partial_{\mathbf{n}}|\mathbf{w}|^\alpha = 0$ on $\partial\Omega$ and using $-\Delta\mathbf{w} = \mathbf{f} - \mathbf{w}$ in Ω, one has

$$\begin{aligned}(\alpha-2)\int_\Omega |\mathbf{w}|^{\alpha-4}|\mathbf{w}\cdot\nabla\mathbf{w}|^2 + \int_\Omega |\mathbf{w}|^{\alpha-2}|\nabla\mathbf{w}|^2 \\ = -\int_\Omega |\mathbf{w}|^{\alpha-2}\mathbf{w}\cdot\Delta\mathbf{w} = \int_\Omega |\mathbf{w}|^{\alpha-2}\mathbf{w}\cdot(\mathbf{f}-\mathbf{w}),\end{aligned}$$

which together with $\alpha - 2 < 0$ and $\int_\Omega |\mathbf{w}|^{\alpha-4}|\mathbf{w}\cdot\nabla\mathbf{w}|^2 \le \int_\Omega |\mathbf{w}|^{\alpha-2}|\nabla\mathbf{w}|^2$ implies

$$(\alpha-1)\int_\Omega |\mathbf{w}|^{\alpha-2}|\nabla\mathbf{w}|^2 + \int_\Omega |\mathbf{w}|^\alpha \le \int_\Omega |\mathbf{w}|^{\alpha-1}|\mathbf{f}|. \tag{2.13}$$

With (2.11) and (2.13), we are in a position to use the same arguments as in [38, pp. 133–134] to show that

$$\|\mathbf{w}\|_{L^{3\alpha}(\Omega)} \le C\|\mathbf{f}\|_{L^p(\Omega)}.$$

By $3\alpha > 3$ and Hölder's inequality: $L^{3\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 < p < \frac{6}{5}$, we know that (2.8) holds for $1 < p < \frac{6}{5}$. Therefore, we have proved that (2.8) holds for $p \in (1,\infty)$ as desired.

(iii) If $\mathbf{f} \in (C^{k,\alpha}(\bar{\Omega}))^n$ with some $\alpha \in (0,1)$ and $k \in \{0,1,2\}$, then (2.4) implies

$$\|\mathbf{w}\|_{H^{k+2}(\Omega)} \le C(k,n,\Omega)\|\mathbf{f}\|_{H^k(\Omega)} \le C(k,n,\Omega),$$

which together with the Sobolev embedding $H^{k+2}(\Omega) \hookrightarrow C^{k,\theta}(\bar{\Omega})$ for any $\theta \in \left(0, 2 - \frac{n}{2}\right)$ shows that

$$\mathbf{f} - \mathbf{w} \in (C^{k,\alpha'}(\bar{\Omega}))^n \tag{2.14}$$

for any $\alpha' \in (0, \min\left\{\alpha, \frac{1}{2}\right\})$. In view of (2.3), (2.7) and (2.14), (2.6) is proved. $\square$

We are now in a position to show the local existence of the unique classical solution to (1.4).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Assume $u_{10}, u_{20} \in W^{1,p}(\Omega)$ with $p > 2$ and $u_{10}, u_{20} \gneqq 0$. Then there exists $T_{max} \in (0,\infty]$ such that the system (1.4) has a unique classical solution $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ satisfying*

$$\begin{cases} u_i \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ \mathbf{w}_i \in \left[C^{2,1}(\bar{\Omega} \times (0, T_{max}))\right]^2, \end{cases} \qquad i = 1, 2,$$

and $u_1, u_2 > 0$ in $\Omega \times (0, T_{max})$. Moreover,

$$\text{if} \quad T_{max} < \infty, \quad \text{then} \quad \lim_{t \to T_{max}} \left(\|u_1(\cdot,t)\|_{W^{1,p}(\Omega)} + \|u_2(\cdot,t)\|_{W^{1,p}(\Omega)}\right) = \infty. \tag{2.15}$$

Proof. Fix $R > 0$, and define for $T \in (0,1)$

$$X_R(T) := \left\{ (u_1, u_2) \in C\left([0,T]; W^{1,p}(\Omega)\right) \;\middle|\; \sup_{t\in[0,T]} \|u_1(t)\|_{W^{1,p}} + \sup_{t\in[0,T]} \|u_2(t)\|_{W^{1,p}} \le R \right\},$$

which is a complete metric space with the metric

$$d_X(\mathbf{u}, \mathbf{v}) = \sup_{t\in[0,T]} \|u_1(t) - v_1(t)\|_{W^{1,p}} + \sup_{t\in[0,T]} \|u_2(t) - v_2(t)\|_{W^{1,p}}$$

for $\mathbf{u} = (u_1(t), u_2(t)) \in X_R(T)$ and $\mathbf{v} = (v_1(t), v_2(t)) \in X_R(T)$. For any $\mathbf{u} = (u_1, u_2) \in X_R(T)$ and $t \in [0,T]$, by $p > 2$ and Lemma 2.2 we know that there exists a unique solution $(\mathbf{w}_1, \mathbf{w}_2) \in \left(H^2(\Omega)\right)^2 \times \left(H^2(\Omega)\right)^2$ to the following system

$$\begin{cases} -\varepsilon_i \Delta \mathbf{w}_i(t) + \mathbf{w}_i(t) = \nabla E_i(u_1, u_2), \; i = 1, 2, \; \text{in } \Omega, \\ \mathbf{w}_i \cdot \mathbf{n} = 0, \; \partial_{\mathbf{n}} \mathbf{w}_i \times \mathbf{n} = 0, \qquad i = 1, 2, \; \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Letting $\left(e^{t\Delta}\right)_{t\ge 0}$ denote the Neumann heat semigroup on Ω, we introduce a mapping $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ on $X_R(T)$ by defining

$$\tilde{u}_i(t) = \Phi_i(u_1, u_2)(\cdot, t) := e^{td_i\Delta} u_{i0} + \int_0^t e^{(t-s)d_i\Delta} \left\{-\nabla \cdot (\chi_i u_i \mathbf{w}_i) + u_i E_i(u_1, u_2)\right\}(\cdot, s) ds$$

for $i = 1, 2$, where $(\mathbf{w}_1, \mathbf{w}_2) \in \left(H^2(\Omega)\right)^2 \times \left(H^2(\Omega)\right)^2$ is the solution of (2.16) uniquely determined by the given $\mathbf{u} = (u_1, u_2) \in X_R(T)$ according to Lemma 2.2. Then one can show that $\mathbf{\Phi}$ is a contraction map from $X_R(T)$ into itself if T is sufficiently small by a standard argument (see e.g. [34, Theorem 2.2]). Therefore, for sufficiently small T, by the Banach fixed point theorem, there is a unique $(u_1, u_2) \in X_R(T)$ such that

$$(u_1, u_2) = (\tilde{u}_1, \tilde{u}_2) = \mathbf{\Phi}(u_1, u_2) = (\Phi_1(u_1, u_2), \Phi_2(u_1, u_2)),$$

and $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ is a unique strong solution of the system (1.4) satisfying

$$\begin{cases} u_i \in C\left([0,T), W^{1,p}(\Omega)\right) \cap C\left((0,T), W^{2,p}(\Omega)\right), \\ \mathbf{w}_i(t) \in \left(H^2(\Omega)\right)^2 \quad \text{for all } t \in [0,T), \end{cases} \qquad i = 1, 2. \tag{2.17}$$

Based on a bootstrap argument, we can use Lemma 2.2, the L^p-estimate and the Schauder estimate (cf. [28]) to show that the unique strong solution $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ of the system (1.4) satisfying (2.17) is actually a classical solution. Finally, u_1, $u_2 \geq 0$ follows from the maximum principle. To be precise, we rewrite the first equation of system (1.4) as

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 + \chi_1 \mathbf{w}_1 \cdot \nabla u_1 + Q_1(x,t) u_1 = 0, & x \in \Omega, t \in (0,T), \\ \frac{\partial u_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t \in (0,T), \\ u_1(x,0) = u_{10}(x), & x \in \Omega, \end{cases} \tag{2.18}$$

where $Q_1(x,t) := \chi_1 \nabla \cdot \mathbf{w}_1 - E_1(u_1, u_2)$ for $(x,t) \in \Omega \times (0,T)$. Then one can apply the strong maximum principle to system (2.18) and gets that $u_1(x,t) > 0$ for $(x,t) \in \Omega \times (0,T)$. Similarly, it holds that $u_2(x,t) > 0$ for $(x,t) \in \Omega \times (0,T)$. The proof is completed. □

Next we prove a useful Lemma.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with a smooth boundary, and let $r, q \geq 1$ be two constants satisfying*

$$\frac{1}{q} > \frac{1}{2} - \frac{1}{n} \quad \text{and} \quad \frac{1}{r} > \frac{1}{q} - \frac{1}{n}. \tag{2.19}$$

Then for any $\varphi \in H^2(\Omega)$ satisfying $\frac{\partial \varphi}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0$, there exists a positive constant C depending only on Ω, n, q and r such that

$$\|\nabla \varphi\|_{L^q(\Omega)} \leq C \left(\|\Delta \varphi\|^{\theta}_{L^2(\Omega)} \|\varphi\|^{1-\theta}_{L^r(\Omega)} + \|\varphi\|_{L^r(\Omega)} \right),$$

where

$$\theta = \frac{\frac{1}{r} + \frac{1}{n} - \frac{1}{q}}{\frac{1}{r} + \frac{2}{n} - \frac{1}{2}} \in (0,1). \tag{2.20}$$

Proof. First, one can use (2.19) to check that θ defined by (2.20) satisfies $\theta \in (0,1)$. Using the Gagliardo–Nirenberg inequality, we have

$$\|\nabla \varphi\|_{L^q(\Omega)} \leq C \left(\|D^2 \varphi\|^{\theta}_{L^2(\Omega)} \|\varphi\|^{1-\theta}_{L^r(\Omega)} + \|\varphi\|_{L^r(\Omega)} \right). \tag{2.21}$$

Under the homogeneous Neumann boundary condition $\frac{\partial\varphi}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0$, it follows from [3, Lemma 1] that $\|\nabla\varphi\|_{H^1(\Omega)} \leq C\|\Delta\varphi\|_{L^2(\Omega)}$, which implies

$$\|D^2\varphi\|_{L^2(\Omega)} \leq \|\nabla\varphi\|_{H^1(\Omega)} \leq C\|\Delta\varphi\|_{L^2(\Omega)}. \tag{2.22}$$

The proof is completed by substituting (2.22) into (2.21). □

We recall the following basic result which will be used to investigate the global stability of solutions.

Lemma 2.5. ([43, Lemma 1.1]) *Let $\tau \geq 0$ and $c > 0$ be two constants, $F(t) \geq 0, \int_\tau^\infty H(t)\mathrm{d}t < \infty$. Assume that $E \in C^1([\tau,\infty)), E$ is bounded from below and satisfies*

$$E'(t) \leq -cF(t) + H(t) \quad \text{in } [\tau,\infty).$$

If either $F \in C^1([\tau,\infty))$ and $F'(t) \leq k$ in $[\tau,\infty)$ for some $k > 0$, or $F \in C^\alpha([\tau,\infty))$ and $\|F\|_{C^\alpha([\tau,\infty))} \leq k$ for constants $0 < \alpha < 1$ and $k > 0$, then $\lim_{t\to\infty} F(t) = 0$.

3. Boundedness of solutions

In this section, we focus on the global boundedness of solutions of the system (1.4). Throughout this section, we assume that the conditions in Theorem 1.1 hold and $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ be a local-in-time classical solution of the system (1.4) obtained in Lemma 2.3 with a maximal time T_{max}. First of all, we give the following basic property for the solution components $\mathbf{w}_1$ and $\mathbf{w}_2$.

Lemma 3.1. *For $i, j \in \{1,2\}$, the solution of (1.4) satisfies*

$$\int_\Omega \Delta\mathbf{w}_i \cdot \mathbf{w}_j = -\int_\Omega \nabla\mathbf{w}_i \cdot \nabla\mathbf{w}_j.$$

Proof. Denote $\mathbf{w}_i = (w_1^{(i)}, w_2^{(i)})$, $i = 1,2$. Using integration by parts we get

$$\begin{aligned}\int_\Omega \Delta\mathbf{w}_i \cdot \mathbf{w}_j &= \sum_{k=1}^2 \int_\Omega \Delta w_k^{(i)} \cdot w_k^{(j)} \\ &= -\sum_{k=1}^2 \int_\Omega \nabla w_k^{(i)} \cdot \nabla w_k^{(j)} + \sum_{k=1}^2 \int_{\partial\Omega} \left(\nabla w_k^{(i)} \cdot \mathbf{n}\right) w_k^{(j)} dS. \end{aligned} \tag{3.1}$$

By (1.6) we have

$$\sum_{k=1}^2 \int_{\partial\Omega} \left(\nabla w_k^{(i)} \cdot \mathbf{n}\right) w_k^{(j)} dS = \sum_{k=1}^2 \int_{\partial\Omega} w_k^{(j)} \partial_\mathbf{n} w_k^{(i)} dS = \int_{\partial\Omega} \mathbf{w}_j \cdot \partial_\mathbf{n}\mathbf{w}_i dS = 0.$$

This together with (3.1) completes the proof. □

The following result is a basic property about the uniform-in-time boundedness of u_1 and u_2 in $L^1(\Omega)$.

