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On Fuzzy Simulations for Expected Values of Functions of Fuzzy Numbers and Intervals

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***Abstract*—Based on existing fuzzy simulation algorithms, this paper presents two innovative techniques for approximating the expected values of fuzzy numbers' monotone functions, which is of utmost importance in fuzzy optimization literature. In this regard, the stochastic discretization algorithm of Liu and Liu (2002) is enhanced by updating the discretization procedure for the simulation of the membership function and the calculation formula for the expected values. This is achieved through initiating a novel uniform sampling process and employing a formula for discrete fuzzy numbers, respectively, as the generated membership function in the stochastic discretization algorithm would adversely affect its accuracy to some extent. What is more, considering that the bisection procedure involved in the numerical integration algorithm of Li (2015) is time-consuming and also not necessary for the specified types of fuzzy numbers, a special numerical integration algorithm is proposed, which can simplify the simulation procedure by adopting the analytical expressions of α-optimistic values. Subsequently, concerning the extensive applications of regular fuzzy intervals, several theorems are introduced and proved as an extended effort to apply the improved stochastic discretization algorithm and the special numerical integration algorithm to the issues of fuzzy intervals. Throughout the article, a series of numerical experiments are conducted from which the superiority of both the two novel techniques over others are conspicuously displayed in aspects of accuracy, stability, and efficiency.**

***Index Terms*—Expected value, fuzzy simulation, regular fuzzy number, regular fuzzy interval.**

I. Introduction

INTUITIVELY, the expected value is a well documented measurement of great importance both in academic literature and real-world applications. In particular, for the mathematical study of the mean value of fuzzy numbers, in the relevant literature, several definitions have been proposed by leading researchers in the field. In this direction, Dubois and Prade [1] constructed the expected value on the foundations of possibility theory for a fuzzy number, and it was formulated as an interval bounded by the expected values obtained using the upper and lower distribution functions. Further, Heilpern [2] introduced the concepts of expected interval and expected value of fuzzy numbers, and the latter was calculated as the center of the former. Lower and upper possibilistic mean values were studied by Carlsson and Fullér [3] as well as the relation between the interval-valued possibilistic and probabilistic means. All the definitions above are framed upon the possibility measure. However, the possibility alongside with the necessity measure has been proved to have a lack of self-duality, which might unavoidably lead to counter-intuitive results. Thus, in this regard, Liu and Liu [4] established the credibility measure by taking advantage of the average of the possibility and necessity measurements to compensate for this serious limitation. In addition, they proposed an expected value operator utilizing the credibility measure and Choquet integral.

In real-life projects, it seems reasonable that measuring the expected values for different functions that contain fuzzy parameters to obtain a general evaluation, like the expected value of the wind speed in [5] or the expected value of the lifetime of a certain product in [6], [7]. For a single fuzzy variable, based on the credibility measure, Xue et al. [8] derived a direct formula for calculating the exact expected value of a monotone function of a fuzzy variable with a continuous membership function. However, the existence of a variety of structures for the fuzzy numbers, and particularly for their complex functions, derives further challenges on the analytical calculation of the expected value when it is compared with the single fuzzy number counterpart. Alternatively, the use of fuzzy simulation methods provides us with an effective method to approximate the expected value. In this regard, a *stochastic discretization algorithm* (SDA) was employed by Liu and Liu [4] to simulate the expected value, whose basic idea is first transforming the continuous fuzzy numbers to discrete ones through a stochastical generation of sample points, and then computing the mean values for functions of these discrete counterparts. Since its establishment, the SDA has not only gained extensive support in the fuzzy expected value simulation literature, but also played a critical role in solving fuzzy expected value models whose target is to optimize the expected objectives with respect to several expected constraints. The SDA along with the SDA-based heuristic algorithms has been widely employed in handling fuzzy expected value models in various areas like portfolio selection with fuzzy returns [9], [10], system reliability analysis [11], project scheduling problem [12], amongst others.