Lemma 3.2. *Suppose that the conditions in Theorem 1.1 hold. Then for $i = 1, 2$, it holds that*

$$\|u_i(\cdot,t)\|_{L^1(\Omega)} \le \max\left\{\|u_{i0}\|_{L^1(\Omega)}, \frac{(1+\gamma_i)^2}{4}|\Omega|\right\} \quad \text{for all } t \in (0, T_{max}). \tag{3.2}$$

Proof. Integrating the first equation of (1.4) with respect to $x \in \Omega$, and using $\nabla u_1 \cdot \mathbf{n} \mid_{\partial\Omega} = \mathbf{w}_1 \cdot \mathbf{n} \mid_{\partial\Omega} = 0$, u_1, $u_2 > 0$ and the Young's inequality $s \le \frac{1}{1+\gamma_1}s^2 + \frac{1+\gamma_1}{4}$ for $s \ge 0$, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_1 &\le \gamma_1 \int_\Omega u_1 - \int_\Omega u_1^2 \\ &\le \gamma_1 \int_\Omega u_1 - \int_\Omega \left[(1+\gamma_1)\, u_1 - \frac{(1+\gamma_1)^2}{4}\right] \\ &= -\int_\Omega u_1 + \frac{(1+\gamma_1)^2}{4}|\Omega| \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.3}$$

An application of Grönwall's inequality to (3.3) yields (3.2) for $i = 1$. Similarly, (3.2) holds for $i = 2$. This completes the proof. □

Now we are in a position to derive the following estimates.

Lemma 3.3. *Suppose that the conditions in Theorem 1.1 hold. Then there exist two constants $k > 0$ and $C > 0$ independent of t such that for all $\tau \in [0, T_{max})$ and $t \in (0, T_{max} - \tau)$, it hold that*

$$\int_\Omega u_1 \log u_1 + \int_\Omega u_2 \log u_2 \le C \tag{3.4}$$

and

$$\sum_{i=1}^{2} \int_t^{t+\tau} \left(\|\nabla\sqrt{u_i}\|_{L^2(\Omega)}^2 + \|u_i\|_{L^2(\Omega)}^2 + \|\mathbf{w}_i\|_{W^{1,2}(\Omega)}^2 + \int_\Omega u_i^2 \log u_i\right) ds \le k\tau + C. \tag{3.5}$$

Proof. Multiplying the first equation in (1.4) by $(\log u_1 + 1)$ and integrating the resulting equation by parts along with the boundary conditions in (1.4), we have

$$\frac{d}{dt}\int_\Omega u_1 \log u_1 + d_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1} = \chi_1 \int_\Omega \mathbf{w}_1 \cdot \nabla u_1 + \int_\Omega u_1 E_1(u_1,u_2)(\log u_1 + 1) \tag{3.6}$$

for all $t \in (0, T_{max})$. For the first term on the right hand side of (3.6), using integration by parts subject to the boundary conditions $\mathbf{w}_1 \cdot \mathbf{n} \mid_{\partial\Omega} = 0$, one has

$$\begin{aligned} \chi_1 \int_\Omega \mathbf{w}_1 \cdot \nabla u_1 = -\chi_1 \int_\Omega u_1 \nabla \cdot \mathbf{w}_1 &\le \frac{\varepsilon_1}{4}\|\nabla \mathbf{w}_1\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\chi_1^2}{\varepsilon_1}\|u_1\|_{L^2(\Omega)}^2 \quad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Multiplying the third equation of (1.4) by $\mathbf{w}_1$, integrating the resulting equation by parts along with the boundary conditions in (1.4), and using Lemma 3.1 and Young's inequality, for all $t \in (0, T_{max})$ one has

$$\begin{aligned} \varepsilon_1 \|\nabla \mathbf{w}_1\|^2_{L^2(\Omega)} + \|\mathbf{w}_1\|^2_{L^2(\Omega)} &= \int_\Omega (u_1 + cu_2) \nabla \cdot \mathbf{w}_1 \\ &\le \frac{\varepsilon_1}{4} \|\nabla \mathbf{w}_1\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon_1} \|u_1\|^2_{L^2(\Omega)} + \frac{2c^2}{\varepsilon_1} \|u_2\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.7}$$

The combination of (3.6)–(3.7) shows that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u_1 \log u_1 + d_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1} + \frac{\varepsilon_1}{2} \|\nabla \mathbf{w}_1\|^2_{L^2(\Omega)} + \|\mathbf{w}_1\|^2_{L^2(\Omega)} \\ &\quad \le \int_\Omega u_1 E_1(u_1, u_2)(\log u_1 + 1) + \frac{(2+\chi_1^2)}{\varepsilon_1} \|u_1\|^2_{L^2(\Omega)} + \frac{2c^2}{\varepsilon_1} \|u_2\|^2_{L^2(\Omega)} \end{aligned} \tag{3.8}$$

for all $t \in (0, T_{max})$. Similarly, for u_2, it holds that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u_2 \log u_2 + d_2 \int_\Omega \frac{|\nabla u_2|^2}{u_2} + \frac{\varepsilon_2}{2} \|\nabla \mathbf{w}_2\|^2_{L^2(\Omega)} + \|\mathbf{w}_2\|^2_{L^2(\Omega)} \\ &\quad \le \int_\Omega u_2 E_2(u_1, u_2)(\log u_2 + 1) + \frac{(2+\chi_2^2)}{\varepsilon_2} \|u_2\|^2_{L^2(\Omega)} + \frac{2b^2}{\varepsilon_2} \|u_1\|^2_{L^2(\Omega)} \end{aligned} \tag{3.9}$$

for all $t \in (0, T_{max})$. Using (3.8) and (3.9), for all $t \in (0, T_{max})$ one has

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (u_1 \log u_1 + u_2 \log u_2) + \int_\Omega (u_1 \log u_1 + u_2 \log u_2) + d_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1} \\ &\quad + d_2 \int_\Omega \frac{|\nabla u_2|^2}{u_2} + \frac{\varepsilon_1}{2} \|\nabla \mathbf{w}_1\|^2_{L^2(\Omega)} + \|\mathbf{w}_1\|^2_{L^2(\Omega)} + \frac{\varepsilon_2}{2} \|\nabla \mathbf{w}_2\|^2_{L^2(\Omega)} \\ &\quad + \|\mathbf{w}_2\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)} + \frac{1}{2} \int_\Omega u_1^2 \log u_1 + \frac{1}{2} \int_\Omega u_2^2 \log u_2 \\ &\le \underbrace{\int_\Omega u_1 E_1(u_1, u_2)(\log u_1 + 1) + q_1 \|u_1\|^2_{L^2(\Omega)} + \int_\Omega u_1 \log u_1 + \frac{1}{2} \int_\Omega u_1^2 \log u_1}_{=:I_1} \\ &\quad + \underbrace{\int_\Omega u_2 E_2(u_1, u_2)(\log u_2 + 1) + q_2 \|u_2\|^2_{L^2(\Omega)} + \int_\Omega u_2 \log u_2 + \frac{1}{2} \int_\Omega u_2^2 \log u_2}_{=:I_2}, \end{aligned} \tag{3.10}$$

where $q_1 := \frac{(2+\chi_1^2)}{\varepsilon_1} + \frac{2b^2}{\varepsilon_2} + 1$ and $q_2 := \frac{(2+\chi_2^2)}{\varepsilon_2} + \frac{2c^2}{\varepsilon_1} + 1$. Clearly, the following results hold:

$$s \log s \le s^2 \quad \text{and} \quad -s(\log s + 1) \le -s \log s \le \frac{1}{e} \le 1 \quad \text{for all } s \ge 0. \tag{3.11}$$

Making use of (1.5), (3.2) and (3.11), we obtain

$$\begin{aligned} I_1 &= \int_\Omega u_1 (\gamma_1 - u_1 - cu_2)(\log u_1 + 1) + q_1 \int_\Omega u_1^2 + \int_\Omega u_1 \log u_1 + \frac{1}{2}\int_\Omega u_1^2 \log u_1, \\ &= (1+\gamma_1)\int_\Omega u_1 \log u_1 + \gamma_1 \int_\Omega u_1 - \frac{1}{2}\int_\Omega u_1^2 \log u_1 \\ &\quad - c\int_\Omega u_1 u_2 (\log u_1 + 1) + (q_1 - 1)\int_\Omega u_1^2, \\ &\leq (1+\gamma_1)\int_\Omega u_1^2 + \gamma_1 \int_\Omega u_1 - \frac{1}{2}\int_\Omega u_1^2 \log u_1 + c\int_\Omega u_2 + (q_1-1)\int_\Omega u_1^2, \\ &\leq -\frac{1}{2}\int_\Omega u_1^2 \left[\log u_1 - 2(\gamma_1 + q_1)\right] + C \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.12}$$

The continuous function $\phi(s) := s^2 [\log s - 2(\gamma_1 + q_1)]$ for $s > 0$ is bounded from below. In fact, $\phi(s) > 0$ for $s \in (s_*, \infty)$ with $s_* := e^{2(\gamma_1+q_1)} > 0$ and

$$\phi(s) \geq s^2 \log s - 2(\gamma_1 + q_1)s_*^2 \geq -\frac{1}{2e} - 2(\gamma_1 + q_1)s_*^2 \quad \text{for } s \in (0, s_*],$$

where we have used the fact that $s^2 \log s \geq -\frac{1}{2e}$ for $s \in (0, \infty)$. This along with (3.12) shows that

$$I_1 \leq C \quad \text{for all } t \in (0, T_{max}). \tag{3.13}$$

Similarly, one can deduce that

$$I_2 \leq C \quad \text{for all } t \in (0, T_{max}). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.10), one has

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u_1 \log u_1 + u_2 \log u_2) + \int_\Omega (u_1 \log u_1 + u_2 \log u_2) + d_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1} \\ &\quad + d_2 \int_\Omega \frac{|\nabla u_2|^2}{u_2} + \frac{\varepsilon_1}{2}\|\nabla \mathbf{w}_1\|_{L^2(\Omega)}^2 + \|\mathbf{w}_1\|_{L^2(\Omega)}^2 + \frac{\varepsilon_2}{2}\|\nabla \mathbf{w}_2\|_{L^2(\Omega)}^2 \\ &\quad + \|\mathbf{w}_2\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega u_1^2 \log u_1 + \frac{1}{2}\int_\Omega u_2^2 \log u_2 \\ &\leq C \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.15}$$

Finally, (3.4) is achieved by applying Grönwall's inequality to (3.15), and (3.5) is obtained by (3.4) and an integration of (3.15) with respect to t. □

With (3.5), we can obtain the following uniform-in-time estimates of $\|u_1\|_{L^q(\Omega)}$ and $\|u_2\|_{L^q(\Omega)}$ for $q \in (1, \infty)$.

Lemma 3.4. *Suppose that the conditions in Theorem 1.1 hold. For any $1 < q < \infty$, there exists a positive constant $C(q)$ independent of t such that*

$$\|u_1\|_{L^q(\Omega)}^q + \|u_2\|_{L^q(\Omega)}^q \leq C(q) \quad \textit{for all } t \in (0, T_{max}). \tag{3.16}$$

Proof. Multiplying the first equation of (1.4) by qu_1^{q-1} for $q > 1$ and integrating the resulting equation by parts subject to the boundary condition $\nabla u_1 \cdot \mathbf{n}\,|_{\partial\Omega} = \mathbf{w}_1 \cdot \mathbf{n}\,|_{\partial\Omega} = 0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\|u_1\|^q_{L^q(\Omega)} + \frac{4(q-1)d_1}{q}\|\nabla u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)} \\
&\quad = (q-1)\chi_1\int_\Omega \mathbf{w}_1\cdot\nabla u_1^q + q\int_\Omega u_1^q(\gamma_1 - u_1 - cu_2) \\
&\quad = -(q-1)\chi_1\int_\Omega u_1^q\nabla\cdot\mathbf{w}_1 + q\int_\Omega u_1^q(\gamma_1 - u_1) - cq\int_\Omega u_2u_1^q \\
&\quad =: I_3 + I_4 - cq\int_\Omega u_2u_1^q \quad \text{for all } t\in(0,T_{max}). \qquad (3.17)
\end{aligned}
$$

By Hölder's inequality we have

$$
I_3 \le (q-1)\chi_1\|u_1^q\|_{L^2(\Omega)}\|\nabla\cdot\mathbf{w}_1\|_{L^2(\Omega)} \quad \text{for all } t\in(0,T_{max}), \qquad (3.18)
$$

where it follows from the Gagliardo–Nirenberg inequality that

$$
\|u_1^q\|_{L^2(\Omega)} = \|u_1^{\frac{q}{2}}\|^2_{L^4(\Omega)} \le C(\|\nabla u_1^{\frac{q}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}\|u_1^{\frac{q}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_1^{\frac{q}{2}}\|_{L^2(\Omega)})^2. \qquad (3.19)
$$

Substituting (3.19) into (3.18), and using Young's inequality and Hölder's inequality, for all $t\in(0,T_{max})$, one has

$$
\begin{aligned}
I_3 &\le C(q)\left(\|\nabla u_1^{\frac{q}{2}}\|_{L^2(\Omega)}\|u_1^{\frac{q}{2}}\|_{L^2(\Omega)} + \|u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)}\right)\|\nabla\cdot\mathbf{w}_1\|_{L^2(\Omega)} \\
&\le \frac{2(q-1)d_1}{q}\|\nabla u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)} + C(q)\|u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)}\|\nabla\cdot\mathbf{w}_1\|^2_{L^2(\Omega)} \\
&\quad + C(q)\|u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)}\|\nabla\cdot\mathbf{w}_1\|_{L^2(\Omega)} \\
&\le \frac{2(q-1)d_1}{q}\|\nabla u_1^{\frac{q}{2}}\|^2_{L^2(\Omega)} + C(q)\|u_1\|^q_{L^q(\Omega)}\left(\|\nabla\cdot\mathbf{w}_1\|^2_{L^2(\Omega)} + 1\right). \qquad (3.20)
\end{aligned}
$$

Using Hölder's inequality: $\|u_1\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q(q+1)}}\|u_1\|_{L^{q+1}(\Omega)}$ and Young's inequality: $q\gamma_1 u_1^q \le (q-1)u_1^{q+1} + \gamma_1^{q+1}C(q)$, we have

$$
\begin{aligned}
I_4 &\le -\|u_1\|^{q+1}_{L^{q+1}(\Omega)} + \gamma_1^{q+1}C(q)|\Omega| \\
&\le -|\Omega|^{-\frac{1}{q}}\|u_1\|^{q+1}_{L^q(\Omega)} + C(q) \quad \text{for all } t\in(0,T_{max}). \qquad (3.21)
\end{aligned}
$$

Then the combination of (3.17), (3.20) and (3.21) shows that

$$
\frac{d}{dt}\|u_1\|^q_{L^q(\Omega)} - C(q)\|u_1\|^q_{L^q(\Omega)}\left(\|\nabla\cdot\mathbf{w}_1\|^2_{L^2(\Omega)} + 1\right) + |\Omega|^{-\frac{1}{q}}\|u_1\|^{q+1}_{L^q(\Omega)} \le 0 \qquad (3.22)
$$

for all $t\in(0,T_{max})$. Using (3.5), for all $\tau\in(0,T_{max})$ and $t\in(0,T_{max}-\tau)$, we have

$$
\int_t^{t+\tau}\|\nabla\cdot\mathbf{w}_1\|^2_{L^2(\Omega)} \le \int_t^{t+\tau}\|\nabla\mathbf{w}_1\|^2_{L^2(\Omega)} \le k\tau + C. \qquad (3.23)
$$

With (3.23), one can apply a nonlinear Gronwall's inequality shown in [22, Lemma 2.3] to (3.22) to obtain

$$
\sup_{t\in(0,T_{max})}\|u_1\|^q_{L^q(\Omega)} \le C(q). \qquad (3.24)
$$

Performing the same procedures as u_1, one can deduce the following estimate for u_2 as

$$\sup_{t\in(0,T_{max})} \|u_2\|^q_{L^q(\Omega)} \le C(q),$$

which along with (3.24) completes the proof. □

The following uniform-in-time estimates of $\mathbf{w}_1$ and $\mathbf{w}_2$ in $W^{1,2}(\Omega)$ can be easily obtained.

Lemma 3.5. *Suppose that the conditions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|\mathbf{w}_1\|_{W^{1,2}(\Omega)} + \|\mathbf{w}_2\|_{W^{1,2}(\Omega)} \le C \quad \text{for all } t \in (0,T_{max}). \tag{3.25}$$

Proof. In view of (3.7) and (3.16), we get $\|\mathbf{w}_1\|_{W^{1,2}(\Omega)} \le C$ for all $t \in (0,T_{max})$. The estimate for $\mathbf{w}_2$ follows similarly. □

Now we are in a position to derive the L^∞-boundedness of u_1 and u_2 by the L^p-L^q estimates of the Neumann heat semigroup.

Lemma 3.6. *Suppose that the conditions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0,T_{max}). \tag{3.26}$$

Proof. It follows from (3.16), (3.25) and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ that

$$\|u_1\mathbf{w}_1\|_{L^4(\Omega)} \le \|u_1\|_{L^{12}(\Omega)}\|\mathbf{w}_1\|_{L^6(\Omega)} \le C \quad \text{for all } t \in (0,T_{max}), \tag{3.27}$$

where Hölder's inequality was used. Now given $t \in (0,T_{max})$, we let $t_0 := (t-1)_+$. Applying the variation-of-constants formula, using u_1, $u_2 \ge 0$ and the comparison principle, for all $t \in (0,T_{max})$, one has

$$\begin{aligned} u_1(\cdot,t) \le{}& e^{(t-t_0)d_1\Delta}u_1(\cdot,t_0) - \chi_1\int_{t_0}^t e^{(t-s)d_1\Delta}\nabla\cdot(u_1(\cdot,s)\mathbf{w}_1(\cdot,s))\,ds \\ &+ \gamma_1\int_{t_0}^t e^{(t-s)d_1\Delta}u_1(\cdot,s)ds, \end{aligned}$$

which implies

$$\begin{aligned} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le{}& \|e^{(t-t_0)d_1\Delta}u_1(\cdot,t_0)\|_{L^\infty(\Omega)} \\ &+ \chi_1\int_{t_0}^t \|e^{(t-s)d_1\Delta}\nabla\cdot(u_1(\cdot,s)\mathbf{w}_1(\cdot,s))\|_{L^\infty(\Omega)}ds \\ &+ \gamma_1\int_{t_0}^t \|e^{(t-s)d_1\Delta}u_1(\cdot,s)\|_{L^\infty(\Omega)}ds \\ =:{}& I_5 + I_6 + I_7 \quad \text{for all } t \in (0,T_{max}). \end{aligned} \tag{3.28}$$

It follows from the well-known Neumann heat semigroup (cf. [32, Lemma 2.2], see also [46, formula (1.8)]) that

$$\|e^{td_1\Delta}u_1(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C}{t}\|u_1(\cdot,t)\|_{L^1(\Omega)} \quad \text{for all } t \in (0,2)\cap(0,T_{max}).$$

For $t \geq 1$, $t_0 = t-1$ and hence we have along with (3.2)

$$I_5 = \|e^{d_1\Delta}u_1(\cdot,t-1)\|_{L^\infty(\Omega)} \leq C\|u_1(\cdot,t-1)\|_{L^1(\Omega)} \\ \leq C \quad \text{for all } t \in [1,\infty) \cap (0,T_{max}). \tag{3.29}$$

Since $u_{10} \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to $p > 2$, it follows from the parabolic maximum principle that

$$I_5 = \|e^{td_1\Delta}u_{10}\|_{L^\infty(\Omega)} \leq \|u_{10}\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0,1) \cap (0,T_{max}). \tag{3.30}$$

For I_6, using (3.27) and the smoothing property of the Neumann heat semigroup [46, Lemma 1.3 (ii)], we get

$$I_6 \leq C\int_{t_0}^{t}(1+(t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}\|u_1(\cdot,s)\mathbf{w}_1(\cdot,s)\|_{L^4(\Omega)}ds \\ \leq C \quad \text{for all } t \in (0,T_{max}), \tag{3.31}$$

where λ_1 denotes the smallest positive eigenvalue of $-\Delta$ in Ω. Now it remains to estimate the term I_7. Let $\overline{u}_1(s) := \frac{1}{|\Omega|}\int_\Omega u_1(x,t)$. Then using $t - t_0 \leq 1$, the smoothing property of the Neumann heat semigroup [46, Lemma 1.3 (i)], (3.2) and Lemma 3.4 with $q = 2$, we obtain

$$\begin{aligned} I_7 &\leq \gamma_1\int_{t_0}^{t}\|e^{(t-s)d_1\Delta}(u_1(\cdot,s)-\overline{u}_1(s))\|_{L^\infty(\Omega)}ds + \gamma_1\int_{t_0}^{t}\|e^{(t-s)d_1\Delta}\overline{u}_1(s)\|_{L^\infty(\Omega)}ds \\ &\leq C\int_{t_0}^{t}(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|u_1(\cdot,s)-\overline{u}_1(s)\|_{L^2(\Omega)}ds + C\int_{t_0}^{t}\|\overline{u}_1(s)\|_{L^\infty(\Omega)} \\ &\leq C\int_{t_0}^{t}(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\left(\|u_1(\cdot,s)\|_{L^2(\Omega)} + \|\overline{u}_1(s)\|_{L^2(\Omega)}\right) + C(t-t_0) \\ &\leq C \quad \text{for all } t \in (0,T_{max}). \end{aligned} \tag{3.32}$$

Now the combination of (3.28)–(3.32) yields

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0,T_{max}).$$

Similar arguments applied to u_2 give

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0,T_{max}).$$

The proof is completed. □

We next deduce the uniform-in-time estimates for ∇u_1 and ∇u_2 in $L^2(\Omega)$.

Lemma 3.7. *Suppose that the conditions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|\nabla u_1(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u_2(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \textit{for all } t \in (0,T_{max}). \tag{3.33}$$

Proof. Multiplying the first equation of (1.4) by $-\Delta u_1$ and integrating the resulting equation by parts along with the boundary conditions in (1.4), one has

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_1\|^2_{L^2(\Omega)} = - d_1\|\Delta u_1\|^2_{L^2(\Omega)} + \chi_1\int_\Omega \Delta u_1\nabla\cdot(u_1\mathbf{w}_1) - \int_\Omega u_1E_1(u_1,u_2)\Delta u_1 \tag{3.34}$$

for all $t\in(0,T_{max})$. With the boundary condition $\nabla u_1\cdot\mathbf{n}\,|_{\partial\Omega}=0$ and (3.26), we further have

$$-\int_\Omega u_1E_1(u_1,u_2)\Delta u_1 = \int_\Omega \left(\gamma_1|\nabla u_1|^2 - 2u_1|\nabla u_1|^2 - c(u_1\nabla u_2 + u_2\nabla u_1)\nabla u_1\right) \le C_1(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}) \quad \text{for all } t\in(0,T_{max}). \tag{3.35}$$

Clearly, for all $t\in(0,T_{max})$, it holds that

$$\int_\Omega \Delta u_1\nabla\cdot(u_1\mathbf{w}_1) = \int_\Omega u_1(\nabla\cdot\mathbf{w}_1)\Delta u_1 + \int_\Omega(\mathbf{w}_1\cdot\nabla u_1)\Delta u_1 =: I_8 + I_9. \tag{3.36}$$

Making use of (3.25) and (3.26), for all $t\in(0,T_{max})$, we obtain

$$I_8 \le \frac{d_1}{4\chi_1}\|\Delta u_1\|^2_{L^2(\Omega)} + \frac{\chi_1}{d_1}\|u_1\nabla\cdot\mathbf{w}_1\|^2_{L^2(\Omega)} \le \frac{d_1}{4\chi_1}\|\Delta u_1\|^2_{L^2(\Omega)} + C_2. \tag{3.37}$$

Noticing that $\nabla|\nabla z|^2 = 2D^2z\cdot\nabla z$ for $z\in C^2(\bar\Omega)$ with $D^2z=\nabla(\nabla z)$, we arrive at

$$\begin{aligned} I_9 &= -\int_\Omega \left(\mathbf{w}_1\cdot D^2u_1 + \nabla\mathbf{w}_1\cdot\nabla u_1\right)\cdot\nabla u_1 \\ &= -\int_\Omega \mathbf{w}_1\cdot(\nabla u_1\cdot D^2u_1) - \int_\Omega(\nabla\mathbf{w}_1\cdot\nabla u_1)\cdot\nabla u_1 \\ &= -\frac{1}{2}\int_\Omega \mathbf{w}_1\cdot\nabla|\nabla u_1|^2 - \int_\Omega(\nabla\mathbf{w}_1\cdot\nabla u_1)\cdot\nabla u_1 \\ &= \frac{1}{2}\int_\Omega(\nabla\cdot\mathbf{w}_1)|\nabla u_1|^2 - \int_\Omega(\nabla\mathbf{w}_1\cdot\nabla u_1)\cdot\nabla u_1 \\ &\le \frac{3}{2}\|\nabla\mathbf{w}_1\|_{L^2(\Omega)}\|\nabla u_1\|^2_{L^4(\Omega)} \\ &\le C_3\|\nabla u_1\|^2_{L^4(\Omega)} \quad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{3.38}$$

where Hölder's inequality and Lemma 3.5 have been used. Applying Lemma 2.4 (with $q=4$, $r=n=2$ and $\theta=\frac{3}{4}$) to $\|\nabla u_1\|_{L^4(\Omega)}$ and using (3.26), (3.38) and Young's inequality, for all $t\in(0,T_{max})$, we have

$$I_9 \le C_4(\|\Delta u_1\|^{\frac{3}{2}}_{L^2}\|u_1\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) \le \frac{d_1}{4\chi_1}\|\Delta u_1\|^2_{L^2(\Omega)} + C_5. \tag{3.39}$$

The substitution of (3.37) and (3.39) into (3.36) yields

$$\chi_1\int_\Omega \Delta u_1\nabla\cdot(u_1\mathbf{w}_1) \le \frac{d_1}{2}\|\Delta u_1\|^2_{L^2(\Omega)} + (C_2+C_5)\chi_1 \quad \text{for all } t\in(0,T_{max}). \tag{3.40}$$

Using (3.26), (3.34), (3.35) and (3.40), for all $t \in (0, T_{max})$, one can see that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_1\|^2_{L^2(\Omega)} + \frac{d_1}{2}\|\Delta u_1\|^2_{L^2(\Omega)} \le C_6(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)} + 1). \tag{3.41}$$

Similar procedures applied to u_2 yield that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_2\|^2_{L^2(\Omega)} + \frac{d_2}{2}\|\Delta u_2\|^2_{L^2(\Omega)} \le C_7(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)} + 1)$$

for all $t \in (0, T_{max})$. This along with (3.41) shows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}\right) + \frac{d_1}{2}\|\Delta u_1\|^2_{L^2(\Omega)} + \frac{d_2}{2}\|\Delta u_2\|^2_{L^2(\Omega)} \\ &\quad \le C_8\left(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)} + 1\right) \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.42}$$

Applying Lemma 2.4 (with $q = r = n = 2$ and $\theta = \frac{1}{2}$) to $\|\nabla u_i\|_{L^2(\Omega)}$ for $i = 1, 2$ and using (3.26) and Young's inequality, for all $t \in (0, T_{max})$, we obtain

$$\begin{aligned} \left(C_8 + \frac{1}{2}\right)\|\nabla u_i\|^2_{L^2(\Omega)} &\le C_9(\|\Delta u_i\|^{\frac{1}{2}}_{L^2}\|u_i\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_i\|_{L^2(\Omega)})^2 \\ &\le \frac{d_i}{4}\|\Delta u_i\|^2_{L^2(\Omega)} + C_{10}. \end{aligned}$$

This together with (3.42) shows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}) + \frac{1}{2}(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}) \\ &\quad + \frac{d_1}{4}\|\Delta u_1\|^2_{L^2(\Omega)} + \frac{d_2}{4}\|\Delta u_2\|^2_{L^2(\Omega)} \\ &\quad \le C_{11} \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.43}$$

Applying Grönwall's inequality to (3.43) leads to

$$\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)} \le C_{12} \quad \text{for all } t \in (0, T_{max}),$$

which proves (3.33). □

With (3.33), we can use Lemma 2.2 to derive the uniform-in-time estimates of $\mathbf{w}_1$ and $\mathbf{w}_2$ in $H^2(\Omega)$ (Lemma 3.8), which along with the L^p-L^q-estimates of the Neumann heat semigroup enables us to further obtain the boundedness of ∇u_1 and ∇u_2 as stated in Lemma 3.9.