Analogous to the SDA, Liu [13] proposed the *uniform*

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discretization algorithm (UDA) from the perspective of uniformly generating sample points, whose guiding principle is the convergence concept of sequences for fuzzy numbers. In practice, the UDA appears to be far more complex both as a concept and calculation procedure. Li [14] commented that both the SDA and UDA demonstrate good performance of accuracy and computational time when it comes to functions of fuzzy numbers with low dimensions, but they fail to return satisfactory approximation values as the dimension increases substantially. Therefore, Li [14] introduced a *numerical integration algorithm* (NIA) to calculate the expected values by means of α-optimistic values of strictly monotone functions of regular fuzzy numbers (a special type of LR fuzzy numbers with continuous and strictly decreasing shape functions, e.g., triangular, normal and Gaussian fuzzy numbers in [14], [15]), which was proved to be stable and reliable.

The fact that the SDA would return inaccurate results when high-dimensional functions occurred was reflected by the comparative results of numerical experiments between the SDA and NIA in Li [14]'s work. Miao et al. [16] further explained the reasoning behind it, indicating that the membership degrees utilized in the SDA were not obtained by sticking strictly to the Zadeh's extension principle. In addition, it is known that the SDA is not merely designed for singular use, but can also be served as a significant step in solving fuzzy expected value models where the SDA is incorporated in a *hybrid intelligent algorithm* (HIA). This sophisticated algorithm was first proposed by Liu [17] and later gained great popularity in applications (see [6], [9], [12]). However, as pointed above, the computation of SDA was proved to be not accurate both from the scenarios of theory and practice. In order to better facilitate the integration of expected value simulation to HIA for more precise solutions of fuzzy expected value models, we try to rectify the inherent deficiencies of the SDA in this research. Meanwhile, our work can also be viewed as a little demonstration for the subsequent fuzzy simulation related research.

Therefore, on the basis of Miao et al. [16], this paper proposes an *improved stochastic discretization algorithm* (iSDA) to generate the expected value simulation for strictly monotone functions involving regular fuzzy numbers, in which not only the stochastic sampling process in the SDA is substituted by a novel uniform sampling process, but also the original calculation formula of the expected value is replaced by another discrete calculation formula. More specifically, a novel simulation method of sampling and fitting membership functions of strictly monotone functions that contain regular fuzzy numbers is proposed, through which simulated membership functions of higher accuracy are obtained comparing with those attained from the SDA. Afterwards, some analytical supplementaries for the NIA are carried out and a special NIA (NIA-S) is thereby proposed so as to further simplify the NIA when the analytical expressions of α-optimistic values of regular fuzzy numbers appear not to be complicated enough to derive. In addition, due to the vast number of real-world applications for regular fuzzy intervals (i.e., a special type of LR fuzzy intervals with continuous and strictly decreasing shape functions, such as the trapezoidal fuzzy numbers), some theorems about α-optimistic and α-pessimistic values, and the expected values of strictly monotone functions of regular fuzzy intervals are proposed and proved. On this basis, for fuzzy intervals, the extension algorithms of the iSDA and NIA are introduced respectively. It should be noted that the discussions in this paper mainly focus on fuzzy numbers and fuzzy intervals, while fuzzy variables speak for a larger range.

The rest of the paper is organized as follows. In Section II, the concepts of the SDA and iSDA are expounded, whose performances are demonstrated by three numerical experiments. Subsequently, in Section III, the algorithm designs of the NIA and NIA-S together with some connections and differences between the iSDA, NIA, and NIA-S are elaborated through other three numerical examples. Section IV introduces regular fuzzy intervals, related theorems, and algorithms along with the conduction of two illustrative examples of four kinds of functions. Finally, Section V concludes the whole discussion and provides the direction of future research.

II. Improved Stochastic Discretization Algorithm

In 1998, Liu and Iwamura [18], [19] firstly proposed a fuzzy simulation technique, known as *stochastic discretization simulation* (SDS), which aims at calculating the possibility of a fuzzy event. Later, SDS was extended to the SDA to simulate the expected value, where the credibility measure of Liu and Liu [4] is employed.

In this section, the specific contents of the SDA including its basic principle and algorithm steps are reviewed first together with two derived deficiencies. Then, a novel uniform sampling method of generating membership functions of regular fuzzy numbers, and the expected value calculation formula for a discrete fuzzy number are successively elaborated. Based on them, we put forward the iSDA to handle the deficiencies derived by the SDA approach.

A. Stochastic discretization algorithm

Liu and Liu [4] defined the expected value of fuzzy numbers in light of the credibility measure as follows.