Lemma 3.8. *Suppose that the conditions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|\mathbf{w}_1\|_{H^2(\Omega)} + \|\mathbf{w}_2\|_{H^2(\Omega)} \le C \quad \textit{for all } t \in (0, T_{max}). \tag{3.44}$$

Proof. From Lemma 2.2 and (3.33), (3.44) follows immediately. □

Lemma 3.9. *Suppose that the conditions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|u_1(\cdot, t)\|_{W^{1,p}(\Omega)} + \|u_2(\cdot, t)\|_{W^{1,p}(\Omega)} \le C \quad \textit{for all } t \in (0, T_{max}). \tag{3.45}$$

Proof. Applying the variation-of-constants formula, one has

$$u_1(\cdot,t) = e^{td_1(\Delta-1)}u_{10} + \int_0^t e^{(t-s)d_1(\Delta-1)}\varphi(u_1,u_2,\mathbf{w}_1)(\cdot,s)ds,$$

where

$$\varphi(u_1,u_2,\mathbf{w}_1) := -\nabla\cdot(\chi_1 u_1\mathbf{w}_1) + u_1(\gamma_1 + d_1 - u_1 - cu_2).$$

It follows from (3.26), (3.33), (3.44) and the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ that

$$\begin{aligned}\|-\nabla\cdot(\chi u_1\mathbf{w}_1)\|_{L^2(\Omega)} &\le C\left(\|\nabla u_1\|_{L^2(\Omega)} + \|\nabla\mathbf{w}_1\|_{L^2(\Omega)}\right)\\ &\le C \quad \text{for all } t\in(0,T_{max}),\end{aligned}$$

which along with (3.26) shows that

$$\|\varphi(u_1,u_2,\mathbf{w}_1)\|_{L^2(\Omega)} \le C \quad \text{for all } t\in(0,T_{max}).$$

Now using the smoothing property of the Neumann heat semigroup [46, Lemma 1.3 (ii)] again, we obtain

$$\begin{aligned}\|\nabla u_1(\cdot,t)\|_{L^p(\Omega)} \le{}& C\,\|u_{10}\|_{W^{1,p}(\Omega)}\\ &+ C\int_0^t\left(1+(t-s)^{\frac{1}{p}-1}\right)e^{-\lambda_1(t-s)}\\ &\times\|\varphi(u_1,u_2,\mathbf{w}_1)(\cdot,s)\|_{L^2(\Omega)}ds\\ \le{}& C \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{3.46}$$

In a similar manner, we have

$$\|\nabla u_2(\cdot,t)\|_{W^{1,p}(\Omega)} \le C \quad \text{for all } t\in(0,T_{max}),$$

which along with (3.26) and (3.46) completes the proof. □

Proof of Theorem 1.1. $T_{max} = \infty$ is a direct consequence (2.15) and (3.45). By (3.45) and Lemma 2.2, one can obtain (1.7) directly. □

4. Global Stability

In this section, we shall investigate the asymptotic behavior of solutions to the system (1.4) and prove Theorem 1.2 by the Lyapunov functional method alongside compactness arguments. To begin with, we derive the following higher-order estimates of solutions when time t is away from 0.

Lemma 4.1. *Suppose that the conditions in Theorem 1.1 hold. Then for any $\theta\in(0,1)$, there exists a positive constant $C(\theta)$ such that*

$$\sum_{i=1}^{2}\left(\|u_i\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[1,\infty))} + \|\mathbf{w}_i\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[1,\infty))}\right) \le C(\theta). \tag{4.1}$$

Proof. It follows from (1.7) that

$$\|u_1(\cdot,t)\|_{W^{1,p}(\Omega)}+\|u_2(\cdot,t)\|_{W^{1,p}(\Omega)}\le C \quad \text{for all } t>0.$$

Taking $t_0=\frac{1}{8}$ as the initial time, then $u_1(\cdot,t_0),u_2(\cdot,t_0)\in W^{1,p}(\Omega)$. Using a similar argument as in the proof of Lemma 3.9, for any $q\in(1,\infty)$, one can find a positive constant $C(q)$ such that

$$\|u_1(\cdot,t)\|_{W^{1,q}(\Omega)}+\|u_2(\cdot,t)\|_{W^{1,q}(\Omega)}\le C(q) \quad \text{for all } t>t_0. \tag{4.2}$$

Then using Lemma 2.2 and (4.2) one has

$$\|\mathbf{w}_1(\cdot,t)\|_{W^{2,q}(\Omega)}+\|\mathbf{w}_2(\cdot,t)\|_{W^{2,q}(\Omega)}\le C(q) \quad \text{for all } t>t_0. \tag{4.3}$$

For any $\theta\in(0,1)$, using (4.3) and the Sobolev embedding $W^{2,r}(\Omega)\hookrightarrow C^{1,1-\frac{2}{r}}(\bar\Omega)\hookrightarrow C^{1,\theta}(\bar\Omega)$ for $r>\frac{2}{1-\theta}$, we can find some $r_0>\frac{2}{1-\theta}$ and $C(\theta)>0$ such that

$$\begin{aligned}&\|\mathbf{w}_1(\cdot,t)\|_{C^{1,\theta}(\bar\Omega)}+\|\mathbf{w}_2(\cdot,t)\|_{C^{1,\theta}(\bar\Omega)}\\&\quad\le C\,\|\mathbf{w}_1(\cdot,t)\|_{W^{2,r_0}(\Omega)}+\|\mathbf{w}_2(\cdot,t)\|_{W^{2,r_0}(\Omega)}\le C(\theta)\end{aligned} \tag{4.4}$$

for all $t>t_0$. From (1.4), we know that u_1 satisfies

$$\begin{cases}\partial_t u_1-d_1\Delta u_1+\chi_1\mathbf{w}_1\cdot\nabla u_1+Q_1(x,t)u_1=0, & x\in\Omega,\quad t>t_0,\\ \frac{\partial u_1}{\partial \mathbf{n}}=0, & x\in\partial\Omega,\, t>t_0,\\ u_1(x,t)\mid_{t=t_0}=u_1(x,t_0), & x\in\Omega,\end{cases} \tag{4.5}$$

where $Q_1(x,t)=\chi_1\nabla\cdot\mathbf{w}_1-E_1(u_1,u_2)$ for $(x,t)\in\Omega\times(t_0,\infty)$. Using (4.2) and (4.4), one has

$$\|\chi_1\mathbf{w}_1\|_{L^\infty\left(\Omega\times\left[j+\frac14,j+2\right]\right)}+\|Q_1\|_{L^\infty\left(\Omega\times\left[j+\frac14,j+2\right]\right)}\le C \quad \text{for all } j\ge 0. \tag{4.6}$$

In view of (4.2) and (4.6), one can apply the interior L^p estimate [28, Theorems 7.22 and 7.35] to (4.5) to obtain

$$\|u_1\|_{W^{2,1}_q\left(\Omega\times\left[j+\frac12,j+2\right]\right)}\le C(q) \quad \text{for all } j\ge 0,$$

where $W^{2,1}_q(\Omega\times[t_1,t_2]):=\left\{u\mid Du,D^2u,u_t\in L^q(\Omega\times[t_1,t_2])\right\}$ for $t_2>t_1>0$. By taking q appropriately large and using the Sobolev embedding theorem we have

$$\|u_1\|_{C^{1+\theta,\frac{1+\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}\le C(\theta) \quad \text{for all } j\ge 0. \tag{4.7}$$

Similarly, it follows that

$$\|u_2\|_{C^{1+\theta,\frac{1+\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}\le C(\theta) \quad \text{for all } j\ge 0. \tag{4.8}$$

Using (4.4), (4.7) and (4.8), we obtain that the solution $(\mathbf{w}_1,\mathbf{w}_2)$ of the elliptic system (2.16) satisfies

$$\|\mathbf{w}_1\|_{C^{1+\theta,\frac{1+\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}+\|\mathbf{w}_2\|_{C^{1+\theta,\frac{1+\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}\le C(\theta) \quad \text{for all } j\ge 0. \tag{4.9}$$

Clearly, it follows from (4.7)–(4.9) that

$$\|\chi_1\mathbf{w}_1\|_{C^{\theta,\frac{\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}+\|Q_1\|_{C^{\theta,\frac{\theta}{2}}\left(\bar\Omega\times\left[j+\frac12,j+2\right]\right)}\le C(\theta) \quad \text{for all } j\ge 0.$$

An application of the Schauder estimate to (4.5) shows that

$$\|u_1\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar\Omega\times[j+1,j+2]\right)} \le C(\theta) \quad \text{for all } j \ge 0. \tag{4.10}$$

Similarly, we have

$$\|u_2\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar\Omega\times[j+1,j+2]\right)} \le C(\theta) \quad \text{for all } j \ge 0. \tag{4.11}$$

In view of (4.10) and (4.11), we can apply Lemma 2.2 to (2.16) to obtain

$$\|\mathbf{w}_1\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar\Omega\times[j+1,j+2]\right)} + \|\mathbf{w}_2\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar\Omega\times[j+1,j+2]\right)} \le C(\theta) \quad \text{for all } j \ge 0. \tag{4.12}$$

Noting that the constant $C(\theta)$ is independent of $j \ge 0$, we get (4.1) directly from (4.10)-(4.12). □

4.1. Weak competition: $c < \frac{\gamma_1}{\gamma_2} < \frac{1}{b}$

To proceed, we define the positive constants

$$\eta_0 := \frac{K_1 - K_1^*}{2(2u_1^* + K_1)} \quad \text{and} \quad \eta := \frac{1-bc}{2c^2}(1-\eta_0)\left(\frac{K_2}{u_2^*} - f(b,c)\right) \tag{4.13}$$

under the condition (1.14).

Lemma 4.2. *Let η_0 and η be defined by (4.13). If $bc \in (0,1)$ and (1.14) holds, then the positive constants*

$$\Gamma_1 := \frac{\eta_0}{2} f(b,c) + \left(1 - \frac{\eta_0}{2}\right)\frac{K_2}{u_2^*} \quad \textit{and} \quad \Gamma_2 := \frac{1}{2}\min\left\{\Gamma_{2*} + \frac{K_1}{u_1^*}\eta, \Gamma_{2*} + \Gamma_2^*\right\}$$

satisfy

$$b^2 + \eta < \Gamma_2 < \frac{K_1}{u_1^*}\eta, \quad \psi_1(\Gamma_2) > 0 \quad \textit{and} \quad \Gamma_1 < \frac{K_2}{u_2^*}, \tag{4.14}$$

where

$$\Gamma_{2*} = \frac{2\left(\alpha_1 - \sqrt{\alpha_1^2 + \alpha_2 c^2}\right)}{c^2}, \quad \Gamma_2^* := \frac{2\left(\alpha_1 + \sqrt{\alpha_1^2 + \alpha_2 c^2}\right)}{c^2},$$

and

$$\psi_1(s) := -\frac{c^2}{4}s^2 + \alpha_1 s + \alpha_2 \quad \textit{for } s > 0,$$

with

$$\alpha_1 := \left(1 - \frac{bc}{2}\right)\Gamma_1 - \eta c^2 - 1 \quad \textit{and}$$

$$\alpha_2 := -\frac{1}{4}b^2\Gamma_1^2 - \left(b^2 + \eta\right)\Gamma_1 + b^2\left(c^2\eta + 1\right) + c^2\eta^2 + \eta.$$

Proof. This proof is straightforward and tedious, and we give the detailed proof in Appendix B. □

Now we are in a position to derive the following result.