Definition 1: (Liu and Liu [4], Liu [20]) Let ξ be a fuzzy variable with membership function μ. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{1}$$

provided that at least one of the two integrals is finite, in which Cr is the credibility measure (details see Appendix B) with

$$\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}\Big(\sup_{x\geq r}\mu(x) + 1 - \sup_{x<r}\mu(x)\Big),$$

$$\mathrm{Cr}\{\xi \leq r\} = \frac{1}{2}\Big(\sup_{x\leq r}\mu(x) + 1 - \sup_{x>r}\mu(x)\Big).$$

Suppose that f is an n-ary real-valued function, and ξ_i are fuzzy numbers with respective membership functions μ_i, $i = 1, 2, \cdots, n$. Then $f(\boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, is also a fuzzy variable (Liu [17]), whose expected value is given by

$$E[f(\boldsymbol{\xi})] = \int_0^{+\infty} \mathrm{Cr}\{f(\boldsymbol{\xi}) \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{f(\boldsymbol{\xi}) \leq r\}\mathrm{d}r. \tag{2}$$

For the purpose of estimating $E[f(\boldsymbol{\xi})]$ as well as for solving a fuzzy expected value model, the following process was proposed by Liu and Liu [4]. Randomly generate u_1^j, u_2^j, $\cdots, u_n^j (j = 1, 2, \cdots, m)$ from the ϵ-level sets of ξ_1, ξ_2, $\cdots, \xi_n$, respectively, in which m is a sufficiently large integer, while ϵ is a sufficiently small number. Denote $\boldsymbol{u}_j = (u_1^j, u_2^j, \cdots, u_n^j)$ and $v_j = \mu_1(u_1^j) \wedge \mu_2(u_2^j) \wedge \cdots \wedge \mu_n(u_n^j)$ for $j = 1, 2, \cdots, m$. Accordingly, for any $r \in \mathbb{R}$, the credibilities $\mathrm{Cr}\{f(\boldsymbol{\xi}) \geq r\}$ and $\mathrm{Cr}\{f(\boldsymbol{\xi}) \leq r\}$ can be respectively estimated by

$$\begin{aligned} E^R(r) &= \frac{1}{2}\Big(\max_{j=1,2,\cdots,m} \{v_j \mid f(\boldsymbol{u}_j) \geq r\} \\ &\quad +1 - \max_{j=1,2,\cdots,m} \{v_j \mid f(\boldsymbol{u}_j) < r\}\Big), \\ E^L(r) &= \frac{1}{2}\Big(\max_{j=1,2,\cdots,m} \{v_j \mid f(\boldsymbol{u}_j) \leq r\} \\ &\quad +1 - \max_{j=1,2,\cdots,m} \{v_j \mid f(\boldsymbol{u}_j) > r\}\Big). \end{aligned} \tag{3}$$

In equation (3), if one of the set is empty, then the maximal value is 0. Applying the SDA, continuous fuzzy numbers are converted to discrete counterparts. Thus, the expected value of the function with respect to these discrete fuzzy numbers can be derived by Eq. (2). To summarize, the steps of the SDA are given in Algorithm 1 (see Appendix D).

Except for the initialization in Step 1 of Algorithm 1, the SDA mainly contains two parts. The first part (see Step 2) targets on transforming continuous fuzzy numbers to discrete counterparts through random generation of sample points, while the second part (see Steps 3 to 8) intends to attain the mean value based on Eqs. (2)-(3) via the integration simulation. Recently, some drawbacks of SDS were found and explicitly proved in Miao et al. [16], and meanwhile, SDS and SDA share the same stochastic sampling process, whose membership degree $\mu(a)^*$ for $f(\boldsymbol{\xi})$ at a real number a is expressed as

$$\mu(a)^* = \max_{1\leq j\leq m} \{ \min_{1\leq i\leq n} \mu_i(u_i^j) \mid f(u_1^j, u_2^j, \cdots, u_n^j) = a\}.$$

Technically, the above equation is capable of obtaining a satisfactory membership degree when the number of sample points m is large enough. However, from the aspect of actual operation of this stochastic sampling process, the general setting of m is a relatively small quantity of 10^3 or 10^4 level regardless of the dimension n, which does not strictly follow the Zadeh's extension principle in [21].