Lemma 4.3. *Let $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ be the global classical solution of (1.4) obtained in Theorem 1.1. Assume $c < \frac{\gamma_1}{\gamma_2} < \frac{1}{b}$ and let (u_1^*, u_2^*) be defined in (1.8). If (1.13) or (1.14) holds, then there exist positive constants Γ_1 and Γ_2 such that the energy functional*

$$\mathcal{E}_1(t) := \Gamma_2 \int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*} \right) + \Gamma_1 \int_\Omega \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*} \right) \quad \text{for all } t > 0$$

satisfies

$$\mathcal{E}_1(t) \geq 0 \quad \text{for all } t \geq 0, \tag{4.15}$$

and

$$\frac{d}{dt}\mathcal{E}_1(t) \leq -\theta_1 \mathcal{F}_1(t) \quad \text{for all } t > 0 \tag{4.16}$$

for a positive constant θ_1, where

$$\mathcal{F}_1(t) := \int_\Omega (u_1 - u_1^*)^2 + \int_\Omega (u_2 - u_2^*)^2 \quad \text{for all } t > 0. \tag{4.17}$$

Proof. By the symmetry of the equations satisfied by u_1 and u_2 in (1.4), we only need to prove the conclusion under the condition (1.14). In the rest of this proof, we let the positive constants Γ_1 and Γ_2 be given by Lemma 4.2.

We first prove (4.15). Define the function $\psi(s) := s - u_1^* - u_1^* \ln \frac{s}{u_1^*}$ for $s > 0$, then $\psi'(s) = 1 - \frac{u_1^*}{s}$ with $\psi'(u_1^*) = 0$ and $\psi''(s) = \frac{u_1^*}{s^2} > 0$. Hence we have $\psi(s) \geq \psi(u_1^*) = 0$ for $s > 0$, which implies $u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*} \geq 0$. Similar arguments for u_2 yield $u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*} \geq 0$. Therefore, (4.15) is proved. It remains to prove (4.16). To this end, we multiply the first equation in (1.4) by $1 - \frac{u_1^*}{u_1}$, integrate the resulting equation by parts along with the boundary conditions in (1.4) and use $\gamma_1 - u_1^* - cu_2^* = 0$ to get

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*} \right) &= \int_\Omega \left(1 - \frac{u_1^*}{u_1} \right) (d_1 \Delta u_1 - \nabla \cdot (\chi_1 u_1 \mathbf{w}_1) + u_1 E_1(u_1, u_2)) \\
&= -d_1 u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} + \chi_1 u_1^* \int_\Omega \mathbf{w}_1 \cdot \frac{\nabla u_1}{u_1} \\
&\quad + \int_\Omega (u_1 - u_1^*)(-u_1 - cu_2 + u_1^* + cu_2^*) \\
&= -d_1 u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} + \chi_1 u_1^* \int_\Omega \mathbf{w}_1 \cdot \frac{\nabla u_1}{u_1} \\
&\quad - \int_\Omega (u_1 - u_1^*)^2 - c \int_\Omega (u_1 - u_1^*)(u_2 - u_2^*)
\end{aligned} \tag{4.18}$$

for all $t > 0$. Similarly, it holds that

$$\frac{d}{dt}\int_\Omega \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*}\right) = -d_2 u_2^* \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} + \chi_2 u_2^* \int_\Omega \mathbf{w}_2 \cdot \frac{\nabla u_2}{u_2} - \int_\Omega (u_2 - u_2^*)^2 - b\int_\Omega (u_1 - u_1^*)(u_2 - u_2^*) \tag{4.19}$$

for all $t > 0$. As deriving (3.7), by Young's inequality we can obtain

$$\begin{aligned}
&\varepsilon_1 \|\nabla \mathbf{w}_1\|_{L^2(\Omega)}^2 + \|\mathbf{w}_1\|_{L^2(\Omega)}^2 \\
&\quad = \int_\Omega \nabla \left(u_1 - u_1^* + c(u_2 - u_2^*)\right) \cdot \mathbf{w}_1 = \int_\Omega \left(u_1 - u_1^* + c(u_2 - u_2^*)\right) \nabla \cdot \mathbf{w}_1 \\
&\quad \le \varepsilon_1 \|\nabla \mathbf{w}_1\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon_1}\left(\|u_1 - u_1^*\|_{L^2(\Omega)}^2 + c^2 \|u_2 - u_2^*\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

which implies

$$\|\mathbf{w}_1\|_{L^2(\Omega)}^2 \le \frac{1}{4\varepsilon_1}\left(\|u_1 - u_1^*\|_{L^2(\Omega)}^2 + c^2 \|u_2 - u_2^*\|_{L^2(\Omega)}^2\right) \quad \text{for all } t > 0. \tag{4.20}$$

Similarly, we can obtain

$$\|\mathbf{w}_2\|_{L^2(\Omega)}^2 \le \frac{1}{4\varepsilon_2}\left(b^2\|u_1 - u_1^*\|_{L^2(\Omega)}^2 + \|u_2 - u_2^*\|_{L^2(\Omega)}^2\right) \quad \text{for all } t > 0. \tag{4.21}$$

For η given by (4.13), the combination of (4.18)–(4.21) shows that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_1(t) &= -d_1 \Gamma_2 u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} - d_2 \Gamma_1 u_2^* \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} - \Gamma_2 \int_\Omega (u_1 - u_1^*)^2 \\
&\quad - \Gamma_1 \int_\Omega (u_2 - u_2^*)^2 + \chi_1 \Gamma_2 u_1^* \int_\Omega \mathbf{w}_1 \cdot \frac{\nabla u_1}{u_1} + \chi_2 \Gamma_1 u_2^* \int_\Omega \mathbf{w}_2 \cdot \frac{\nabla u_2}{u_2} \\
&\quad - (c\Gamma_2 + b\Gamma_1) \int_\Omega (u_1 - u_1^*)(u_2 - u_2^*) \\
&\le -d_1 \Gamma_2 u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} - d_2 \Gamma_1 u_2^* \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} \\
&\quad - 4\varepsilon_2 \|\mathbf{w}_2\|_{L^2(\Omega)}^2 - 4\eta\varepsilon_1 \|\mathbf{w}_1\|_{L^2(\Omega)}^2 \\
&\quad - \left(\Gamma_2 - b^2 - \eta\right) \int_\Omega (u_1 - u_1^*)^2 - \left(\Gamma_1 - 1 - \eta c^2\right) \int_\Omega (u_2 - u_2^*)^2 \\
&\quad + \chi_1 \Gamma_2 u_1^* \int_\Omega \mathbf{w}_1 \cdot \frac{\nabla u_1}{u_1} + \chi_2 \Gamma_1 u_2^* \int_\Omega \mathbf{w}_2 \cdot \frac{\nabla u_2}{u_2} \\
&\quad - (c\Gamma_2 + b\Gamma_1) \int_\Omega (u_1 - u_1^*)(u_2 - u_2^*) \\
&\le -\int_\Omega X_1 A_1 X_1^T - \int_\Omega Y_1 B_1 Y_1^T \quad \text{for all } t > 0,
\end{aligned} \tag{4.22}$$

where $X_1 := (u_1 - u_1^*, u_2 - u_2^*)$, $Y_1 := \left(\frac{\nabla u_1}{u_1}, \frac{\nabla u_2}{u_2}, \mathbf{w}_1, \mathbf{w}_2\right)$ and A_1, B_1 are matrices denoted by

$$A_1 := \begin{pmatrix} \Gamma_2 - b^2 - \eta & \frac{c\Gamma_2 + b\Gamma_1}{2} \\ \frac{c\Gamma_2 + b\Gamma_1}{2} & \Gamma_1 - 1 - \eta c^2 \end{pmatrix},$$

$$B_1 := \begin{pmatrix} d_1\Gamma_2 u_1^* & 0 & -\frac{\chi_1\Gamma_2 u_1^*}{2} & 0 \\ 0 & d_2\Gamma_1 u_2^* & 0 & -\frac{\chi_2\Gamma_1 u_2^*}{2} \\ -\frac{\chi_1\Gamma_2 u_1^*}{2} & 0 & 4\eta\varepsilon_1 & 0 \\ 0 & -\frac{\chi_2\Gamma_1 u_2^*}{2} & 0 & 4\varepsilon_2 \end{pmatrix}.$$

We next prove that the matrices A_1 and B_1 are both positive definite. Denoting the determinant of a general square matrix X by $|X|$ and denote the upper left k-by-k $(k = 1, 2, 3)$ corner of B_1 by $B_1^{(k)}$, then by (4.14) we have $|B_1^{(1)}| = d_1\Gamma_2 u_1^* > 0$, $|B_1^{(2)}| = d_1 d_2 \Gamma_2\Gamma_1 u_1^* u_2^* > 0$,

$$\begin{aligned} |B_1^{(3)}| &= \begin{vmatrix} d_1\Gamma_2 u_1^* & 0 & -\frac{\chi_1\Gamma_2 u_1^*}{2} \\ 0 & d_2\Gamma_1 u_2^* & 0 \\ -\frac{\chi_1\Gamma_2 u_1^*}{2} & 0 & 4\eta\varepsilon_1 \end{vmatrix} \\ &= \frac{1}{4} d_2\Gamma_2\Gamma_1 \left(\chi_1 u_1^*\right)^2 u_2^* \left(\frac{K_1}{u_1^*}\eta - \Gamma_2\right) > 0, \end{aligned}$$

and

$$|B_1| = \frac{1}{16}\Gamma_2\Gamma_1 \left(\chi_1\chi_2 u_1^* u_2^*\right)^2 \left(\frac{K_1}{u_1^*}\eta - \Gamma_2\right)\left(\frac{K_2}{u_2^*} - \Gamma_1\right) > 0.$$

Sylvester's criterion thus entails that the matrix B_1 is positive definite. For the matrix A_1, we know from (4.14) that $\Gamma_2 - b^2 - \eta > 0$ and

$$|A_1| = \begin{vmatrix} \Gamma_2 - b^2 - \eta & \frac{c\Gamma_2 + b\Gamma_1}{2} \\ \frac{c\Gamma_2 + b\Gamma_1}{2} & \Gamma_1 - 1 - \eta c^2 \end{vmatrix} = \psi_1(\Gamma_2) > 0,$$

where the function ψ_1 is defined in Lemma 4.2. Again, it follows from Sylvester's criterion that the matrix A_1 is positive definite. Therefore, we can find a positive constant θ_1 such that

$$X_1 A_1 X_1^T \geq \theta_1 |X_1|^2 \quad \text{and} \quad Y_1 B_1 Y_1^T \geq \theta_1 |Y_1|^2 \quad \text{for all } t > 0. \tag{4.23}$$

The combination of (4.17), (4.22) and (4.23) proves (4.16). □

With Lemmas 2.5, 4.1 and 4.3, we can use a similar argument as in the proof of [42, Lemma 3.4] to prove the following result.

Lemma 4.4. *Under the conditions of Lemma 4.3, for any $\theta \in (0, 1)$, it holds that*

$$\|u_1 - u_1^*\|_{C^{2+\theta}(\bar\Omega)} + \|u_2 - u_2^*\|_{C^{2+\theta}(\bar\Omega)} + \|\mathbf{w}_1\|_{C^{2+\theta}(\bar\Omega)} + \|\mathbf{w}_2\|_{C^{2+\theta}(\bar\Omega)} \to 0 \text{ as } t \to \infty.$$

Proof. Let $\mathcal{E}_1(t), \mathcal{F}_1(t)$ be given in Lemma 4.3. Clearly, $\mathcal{E}_1(t)$ is bounded from below according to (4.15). Thanks to Lemma 4.1, it can be seen that $\mathcal{F}_1(t) \in$

$C^{\theta/2}([1,\infty))$ and $\|\mathcal{F}_1\|_{C^{\theta/2}([1,\infty))} \le k$ in $[1,\infty)$ for some $k>0$. In view of (4.16), we can apply Lemma 2.5 to deduce $\lim_{t\to\infty}\mathcal{F}_1(t)=0$. That is

$$\lim_{t\to\infty}\left(\|u_1-u_1^*\|_2+\|u_2-u_2^*\|_2\right)=0.$$

Take $0<\theta<\theta'<1$. According to Theorem 2.1, in the space $C^{2+\theta'}(\bar\Omega)$, $u_1(\cdot,t)$ and $u_2(\cdot,t)$ are bounded for $t\ge 1$. Using the compact arguments and the uniqueness of limits (cf. [42, (3.12)], see also [20, Remark 6.2]) we can show that

$$\|u_1-u_1^*\|_{C^{2+\theta}(\bar\Omega)}+\|u_2-u_2^*\|_{C^{2+\theta}(\bar\Omega)}\to 0 \text{ as } t\to\infty. \tag{4.24}$$

Using (4.20), (4.21) and (4.24), one has

$$\|\mathbf{w}_1\|_{W^{1,2}(\Omega)}+\|\mathbf{w}_2\|_{W^{1,2}(\Omega)}\to 0 \quad \text{as } t\to\infty.$$

This together with (4.1), the compact arguments and the uniqueness of limits shows that

$$\|\mathbf{w}_1\|_{C^{1+\theta}(\bar\Omega)}+\|\mathbf{w}_2\|_{C^{1+\theta}(\bar\Omega)}\to \infty \quad \text{as } t\to\infty,$$

which along with (4.24) completes the proof. □

We are now in a position to investigate the convergence rate.

Lemma 4.5. *Under the conditions of Lemma 4.3. There exist two constants $\sigma>0$ and $C>0$ independent of t such that*

$$\sum_{i=1}^{2}\left(\|u_i(\cdot,t)-u_i^*\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}_i\|_{W^{1,\infty}(\Omega)}\right)\le Ce^{-\sigma t} \tag{4.25}$$

holds for all $t>T_0$ with some $T_0>1$, where u_1^* and u_2^* are given by (1.8).