With respect to Miao et al. [16]'s theorem of fuzzy arithmetic, a novel uniform sampling method and a novel simulation technique, namely iSDA, which concerns LR fuzzy numbers are proposed here to improve the SDA approach. Additionally, in the iSDA, the original calculation formula of the expected value, as illustrated in Step 8 of Algorithm 1, is also substituted. Both the basic principle and the iSDA are explained in details in the following section.

B. Improved stochastic discretization algorithm

For this part, we are concerned about a specialized type of LR fuzzy numbers (see Definitions 6 and 7 in Appendix B) with continuous and strictly decreasing shape functions L and R on the open intervals $\{x|0 < L(x) < 1\}$ and $\{x|0 < R(x) < 1\}$ respectively, which are called regular fuzzy numbers in [15] and utilized in [14], [15]. Three commonly used regular fuzzy numbers are given in Examples 9-11 of Appendix C, including the triangular, normal, and Gaussian fuzzy numbers.

As for regular fuzzy numbers $\xi_i (i = 1, 2, \cdots, n)$ and a continuous and strictly monotone function f defined in [22], the operational law for the membership function of a fuzzy number $f(\boldsymbol{\xi}), \boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, is given in Theorem 1 in accordance with [16].

Theorem 1: (Miao et al. [16]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent regular fuzzy numbers. If the continuous function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing in regard to $x_1, x_2, \cdots, x_h$ and strictly decreasing in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, then the membership function of the fuzzy number $f(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$\mu(x) = \mu_1(x_1) \mid_{x=f(x_1,x_2,\cdots,x_n),\ (x_1,x_2,\cdots,x_n) \in \mathcal{L}\cup\mathcal{R}},$$

where μ_1 is the membership function of ξ_1,

$$\mathcal{L} = \{(\xi_1^L(\alpha),\cdots,\xi_h^L(\alpha),\xi_{h+1}^R(\alpha),\cdots,\xi_n^R(\alpha)) : 0 < \alpha \leq 1\},$$

$$\mathcal{R} = \{(\xi_1^R(\alpha),\cdots,\xi_h^R(\alpha),\xi_{h+1}^L(\alpha),\cdots,\xi_n^L(\alpha)) : 0 < \alpha \leq 1\},$$

and $[\xi_i^L(\alpha), \xi_i^R(\alpha)]$ is the α-level set of ξ_i, $i = 1, 2, \cdots, n$, i.e.,

$$\xi_i^L(\alpha) = \inf\{r \mid \mathrm{Cr}\{\xi_i \leq r\} \geq \alpha\},$$
$$\xi_i^R(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi_i \geq r\} \geq \alpha\}.$$

Based on Theorem 1, we initiate a novel uniform sampling method to approximate the continuous fuzzy number, $f(\boldsymbol{\xi})$, using a discrete counterpart, $f^*(\boldsymbol{\xi}), \boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$. First, denote the closure of the support of ξ_i by $S_i = [a_i, b_i]$ for $i = 1, 2, \cdots, n$ (the support of ξ_i contains all x with $\mu_{\xi_i}(x) > 0$). When the range of S_i is not finite, a set including the most values is utilized to substitute S_i as an alternative. Since ξ_i is regular, it is easy to know that there exists one and only one value $c_i \in S_i$ such that $\mu_{\xi_i}(c_i) = 1$ and $a_i < c_i < b_i$. Second, define

$$\begin{aligned} x_{ij}^L &= a_i + (c_i - a_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \\ x_{ij}^R &= b_i - (b_i - c_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \end{aligned} \tag{4}$$

and write

$$\begin{aligned} &\boldsymbol{X}_j^L = (x_{1j}^L, \cdots, x_{hj}^L, x_{h+1j}^R, \cdots, x_{nj}^R),\ j = 0, 1, \cdots, k-1, \\ &\boldsymbol{X}_j^R = (x_{1j}^R, \cdots, x_{hj}^R, x_{h+1j}^L, \cdots, x_{nj}^L),\ j = 0, 1, \cdots, k-1, \\ &\boldsymbol{c} = (c_1, c_2, \cdots, c_n). \end{aligned} \tag{5}$$