Proof. With Lemmas 4.3 and 4.4, one can use a similar argument as in the proof of [44, Lemma 3.7] (where the $L^\infty(\Omega)$ decay rate are obtained) to obtain that there exist positive constants σ_1 and T_0 such that

$$\|u_1(\cdot,t)-u_1^*\|_{W^{1,\infty}(\Omega)}+\|u_2(\cdot,t)-u_2^*\|_{W^{1,\infty}(\Omega)}\le C_1e^{-\sigma_1 t} \quad \text{for all } t>T_0. \tag{4.26}$$

For the convenience of readers, we shall sketch the proof of (4.26). In view of Lemma 4.4, we can apply L'Hôpital's rule to derive that

$$\lim_{u_i\to u_i^*}\frac{u_i-u_i^*-u_i^*\ln\frac{u_i}{u_i^*}}{(u_i-u_i^*)^2}=\lim_{u_i\to u_i^*}\frac{1-\frac{u_i^*}{u_i}}{2(u_i-u_i^*)}=\lim_{u_i\to u_i^*}\frac{1}{2u_i}=\frac{1}{2u_i^*},\quad i=1,2,$$

which by the continuity yields a constant $T_0>1$ such that

$$\frac{1}{4u_i^*}\int_\Omega (u_i-u_i^*)^2\le\int_\Omega\left(u_i-u_i^*-u_i^*\ln\frac{u}{u_i^*}\right)\le\frac{1}{u_i^*}\int_\Omega(u_i-u_i^*)^2,\quad i=1,2 \tag{4.27}$$

for all $t\ge T_0$. Then, it follows from the definition of $\mathcal{E}_1(t)$ and $\mathcal{F}_1(t)$ that

$$\mathcal{E}_1(t)\le C_2\mathcal{F}_1(t)\quad \text{for all } t\ge T_0,$$

which together with (4.16) implies

$$\frac{d}{dt}\mathcal{E}_1(t) + \frac{\theta_1}{C_2}\mathcal{E}_1(t) \le 0 \quad \text{for all } t \ge T_0.$$

Solving the above inequality, we obtain

$$\mathcal{E}_1(t) \le \mathcal{E}_1(T_0)e^{-\frac{\theta_1}{C_2}t} \quad \text{for all } t \ge T_0. \tag{4.28}$$

By the definition of $\mathcal{E}_1(t)$ and $\mathcal{F}_1(t)$ and (4.27), one can find a constant $C_3 > 0$ such that

$$\mathcal{F}_1(t) \le C_3\mathcal{E}_1(t) \quad \text{for all } t \ge T_0,$$

which along with (4.28) shows that

$$\|u_1(\cdot,t) - u_1^*\|_{L^2(\Omega)} + \|u_2(\cdot,t) - u_2^*\|_{L^2(\Omega)} \le C_4 e^{-\frac{\theta_1}{2C_2}t} \quad \text{for all } t \ge T_0. \tag{4.29}$$

The combination of (4.1), (4.29) and the Gagliardo–Nirenberg inequality

$$\|u_i - u_i^*\|_{W^{1,\infty}(\Omega)} \le C_5\left(\|D^2u_i\|_{L^4}^{\frac{4}{5}}\|u_i - u_i^*\|_{L^2}^{\frac{1}{5}} + \|u_i - u_i^*\|_{L^2}\right) \quad \text{for } i = 1,2 \tag{4.30}$$

yields (4.26) by choosing C_1 large enough and taking $\sigma_1 = \frac{\theta_1}{10C_2}$.

In view of (4.20) and (4.21), it follows from (4.26) immediately that

$$\|\mathbf{w}_1\|_{L^2(\Omega)} + \|\mathbf{w}_2\|_{L^2(\Omega)} \le C_6 e^{-\sigma_1 t} \quad \text{for all } t > T_0. \tag{4.31}$$

With (4.1) and (4.31), we can use a similar argument as deriving (4.30) to show that

$$\|\mathbf{w}_1\|_{W^{1,\infty}(\Omega)} + \|\mathbf{w}_2\|_{W^{1,\infty}(\Omega)} \le C_7 e^{-\sigma_2 t} \quad \text{for all } t > T_0, \tag{4.32}$$

where $\sigma_2 = \frac{\sigma_1}{5}$. Therefore, (4.25) is a direct consequence of (4.26) and (4.32) by taking C appropriately large and $\sigma = \sigma_2$. The proof is completed. □

4.2. Competitive exclusion: $\frac{\gamma_1}{\gamma_2} < \min\{\frac{1}{b}, c\}$

As the results stated for the weak competition case in the above subsection, we have the following conclusions.

Lemma 4.6. *Let $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2)$ be the global classical solution of (1.4) obtained in Theorem 1.1. Assume $\frac{\gamma_1}{\gamma_2} < \min\{\frac{1}{b}, c\}$ and (1.15) holds. Define two constants*

$$\Gamma_3 := \frac{1}{2}\left(\frac{K_2}{\gamma_2} + f\left(b, \frac{\gamma_1}{\gamma_2}\right)\right) \quad \text{and} \quad \Gamma_4 := \frac{\Gamma_3\left(2\gamma_2^2 - b\gamma_1\gamma_2\right) - 2\gamma_2^2}{\gamma_1^2}, \tag{4.33}$$

where the function f is given by (1.9). Then

$$\frac{K_2}{\gamma_2} > \Gamma_3 > f\left(b, \frac{\gamma_1}{\gamma_2}\right) > 1 \quad \text{and} \quad \Gamma_4 > b^2. \tag{4.34}$$

Moreover, the energy functional

$$\mathcal{E}_2(t) := \Gamma_4\int_\Omega u_1 + \Gamma_3\int_\Omega\left(u_2 - \gamma_2 - \gamma_2\ln\frac{u_2}{\gamma_2}\right) \quad \text{for all } t > 0 \tag{4.35}$$

satisfies $\mathcal{E}_2(t) \geq 0$ *for all* $t > 0$*, and*

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -\theta_2 \mathcal{F}_2(t) \quad \textit{for all } t > 0 \tag{4.36}$$

for a positive constant θ_2*, where*

$$\mathcal{F}_2(t) := \int_\Omega u_1^2 + \int_\Omega (u_2 - \gamma_2)^2 \quad \textit{for all } t > 0. \tag{4.37}$$

Proof. First of all, by a similar argument as proving (4.15), we can obtain $\mathcal{E}_2(t) \geq 0$ for all $t > 0$. We then prove (4.34). Using $\gamma_2 - b\gamma_1 > 0$ and (1.15), we obtain the first inequality in (4.34) directly, moreover, we have

$$\begin{aligned}
\Gamma_4 - b^2 &= \frac{1}{2\gamma_1^2(\gamma_2 - b\gamma_1)}\left[(2\gamma_2 - b\gamma_1)(\gamma_2 - b\gamma_1)K_2 + b^3\gamma_1^3 + 3b\gamma_1\gamma_2^2 - 2\gamma_2^3\right] \\
&\geq \frac{1}{2\gamma_1^2(\gamma_2 - b\gamma_1)}\left[(2\gamma_2 - b\gamma_1)\left(\gamma_2^2 + b^2\gamma_1^2\right) + b^3\gamma_1^3 + 3b\gamma_1\gamma_2^2 - 2\gamma_2^3\right] \\
&= \frac{b\gamma_2(\gamma_2 + b\gamma_1)}{\gamma_1(\gamma_2 - b\gamma_1)} > 0,
\end{aligned}$$

which proves the second inequality in (4.34).

It remains to prove (4.36). Integrating the first two equations of (1.4) by parts with $\nabla u_1 \cdot \mathbf{n}\,|_{\partial\Omega} = \mathbf{w}_1 \cdot \mathbf{n}\,|_{\partial\Omega} = 0$ and using $\frac{\gamma_1}{\gamma_2} < c$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_1 &= \int_\Omega u_1(\gamma_1 - u_1 - cu_2) \\
&\leq \int_\Omega u_1\left(\gamma_1 - u_1 - \frac{\gamma_1}{\gamma_2}u_2\right) \\
&= -\int_\Omega u_1^2 - \frac{\gamma_1}{\gamma_2}\int_\Omega u_1(u_2 - \gamma_2) \quad \text{for all } t > 0.
\end{aligned} \tag{4.38}$$

As deriving (4.19), for all $t > 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(u_2 - \gamma_2 - \gamma_2 \ln\frac{u_2}{\gamma_2}\right) \\
&\quad \leq -d_2\gamma_2\int_\Omega \frac{|\nabla u_2|^2}{u_2^2} + \chi_2\gamma_2\int_\Omega \mathbf{w}_2 \cdot \frac{\nabla u_2}{u_2} - \int_\Omega (u_2 - \gamma_2)^2 - b\int_\Omega u_1(u_2 - \gamma_2).
\end{aligned} \tag{4.39}$$

It follows from (4.21), (4.35), (4.38), (4.39) and Young's inequality that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) \leq &-\left(\Gamma_4 - b^2\right)\int_\Omega u_1^2 - (\Gamma_3 - 1)\int_\Omega (u_2 - \gamma_2)^2 - \left(\frac{\gamma_1}{\gamma_2}\Gamma_4 + b\Gamma_3\right) \\
&\times \int_\Omega u_1(u_2 - \gamma_2) - d_2\gamma_2\Gamma_3\int_\Omega \frac{|\nabla u_2|^2}{u_2^2} \\
&+ \chi_2\gamma_2\Gamma_3\int_\Omega \mathbf{w}_2 \cdot \frac{\nabla u_2}{u_2} - 4\varepsilon_2\|\mathbf{w}_2\|_{L^2(\Omega)}^2 \\
\leq &-\int_\Omega X_2A_2X_2^T - \int_\Omega Y_2B_2Y_2^T \quad \text{for all } t > 0,
\end{aligned} \tag{4.40}$$

where $X_2 = (u_1, u_2 - \gamma_2)$, $Y_2 = \left(\frac{\nabla u_2}{u_2}, \mathbf{w}_2\right)$ and A_2, B_2 are matrices denoted by

$$A_2 := \begin{pmatrix} \Gamma_4 - b^2 & \frac{\frac{\gamma_1}{\gamma_2}\Gamma_4 + b\Gamma_3}{2} \\ \frac{\frac{\gamma_1}{\gamma_2}\Gamma_4 + b\Gamma_3}{2} & \Gamma_3 - 1 \end{pmatrix}, \quad B_2 := \begin{pmatrix} d_2\gamma_2\Gamma_3 & -\frac{\chi_2\gamma_2\Gamma_3}{2} \\ -\frac{\chi_2\gamma_2\Gamma_3}{2} & 4\varepsilon_2 \end{pmatrix}.$$

Using $\gamma_2 - b\gamma_1 > 0$, (1.15), (4.33) and (4.34), we obtain $\Gamma_4 - b^2 > 0$ and

$$\begin{aligned} |A_2| &= \frac{\left[(\gamma_2 - b\gamma_1)K_2 + b^2\gamma_1^2 + 2b\gamma_1\gamma_2 - \gamma_2^2\right]\left[(\gamma_2 - b\gamma_1)K_2 - \left(\gamma_2^2 + b^2\gamma_1^2\right)\right]}{4\gamma_1^2\gamma_2(\gamma_2 - b\gamma_1)} \\ &= \frac{\left((\gamma_2 - b\gamma_1)K_2 + b^2\gamma_1^2 + 2b\gamma_1\gamma_2 - \gamma_2^2\right)}{4\gamma_1^2}\left(\frac{K_2}{\gamma_2} - f\left(b, \frac{\gamma_1}{\gamma_2}\right)\right) \\ &\geq \frac{b(b\gamma_1 + \gamma_2)}{2\gamma_1}\left(\frac{K_2}{\gamma_2} - f\left(b, \frac{\gamma_1}{\gamma_2}\right)\right) > 0. \end{aligned} \tag{4.41}$$

Moreover, it is obvious that $d_2\gamma_2\Gamma_3 > 0$, and by (4.34) we have

$$|B_2| = \frac{\Gamma_3\left(\gamma_2\chi_2\right)^2}{4}\left(\frac{K_2}{\gamma_2} - \Gamma_3\right) > 0. \tag{4.42}$$

In view of Sylvester's criterion, it follows from (4.41) and (4.42) that the matrices A_2 and B_2 are positive definite, and hence we can find a positive constant θ_2 such that

$$X_2A_2X_2^T \geq \theta_2|X_2|^2 \quad \text{and} \quad Y_2B_2Y_2^T \geq \theta_2|Y_2|^2 \quad \text{for all } t > 0,$$

which along with $X_2 = (u_1, u_2 - \gamma_2)$, (4.37) and (4.40) proves (4.36). The proof is completed. □

Lemma 4.7. *Under the conditions of Lemma 4.6, there exists a constant $T_1 > 0$ such that*

$$\|u_1(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_2(\cdot,t) - \gamma_2\|_{W^{1,\infty}(\Omega)} + \sum_{i=1}^{2}\|\mathbf{w}_i\|_{W^{1,\infty}(\Omega)} \leq \frac{C}{1+t}$$

for all $t > T_1$, where C is a positive constant independent of t.