Afterwards, a new discrete fuzzy number, $f^*(\boldsymbol{\xi})$, is defined as follows

$$f^*(\boldsymbol{\xi}) = \begin{cases} f(\boldsymbol{X}_j^L), & \text{with membership degree } \mu_1(x_{1j}^L), \\ & \quad j = 0, 1, \cdots, k-1 \\ f(\boldsymbol{X}_j^R), & \text{with membership degree } \mu_1(x_{1j}^R), \\ & \quad j = 0, 1, \cdots, k-1 \\ f(\boldsymbol{c}), & \text{with membership degree } 1. \end{cases} \tag{6}$$

Denote $\mathcal{L}' = \{\boldsymbol{X}_0^L, \boldsymbol{X}_1^L, \cdots, \boldsymbol{X}_{k-1}^L\}$ and $\mathcal{R}' = \{\boldsymbol{X}_0^R, \boldsymbol{X}_1^R, \cdots, \boldsymbol{X}_{k-1}^R\}$. Obviously $\mathcal{L}'$ and $\mathcal{R}'$ are respectively subsets of $\mathcal{L}$ and $\mathcal{R}$ defined in Theorem 1.

It is easy to derive that the discrete fuzzy number $f^*(\boldsymbol{\xi})$ is in close proximity to the continuous fuzzy number $f(\boldsymbol{\xi})$, when k is large enough. As a consequence, the mean value of $f^*(\boldsymbol{\xi})$ can be reasonably viewed to be an approximation of the expected value of $f(\boldsymbol{\xi})$. Subsequently, by taking advantage of the calculation formula of the expected value of discrete fuzzy numbers in [4] and [17], the expected value of $f^*(\boldsymbol{\xi})$ is calculated by

$$E[f^*(\boldsymbol{\xi})] = \sum_{j=0}^{k-1} w_j f(\boldsymbol{X}_j^L) + w_k f(\boldsymbol{c}) + \sum_{j=0}^{k-1} w_{m-j} f(\boldsymbol{X}_j^R), \quad m = 2k, \tag{7}$$

where w_j, $j = 0, 1, \cdots, 2k$, are ascertained by

$$\begin{aligned} w_j &= \frac{1}{2}\Big(\max_{t\le j}\mu(f(\boldsymbol{X}_t^L)) - \max_{t<j}\mu(f(\boldsymbol{X}_t^L)) \\ &\quad + \max_{t\ge j}\mu(f(\boldsymbol{X}_t^L)) - \max_{t>j}\mu(f(\boldsymbol{X}_t^L))\Big), \\ &\quad j = 0, 1, \cdots, k-1, \\ w_k &= \frac{1}{2}\Big(2 - \mu(f(\boldsymbol{X}_{k-1}^L)) - \mu(f(\boldsymbol{X}_{k-1}^R))\Big), \\ w_{m-j} &= \frac{1}{2}\Big(\max_{t\le j}\mu(f(\boldsymbol{X}_t^R)) - \max_{t<j}\mu(f(\boldsymbol{X}_t^R)) \\ &\quad + \max_{t\ge j}\mu(f(\boldsymbol{X}_t^R)) - \max_{t>j}\mu(f(\boldsymbol{X}_t^R))\Big), \\ &\quad j = 0, 1, \cdots, k-1, \end{aligned} \tag{8}$$

and μ represents the membership function of $f^*(\boldsymbol{\xi})$ in Eq. (6). Further, utilizing the strict monotonicity of the shape functions of ξ_1 (that is, $\mu_1(x_{1i}^L) < \mu_1(x_{1j}^L)$ and $\mu_1(x_{1i}^R) > \mu_1(x_{1j}^R)$ hold for all $i < j$), we can simplify Eq. (8) as follows:

$$\begin{aligned} w_0 &= \frac{1}{2}\mu_1(x_{10}^L), \quad w_m = \frac{1}{2}\mu_1(x_{10}^R), \\ w_j &= \frac{1}{2}\big(\mu_1(x_{1j}^L) - \mu_1(x_{1(j-1)}^L)\big), j = 1, 2, \cdots, k-1, \\ w_k &= 1 - \frac{1}{2}\big(\mu_1(x_{1(k-1)}^L) + \mu_1(x_{1(k-1)}^R)\big), \\ w_{m-j} &= \frac{1}{2}\big(\mu_1(x_{1j}^R) - \mu_1(x_{1(j-1)}^R)\big), j = 1, 2, \cdots, k-1. \end{aligned} \tag{9}$$

Therefore, a novel simulation technique, namely iSDA, to simulate the expected value $E[f(\boldsymbol{\xi})]$ is proposed combining the uniform sampling process in Eqs. (4)-(6) and the expected value calculation formula for discrete fuzzy numbers in Eqs. (7)-(9). And the detailed procedure of the iSDA is described as follows.

Algorithm 2 (iSDA)

Step 1. Initialize the number of sample points m. Set $k = m/2$, $E = 0$ and $j = 0$.

Step 2. Calculate $f(\boldsymbol{X}_j^L)$ with Eqs. (4)-(5).

Step 3. Calculate w_j with Eq. (9). Reset $E = E + w_j f(\boldsymbol{X}_j^L)$ and $j = j + 1$.

Step 4. If $j < k$, go to Step 2. Otherwise, reset $j = 0$ and go to Step 5.

Step 5. Calculate $f(\boldsymbol{X}_j^R)$ with Eqs. (4)-(5).

Step 6. Calculate w_{m-j} with Eq. (9). Reset $E = E + w_{m-j} f(\boldsymbol{X}_j^R)$ and $j = j + 1$.

Step 7. If $j < k$, go to Step 5. Otherwise, go to Step 8.

Step 8. Calculate $f(\boldsymbol{c})$ and w_k. Reset $E = E + w_k f(\boldsymbol{c})$.

Step 9. Return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.

Similarly to the SDA, the calculation procedure of the iSDA basically consists of two parts. Steps 2 and 5 indicate the uniform sampling process, and Steps 3, 6 and 8 represent the expected value calculation procedure for the discrete fuzzy number, $f^*(\boldsymbol{\xi})$.

For demonstrating clearly the feasibility and effectiveness of the iSDA, a series of contrast outcomes of the SDA and iSDA considering different fuzzy variables and functions are presented in the following two subsections. Furthermore, since the calculation formula in Step 8 of Algorithm 1 is not easy to be understood, and to observe the efficiency of this formula, we specifically design an intermediate simulation algorithm, SDA*. It employs the same uniform sampling process with the iSDA in Steps 2 and 5 of Algorithm 2 and utilizes the same calculation procedure of the expected value with the SDA from Steps 3 to 8 of Algorithm 1.

C. Comparative study between the SDA and iSDA: The case of triangular fuzzy numbers

The comparative results between the SDA and iSDA as well as for the SDA* facilitating a numerical example are presented in this section, including the simulation accuracy, computational time, and complexity analysis of each algorithm. These algorithms together with the algorithms in the subsequent sections in this paper are coded in C language and operate under an identical computational condition, i.e., using a Personal Computer with 2.27 GHz processor speed and 32 GB memory.

Example 1: Suppose that $\eta_i, i = 1, 2, \cdots, 10$, are independent triangular fuzzy numbers listed in Table I, incorporated in a continuous and strictly increasing function $f_1(x_1, x_2, \cdots, x_{10}) = x_1 + x_2 + \cdots + x_{10}$. This example (see Li [14]) aims at calculating the expected value, $E[\xi]$, of the fuzzy number, $\xi = f_1(\eta_1, \eta_2, \cdots, \eta_{10})$.

Before conducting the simulation, in terms of the linearity towards the expected value operator for independent fuzzy numbers proved in [4], the exact value of $E[\xi]$ can be calculated in a straightforward manner, that is,

$$E[\xi] = E[\eta_1] + E[\eta_2] + \cdots + E[\eta_{10}] = 38.5.$$

With regard to Example 1, we run the SDA, SDA*, and iSDA, ten times and record their simulation results in Table II, accordingly. The quantity of integration points in the SDA is settled as 10000, while those of sample points in the three algorithms are all set as 1000. In order to express the relative error degree of all the results via the three algorithms, an index