Proof. First, by a similar argument as in the proof of Lemma 4.4, one can obtain

$$\|u_1\|_{C^{2+\theta}(\bar{\Omega})} + \|u_2 - \gamma_2\|_{C^{2+\theta}(\bar{\Omega})} + \|\mathbf{w}_1\|_{C^{1+\theta}(\bar{\Omega})} + \|\mathbf{w}_2\|_{C^{1+\theta}(\bar{\Omega})} \to 0 \text{ as } t \to \infty.$$

Recalling the definitions of $\mathcal{E}_2(t)$ and $\mathcal{F}_2(t)$, and using (4.27) and Hölder's inequality, we can find some $T_1 > 0$ such that

$$\begin{aligned} \mathcal{E}_2(t) &\leq C_1\left(\int_\Omega u_1 + \int_\Omega (u_2 - \gamma_2)^2\right) \\ &\leq C_2\left\{\left(\int_\Omega u_1^2\right)^{\frac{1}{2}} + \left(\int_\Omega (u_2 - \gamma_2)^2\right)^{\frac{1}{2}}\right\} \\ &\leq C_3\mathcal{F}_2^{\frac{1}{2}}(t) \quad \text{for all } t > T_1, \end{aligned}$$

which together with (4.36) gives that

$$\frac{d}{dt}\mathcal{E}_2(t) + \frac{\theta_2}{C_3^2}\mathcal{E}_2^2(t) \le 0 \quad \text{for all } t > T_1.$$

Solving the above ordinary differential inequality, we arrive at

$$\mathcal{E}_2(t) \le \frac{C_4}{1+t} \quad \text{for all } t > T_1.$$

The rest of the proof can follow similar arguments as in the proof of Lemma 4.5 and we omit it for brevity. □

Proof of Theorem 1.2.. The assertions in (i) and (ii) of Theorem 1.2 result from Lemma 4.5 and Lemma 4.7, respectively. The assertions in (iii) are essentially similar to those in (ii) and can be proved by simply swapping u_1, b, γ_1 with u_2, c, γ_2, respectively, in the proof of (ii). □

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Appendix A

In this section, we consider the regularity of the vector field $\mathbf{w}$ satisfying:

$$\begin{cases} -\Delta \mathbf{w} + \mathbf{w} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} = 0, \ \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{A1}$$

where $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded domain with a smooth boundary and $\mathbf{f} \in (L^p(\Omega))^n$ for some $1 < p < \infty$. Then the following regularity result holds.

Lemma A1. *If $\mathbf{f} \in \left(L^2(\Omega)\right)^n$, then the problem (A1) has a unique solution $\mathbf{w} \in (H^2(\Omega))^n$ and there is a positive constant $C(\Omega)$ depending only on Ω such that*

$$\|\mathbf{w}\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}.$$

Proof. The existence and uniqueness of the solution $\mathbf{w} \in (H^2(\Omega))^n$ to (A1) is stated in Lemma 2.2. Next we prove the regularity. We only give the details for $n = 3$, and the proof for $n = 2$ is similar and simpler. We divide the proof into two steps.

Step 1. We prove the following estimate for a positive constant C,

$$\|\mathbf{w}\|_{H^1(\Omega)} \le C\|\mathbf{f}\|_{L^2(\Omega)}. \tag{A2}$$

Multiplying the i-th component ($i = 1, 2, 3$) of the first equation in (A1) by w_i and integrating the resulting equation by parts, we get

$$\int_\Omega |\nabla w_i|^2 dx + \int_\Omega |w_i|^2 dx - \int_{\partial\Omega} w_i \cdot \partial_{\mathbf{n}} w_i dx = \int_\Omega f_i w_i dx,$$

which gives

$$\int_\Omega |\nabla \mathbf{w}|^2 dx + \int_\Omega |\mathbf{w}|^2 dx - \int_{\partial\Omega} \mathbf{w} \cdot \partial_{\mathbf{n}} \mathbf{w} dx = \int_\Omega \mathbf{w} \cdot \mathbf{f} dx.$$

Noticing the boundary condition implies

$$\mathbf{w} \cdot \partial_{\mathbf{n}} \mathbf{w} = 0 \quad \text{on } \partial\Omega,$$

we get (A2) by the Cauchy-Schwarz inequality: $\int_\Omega \mathbf{w} \cdot \mathbf{f} dx \le \frac{1}{2}\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(\Omega)}$.

Step 2. Suppose $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ constitutes the Frenet coordinate associated with the boundary $\partial\Omega$. By assuming the domain is sufficiently smooth, we can extend $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ to a C^∞ function defined in Ω such that their C^2 norms are uniformly bounded and any two of them are orthogonal to each other, still denoted by $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$.

Now we shall estimate the H^2-norm of $\mathbf{w}\cdot\mathbf{n}$, $\mathbf{w}\cdot\boldsymbol{\tau}_1$ and $\mathbf{w}\cdot\boldsymbol{\tau}_2$ in the following. We start with $\mathbf{w}\cdot\mathbf{n}$ which satisfies

$$\Delta(\mathbf{w}\cdot\mathbf{n}) = \Delta\mathbf{w}\cdot\mathbf{n} + \mathbf{w}\cdot\Delta\mathbf{n} + 2\sum_{i=1}^{3}\nabla w_i\cdot\nabla n_i.$$

Therefore, we have from (A1) that

$$\begin{cases} -\Delta(\mathbf{w}\cdot\mathbf{n}) = (\mathbf{f}-\mathbf{w})\cdot\mathbf{n} - \mathbf{w}\cdot\Delta\mathbf{n} - 2\sum\limits_{i=1}^{3}\nabla w_i\cdot\nabla n_i & \text{in } \Omega,\\ \mathbf{w}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A3}$$

We denote the right hand side of the first equation in (A3) by $\tilde{\mathbf{f}}_1$. Using the classical elliptic regularity estimate, along with the boundary condition and the conclusion in Step 1, we get

$$\|\mathbf{w}\cdot\mathbf{n}\|_{H^2(\Omega)} \le C(\Omega)\|\tilde{\mathbf{f}}_1\|_{L^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}. \tag{A4}$$

Concerning $\mathbf{w}\cdot\boldsymbol{\tau}_1$, we have

$$\begin{cases} -\Delta(\mathbf{w}\cdot\boldsymbol{\tau}_1) = (\mathbf{f}-\mathbf{w})\cdot\boldsymbol{\tau}_1 - \mathbf{w}\cdot\Delta\boldsymbol{\tau}_1 - 2\sum\limits_{i=1}^{3}\nabla w_i\cdot\nabla\boldsymbol{\tau}_{1i} & \text{in } \Omega,\\ \partial_{\mathbf{n}}(\mathbf{w}\cdot\boldsymbol{\tau}_1) = \mathbf{w}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1 & \text{on } \partial\Omega, \end{cases} \tag{A5}$$

where the boundary condition holds since

$$\partial_{\mathbf{n}}(\mathbf{w}\cdot\boldsymbol{\tau}_1) = \partial_{\mathbf{n}}\mathbf{w}\cdot\boldsymbol{\tau}_1 + \mathbf{w}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1 = \mathbf{w}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1.$$

By the trace theorem and the conclusion in Step 1, we have

$$\|\partial_{\mathbf{n}}(\mathbf{w}\cdot\boldsymbol{\tau}_1)\|_{H^{\frac12}(\partial\Omega)} = \|\mathbf{w}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1\|_{H^{\frac12}(\partial\Omega)} \le C(\Omega)\|\mathbf{w}\|_{H^1(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}.$$

On the other hand, by denoting the right hand side in the first equation of (A5) by $\tilde{\mathbf{f}}_2$, it is easy to see that $\|\tilde{\mathbf{f}}_2\|_{L^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}$. By the results of [17, Section 2.3], we have

$$\|\mathbf{w}\cdot\boldsymbol{\tau}_1\|_{H^2(\Omega)} \le C(\Omega)\left(\|\tilde{\mathbf{f}}_2\|_{L^2(\Omega)} + \|\partial_{\mathbf{n}}(\mathbf{w}\cdot\boldsymbol{\tau}_1)\|_{H^{\frac12}(\partial\Omega)}\right) \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}. \tag{A6}$$

Similarly, we have

$$\|\mathbf{w}\cdot\boldsymbol{\tau}_2\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}. \tag{A7}$$

Collecting (A4), (A6) and (A7), we arrive at

$$\|\mathbf{w}\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{L^2(\Omega)}.$$

Hence the proof is completed. □

Lemma A2. *If* $\mathbf{f} \in \left(H^2(\Omega)\right)^n$, *then the system* (A1) *has a unique solution* $\mathbf{w} \in (H^4(\Omega))^n$ *and there is a positive constant* $C(\Omega)$ *depending only on* Ω *such that*

$$\|\mathbf{w}\|_{H^3(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^1(\Omega)} \quad \textit{and} \quad \|\mathbf{w}\|_{H^4(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)}.$$

Proof. We consider the regularity of solutions to (A1) for $n = 3$ only, and the case for $n = 2$ can be proved similarly. Defining the Frenet coordinate system $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ and taking differentiation of (A1) with respect to x_k, $k \in \{1, 2, 3\}$, we have

$$\begin{cases} -\Delta \mathbf{w}_{x_k} + \mathbf{w}_{x_k} = \mathbf{f}_{x_k} & \text{in } \Omega, \\ \mathbf{w}_{x_k} \cdot \mathbf{n} = -\mathbf{w} \cdot \mathbf{n}_{x_k} & \text{on } \partial\Omega, \\ \partial_{\mathbf{n}} \mathbf{w}_{x_k} \times \mathbf{n} + \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n}_{x_k} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A8}$$

Next, we estimate the functions $\mathbf{w}_{x_k} \cdot \mathbf{n}$, $\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1$ and $\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_2$. By direct computations, we get

$$\begin{cases} -\Delta(\mathbf{w}_{x_k} \cdot \mathbf{n}) = (\mathbf{f}_{x_k} - \mathbf{w}_{x_k}) \cdot \mathbf{n} - \mathbf{w}_{x_k} \cdot \Delta \mathbf{n} - 2 \sum_{i=1}^{3} \nabla(\partial_{x_k} w_i) \cdot \nabla n_i & \text{in } \Omega, \\ \mathbf{w}_{x_k} \cdot \mathbf{n} = -\mathbf{w} \cdot \mathbf{n}_{x_k} & \text{on } \partial\Omega. \end{cases} \tag{A9}$$

Denote the right hand side of (A9) by $\tilde{\mathbf{f}}_3$. Using Lemma A1, we see that

$$\|\tilde{\mathbf{f}}_3\|_{L^2(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{H^1(\Omega)}.$$

By the classical elliptic regularity, we have

$$\|\mathbf{w}_{x_k} \cdot \mathbf{n}\|_{H^2(\Omega)} \leq C(\Omega) \left(\|\tilde{\mathbf{f}}_3\|_{L^2(\Omega)} + \|\mathbf{w} \cdot \mathbf{n}_{x_k}\|_{H^2(\Omega)} \right) \leq C(\Omega) \|\mathbf{f}\|_{H^1(\Omega)}. \tag{A10}$$

Concerning $\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1$, we have

$$\begin{cases} -\Delta(\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1) = (\mathbf{f}_{x_k} - \mathbf{w}_{x_k}) \cdot \boldsymbol{\tau}_1 - \mathbf{w}_{x_k} \cdot \Delta \boldsymbol{\tau}_1 \\ \qquad\qquad -2 \sum_{i=1}^{3} \nabla(\partial_{x_k} w_i) \cdot \nabla \boldsymbol{\tau}_{1i} & \text{in } \Omega, \\ \partial_{\mathbf{n}}(\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1) = \partial_{\mathbf{n}} \mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1 + \mathbf{w}_{x_k} \cdot \partial_{\mathbf{n}} \boldsymbol{\tau}_1 & \text{on } \partial\Omega. \end{cases} \tag{A11}$$

Denoting the right hand side in the first equation of (A11) by $\tilde{\mathbf{f}}_4$, we can use Lemma A1 to get that

$$\|\tilde{\mathbf{f}}_4\|_{L^2(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{H^1(\Omega)}. \tag{A12}$$

From the last two equations in (A8), we notice that on $\partial\Omega$,

$$\partial_{\mathbf{n}} \mathbf{w}_{x_k} \times \mathbf{n} = -\partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n}_{x_k}. \tag{A13}$$

From (A13) we get

$$\partial_{\mathbf{n}} \mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1 = \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n}_{x_k} \times \mathbf{n} \cdot \boldsymbol{\tau}_1. \tag{A14}$$

As a consequence of (A14), we have

$$\partial_{\mathbf{n}}(\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1) = \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n}_{x_k} \times \mathbf{n} \cdot \boldsymbol{\tau}_1 + \mathbf{w}_{x_k} \cdot \partial_{\mathbf{n}} \boldsymbol{\tau}_1.$$

Then it is easy to see that

$$\begin{aligned} \|\partial_{\mathbf{n}}(\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1)\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C(\Omega) \|\partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n}_{x_k} \times \mathbf{n} \cdot \boldsymbol{\tau}_1 + \mathbf{w}_{x_k} \cdot \partial_{\mathbf{n}} \boldsymbol{\tau}_1\|_{H^1(\Omega)} \\ &\leq C(\Omega) \|\mathbf{w}\|_{H^2(\Omega)}. \end{aligned} \tag{A15}$$

With (A12) and (A15), the results of [17, Section 2.3] entail that

$$\|\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1\|_{H^2(\Omega)} \leq C(\Omega) \left(\|\tilde{\mathbf{f}}_4\|_{L^2(\Omega)} + \|\partial_{\mathbf{n}}(\mathbf{w}_{x_k} \cdot \boldsymbol{\tau}_1)\|_{H^{\frac{1}{2}}(\partial\Omega)} \right) \leq C(\Omega) \|\mathbf{f}\|_{H^1(\Omega)}.$$