TABLE I
DIFFERENT KINDS OF REGULAR FUZZY NUMBERS UTILIZED IN EXAMPLES

Index	Triangular Fuzzy Number	Normal Fuzzy Number	Gaussian Fuzzy Number
η_1	$\mathcal{T}(2,3,4)$	$\mathcal{N}(0,1)$	$\mathcal{G}(0,1)$
η_2	$\mathcal{T}(5,6,8)$	$\mathcal{N}(0,2)$	$\mathcal{G}(0,2)$
η_3	$\mathcal{T}(6,7,8)$	$\mathcal{N}(1,2)$	$\mathcal{G}(1,2)$
η_4	$\mathcal{T}(4,5,6)$	$\mathcal{N}(2,4)$	$\mathcal{G}(2,4)$
η_5	$\mathcal{T}(3,4,6)$	$\mathcal{N}(4,6)$	$\mathcal{G}(4,6)$
η_6	$\mathcal{T}(7,9,10)$	$\mathcal{N}(5,8)$	$\mathcal{G}(5,8)$
η_7	$\mathcal{T}(-5,-3,-2)$	$\mathcal{N}(-1,2)$	$\mathcal{G}(-1,2)$
η_8	$\mathcal{T}(5,6,8)$	$\mathcal{N}(-3,6)$	$\mathcal{G}(-3,6)$
η_9	$\mathcal{T}(0,1,2)$	$\mathcal{N}(-5,2)$	$\mathcal{G}(-5,2)$
η_{10}	$\mathcal{T}(-1,0,2)$	$\mathcal{N}(-7,7)$	$\mathcal{G}(-7,7)$

called "*Error*" is displayed on the fourth column of Table II, which is defined as

$$\text{Error} = \frac{|\text{Simulation value} - \text{Exact value}|}{\text{Exact value}} \times 100\%. \quad (10)$$

Note that the simulation value utilized in Eq. (10) during the calculation of Error in Table II is the average value of the ten times simulation results. From Table II, it is clear that both the stability and accuracy of the iSDA are superior to those of the SDA and SDA*.

TABLE II
TEN COMPARATIVE RESULTS AMONG THE SDA, SDA*, AND ISDA FOR EXAMPLE 1.

Algorithm	Simulation Value of $E[\xi]$ for Time 1-10				Deviation	Error
	38.8528	38.8765	38.8738			
SDA	38.8918	38.8617	38.8764		0.02	1.00%
	38.9195	38.8955	38.8751	38.9079		
	38.4900	38.6885	38.2088			
SDA*	38.4986	38.0610	38.4801		0.21	0.26%
	38.2785	38.2741	38.7293	38.3066		
	38.4990	38.4990	38.4990			
iSDA	38.4990	38.4990	38.4990		0.00	0.00%
	38.4990	38.4990	38.4990	38.4990		

For demonstrating further the performance of the SDA, SDA*, and iSDA, their simulation values, deviation, and computational time are obtained through the variation of the quantity of sample points m as well as that of integration points N. Particularly, we alter m in the SDA when N is set to be 10000 or 20000 respectively to test whether the increasing of sample points will positively affect the accuracy of the final results. Accordingly, the detailed results towards the above-mentioned experiment are displayed in Table III and visualized in Fig. 1. It is noted that the simulation value and the computational time listed in this table as well as in any subsequent tables are all the average values of running the corresponding algorithm for ten times.

Below is the detailed analysis of Table III and Fig. 1. First, from the point of view of the derived accuracy, as shown in Fig. 1, the results of the iSDA are steadily converged and almost coincide with the exact value 38.5, which nearly provide no error even when the number of integration points m is small (e.g., $m = 1000$ in Table III). It can also be seen

TABLE III
COMPARATIVE RESULTS AMONG THE SDA, SDA*, AND ISDA FOR EXAMPLE 1.

Algorithm	Number of Sample Points m	Number of Integration Points N	Simulation Value of $E[\xi]$	Error	CPU Time (s)
SDA	1000	10000	38.8831	1.00%	0.180
	3000	10000	38.7269	0.59%	0.527
	5000	10000	38.8252	0.84%	0.843
	10000	10000	38.8618	0.94%	1.907
	15000	10000	38.7456	0.64%	2.866
	20000	10000	38.7887	0.75%	3.898
	1000	20000	38.8902	1.01%	0.328
	3000	20000	38.7277	0.59%	0.845
	5000	20000	38.8238	0.84%	1.950
	10000	20000	38.8492	0.91%	3.643
	15000	20000	38.7456	0.64%	5.938
	20000	20000	38.8043	0.79%	7.800
SDA*	1000	1000	38.4015	0.26%	0.022
	3000	3000	38.4164	0.22%	0.169
	5000	5000	38.4400	0.16%	0.393
	10000	10000	38.4612	0.10%	1.777
	15000	15000	38.4802	0.05%	4.053
	20000	20000	38.4889	0.03%	7.294
iSDA	1000	none	38.4990	0.00%	0.001
	3000	none	38.4997	0.00%	0.002
	5000	none	38.4998	0.00%	0.003
	10000	none	38.4999	0.00%	0.004
	15000	none	38.4999	0.00%	0.005
	20000	none	38.5000	0.00%	0.006