Similarly, we have

$$\|\mathbf{w}_{x_k}\cdot\boldsymbol{\tau}_2\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^1(\Omega)}.$$

Combining the above two estimates with (A10), we get

$$\|\mathbf{w}\|_{H^3(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^1(\Omega)}. \tag{A16}$$

Now we derive the higher regularity based on the assumption that $\mathbf{f}\in H^2(\Omega)$. Differentiating (A1) with respect to x_j and x_k, $j,k\in\{1,2,3\}$, we have

$$\begin{cases} -\Delta\mathbf{w}_{x_jx_k}+\mathbf{w}_{x_jx_k}=\mathbf{f}_{x_jx_k} & \text{in } \Omega,\\ \mathbf{w}_{x_jx_k}\cdot\mathbf{n}=-\mathbf{w}\cdot\mathbf{n}_{x_jx_k}-\mathbf{w}_{x_j}\cdot\mathbf{n}_{x_k}-\mathbf{w}_{x_k}\cdot\mathbf{n}_{x_j} & \text{on } \partial\Omega,\\ \partial_{\mathbf{n}}\mathbf{w}_{x_jx_k}\times\mathbf{n}+\partial_{\mathbf{n}}\mathbf{w}_{x_j}\times\mathbf{n}_{x_k}+\partial_{\mathbf{n}}\mathbf{w}_{x_k}\times\mathbf{n}_{x_j}+\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}_{x_jx_k}=0 & \text{on } \partial\Omega. \end{cases} \tag{A17}$$

Next, we estimates the terms $\mathbf{w}_{x_jx_k}\cdot\mathbf{n}$, $\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1$ and $\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_2$. Direct computations give us that

$$\begin{cases} -\Delta(\mathbf{w}_{x_jx_k}\cdot\mathbf{n})=(\mathbf{f}_{x_jx_k}-\mathbf{w}_{x_jx_k})\cdot\mathbf{n}-\mathbf{w}_{x_jx_k}\cdot\Delta\mathbf{n} & \\ \qquad\qquad -2\sum_{i=1}^{3}\nabla(\partial_{x_jx_k}w_i)\cdot\nabla n_i & \text{in } \Omega,\\ \mathbf{w}_{x_jx_k}\cdot\mathbf{n}=-\mathbf{w}\cdot\mathbf{n}_{x_jx_k}-\mathbf{w}_{x_j}\cdot\mathbf{n}_{x_k}-\mathbf{w}_{x_k}\cdot\mathbf{n}_{x_j} & \text{on } \partial\Omega. \end{cases} \tag{A18}$$

Denote the right hand side of the first and second equations of (A18) by $\tilde{\mathbf{f}}_5$ and $\tilde{\mathbf{f}}_6$, respectively. Using (A16), we see that

$$\|\tilde{\mathbf{f}}_5\|_{L^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)},$$

and

$$\|\tilde{\mathbf{f}}_6\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{w}\|_{H^3(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^1(\Omega)}.$$

By the classical elliptic regularity, we have

$$\|\mathbf{w}_{x_jx_k}\cdot\mathbf{n}\|_{H^2(\Omega)} \le C(\Omega)\left(\|\tilde{\mathbf{f}}_5\|_{L^2(\Omega)}+\|\tilde{\mathbf{f}}_6\|_{H^2(\Omega)}\right) \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)}. \tag{A19}$$

Concerning $\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1$, we find it satisfies

$$\begin{cases} -\Delta(\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1)=(\mathbf{f}_{x_jx_k}-\mathbf{w}_{x_jx_k})\cdot\boldsymbol{\tau}_1-\mathbf{w}_{x_jx_k}\cdot\Delta\boldsymbol{\tau}_1 & \\ \qquad\qquad -2\sum_{i=1}^{3}\nabla(\partial_{x_jx_k}w_i)\cdot\nabla\boldsymbol{\tau}_{1i} & \text{in } \Omega,\\ \partial_{\mathbf{n}}(\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1)=\partial_{\mathbf{n}}\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1+\mathbf{w}_{x_jx_k}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1 & \text{on } \partial\Omega. \end{cases}$$

Denoting the right hand side in the first equation of (A11) by $\tilde{\mathbf{f}}_7$ and using (A15), we get

$$\|\tilde{\mathbf{f}}_7\|_{L^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)}.$$

From the last equation in (A17), we have

$$\partial_{\mathbf{n}}\mathbf{w}_{x_jx_k}\times\mathbf{n}=-\partial_{\mathbf{n}}\mathbf{w}_{x_j}\times\mathbf{n}_{x_k}-\partial_{\mathbf{n}}\mathbf{w}_{x_k}\times\mathbf{n}_{x_j}-\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}_{x_jx_k} \quad \text{on } \partial\Omega,$$

which yields

$$\partial_{\mathbf{n}}\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1=(\partial_{\mathbf{n}}\mathbf{w}_{x_j}\times\mathbf{n}_{x_k}+\partial_{\mathbf{n}}\mathbf{w}_{x_k}\times\mathbf{n}_{x_j}+\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}_{x_jx_k})\times\mathbf{n}\cdot\boldsymbol{\tau}_1 \quad \text{on } \partial\Omega. \tag{A20}$$

As a consequence of (A20), we have

$$\partial_{\mathbf{n}}(\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1) = (\partial_{\mathbf{n}}\mathbf{w}_{x_j}\times\mathbf{n}_{x_k}+\partial_{\mathbf{n}}\mathbf{w}_{x_k}\times\mathbf{n}_{x_j}+\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}_{x_jx_k})\times\mathbf{n}\cdot\boldsymbol{\tau}_1 + \mathbf{w}_{x_jx_k}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1 \quad \text{on } \partial\Omega. \tag{A21}$$

Then it is easy to see that

$$\begin{aligned}\|\partial_{\mathbf{n}}(\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1)\|_{H^{\frac12}(\partial\Omega)} &\le C(\Omega)\|(\partial_{\mathbf{n}}\mathbf{w}_{x_j}\times\mathbf{n}_{x_k}+\partial_{\mathbf{n}}\mathbf{w}_{x_k}\times\mathbf{n}_{x_j}+\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}_{x_jx_k})\times\mathbf{n}\cdot\boldsymbol{\tau}_1\|_{H^1(\Omega)}\\ &\quad + C(\Omega)\|\mathbf{w}_{x_jx_k}\cdot\partial_{\mathbf{n}}\boldsymbol{\tau}_1\|_{H^1(\Omega)}\\ &\le C(\Omega)\|\mathbf{w}\|_{H^3(\Omega)}.\end{aligned} \tag{A22}$$

With (A21) and (A22), we use the results of [17, Section 2.3] again and get that

$$\begin{aligned}\|\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1\|_{H^2(\Omega)} &\le C(\Omega)\left(\|\tilde{\mathbf{f}}_7\|_{L^2(\Omega)}+\|\partial_{\mathbf{n}}(\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_1)\|_{H^{\frac12}(\partial\Omega)}\right)\\ &\le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)}.\end{aligned}$$

Similar procedures give

$$\|\mathbf{w}_{x_jx_k}\cdot\boldsymbol{\tau}_2\|_{H^2(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)}.$$

Combining the above two estimates with (A19), we get

$$\|\mathbf{w}\|_{H^4(\Omega)} \le C(\Omega)\|\mathbf{f}\|_{H^2(\Omega)},$$

which along with (A16) completes the proof. □

Appendix B

In this section, we give the proof of Lemma 4.2.

Proof of Lemma 4.2. It follows from $\eta > 0$,

$$f(b,c)=\frac{1+b^2c^2}{1-bc}>\frac{2+b^2c^2}{2-bc}>1 \quad \text{and} \quad \frac{2c^2}{1-bc}>\frac{3c^2}{2-bc}>c^2$$

that $\beta := \eta_0 f(b,c)+(1-\eta_0)\frac{K_2}{u_2^*}<\frac{K_2}{u_2^*}$ satisfies

$$\beta = f(b,c)+\frac{2c^2}{1-bc}\eta > \frac{2+b^2c^2}{2-bc}+\frac{3c^2}{2-bc}\eta > 1+c^2\eta. \tag{B1}$$

Moreover, it holds that

$$\Gamma_1 = \frac{\eta_0}{2}f(b,c)+\left(1-\frac{\eta_0}{2}\right)\frac{K_2}{u_2^*}=\frac12\left(\beta+\frac{K_2}{u_2^*}\right)\in\left(\beta,\frac{K_2}{u_2^*}\right). \tag{B2}$$

Thus the inequality satisfied by Γ_1 in (4.14) is proved. We next prove the inequalities satisfied by Γ_2 in (4.14). By (B1) and (B2), we have

$$\begin{aligned}\alpha_1^2+\alpha_2c^2 &= \left(\Gamma_1-\eta c^2-1\right)\left(\Gamma_1(1-bc)-\left(b^2c^2+2c^2\eta+1\right)\right)\\ &= \left(\Gamma_1-\eta c^2-1\right)(1-bc)(\Gamma_1-\beta)\\ &> \left(\beta-\eta c^2-1\right)(1-bc)(\Gamma_1-\beta)\\ &> 0.\end{aligned} \tag{B3}$$

Using (1.11), (1.14) and (4.13), we obtain

$$
\begin{aligned}
&\alpha_1 - \frac{c^2 K_1}{2u_1^*}\eta \\
&\quad = \left(1-\frac{bc}{2}\right)\Gamma_1 - \eta c^2\left(1+\frac{K_1}{2u_1^*}\right) - 1 \\
&\quad < \left(1-\frac{bc}{2}\right)\frac{K_2}{u_2^*} - \frac{1-bc}{2}(1-\eta_0)\left(\frac{K_2}{u_2^*} - f(b,c)\right)\left(1+\frac{K_1}{2u_1^*}\right) - 1 \\
&\quad = \frac{1-bc}{4u_1^*u_2^*}\left\{\frac{2(2-bc)}{1-bc}K_2u_1^* - (1-\eta_0)\left(K_2 - f(b,c)u_2^*\right)\left(2u_1^*+K_1\right) - \frac{4u_1^*u_2^*}{1-bc}\right\} \\
&\quad = \frac{1-bc}{4u_1^*u_2^*}\left\{2u_1^*\left(\frac{K_2}{1-bc} - (1+bc)u_2^*\right) - \left(K_2 - f(b,c)u_2^*\right)K_1\right\} \\
&\qquad + \frac{(1-bc)\eta_0}{4u_1^*u_2^*}\left\{2u_1^*K_2 - 2u_1^*u_2^* f(b,c) + \left(K_2 - f(b,c)u_2^*\right)K_1\right\} \\
&\quad = -\frac{1-bc}{4u_1^*u_2^*}\left(K_2 - f(b,c)u_2^*\right)\left((K_1-K_1^*) - \eta_0\left(2u_1^*+K_1\right)\right) \\
&\quad = -\frac{1-bc}{8u_1^*u_2^*}\left(K_2 - f(b,c)u_2^*\right)\left(K_1-K_1^*\right) < 0.
\end{aligned} \tag{B4}
$$

Clearly,

$$
\Gamma_{2*} < \Gamma_2^*, \tag{B5}
$$

and by (B3) and (B4) we get

$$
\begin{aligned}
\frac{K_1}{u_1^*}\eta - \Gamma_{2*} &= \frac{K_1}{u_1^*}\eta - \frac{2\left(\alpha_1 - \sqrt{\alpha_1^2+\alpha_2c^2}\right)}{c^2} \\
&= \frac{2}{c^2}\left(\sqrt{\alpha_1^2+\alpha_2c^2} - \left(\alpha_1 - \frac{c^2K_1}{2u_1^*}\eta\right)\right) \\
&\ge \frac{2}{c^2}\sqrt{\alpha_1^2+\alpha_2c^2} \\
&> 0.
\end{aligned} \tag{B6}
$$

We deduce from (B5) and (B6) that

$$
\Gamma_{2*} < \Gamma_2 < \min\left\{\frac{K_1}{u_1^*}\eta, \Gamma_2^*\right\}. \tag{B7}
$$

Since Γ_{2*} and Γ_2^* are two zeros of $\psi_1(s) = -\frac{c^2}{4}s^2 + \alpha_1 s + \alpha_2$ for $s>0$, by (B7) we have $\psi_1(\Gamma_2) > 0$. It remains to prove $\Gamma_2 > b^2 + \eta$. Indeed, we have

$$\Gamma_{2*} - (b^2+\eta) = \frac{2\left(\alpha_1 - \sqrt{\alpha_1^2+\alpha_2 c^2}\right)}{c^2} - (b^2+\eta)$$
$$= \frac{1}{c^2}\left(\underbrace{(2-bc)\Gamma_1 - \left(2+b^2c^2+3c^2\eta\right)}_{=:J_1} - \underbrace{2\sqrt{\alpha_1^2+\alpha_2 c^2}}_{=:J_2}\right). \tag{B8}$$

By (B1) and (B2) we know that

$$J_1 > (2-bc)\beta - \left(2+b^2c^2+3c^2\eta\right) > 0 \quad \text{and}$$
$$J_1^2 - J_2^2 = c^2\left(b^2c + b\Gamma_1 + c\eta\right)^2 \geq 0,$$

which along with (B7) and (B8) shows that $\Gamma_2 > \Gamma_{2*} \geq b^2+\eta$, and hence the proof is completed. □

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