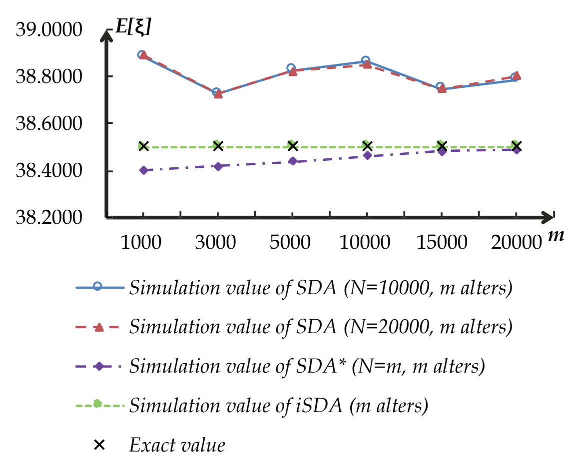

Fig. 1. The visualisation of comparative results in Table III.

that under different combinations of m and N, the biggest error of the SDA is 1.01%, and the changes of m in the SDA do not affect positively the final results. Second, from the stability point of view, the experimental results of the SDA have a larger deviation, while the iSDA is quite stable in returning the simulation values. Third, the computational time of iSDA is hundreds of times faster than SDA. More precisely, in Table III it is noticed that the longest time of the SDA is 7.800s and that of the iSDA is only 0.006s. It is known that the computational time has a strong relationship with the algorithm complexity, whose expressions of the SDA and iSDA are O(mN) and O(N), respectively. To summarize, the iSDA is equipped with prominent advantage speaking of accuracy, stability, and operation speed in contrast to the SDA.

As an intermediate algorithm, the results of the SDA* are listed in Table III and depicted in Fig. 1 as well. It can be seen that both the accuracy and the computational time of the SDA* are not comparable to those of the iSDA. On the one hand, through comparing the SDA* with the SDA, it reveals that the uniform sampling method is reasonable and effective, especially reflected on the convergence of the simulation results in the SDA*. On the other hand, through comparing the SDA* with the iSDA, the effectiveness of the calculation formula of the expected value of discrete fuzzy numbers in Eqs. (7)-(9) utilized in the iSDA can also be validated. These two comparisons demonstrate the feasibility and reliability of the two improvements for the iSDA.

D. Comparative study between different functions and fuzzy numbers

Two more examples are given in this section to further demonstrate the superiority of the iSDA among the other two algorithms.

Example 2: This example targets on simulating the expected values of the same function f_1 of Example 1 using the SDA, SDA*, and iSDA, but the fuzzy variables included are triangular, normal, and Gaussian fuzzy numbers, see Table I, respectively.

The simulation results are illustrated in Table IV, in which (m/N) represents the number of sample or integration points adopted in each algorithm. Since the support S_i of a normal fuzzy number $\mathcal{N}(c_i, \sigma_i)$ or a Gaussian fuzzy number $\mathcal{G}(c_i, b_i)$ is infinite, we respectively set $S_i = [c_i - g\sigma_i, c_i + g\sigma_i]$ or $S_i = [c_i - gb_i, c_i + gb_i]$, where g is a positive integer. The range of the support S_i in a triangular fuzzy number is finite. As it is known that $\pm 6\sigma$ can cover a relative large range of values to 99.99966%, then we assign three values: 1, 3, and 6 to g to observe the differences.

From Table IV, it is obvious that the iSDA is rather accurate, reliable, and fast on the outputs. In addition, the iSDA performs better when $g = 6$ in contrast to $g = 1$, which reflects its sensitivity to the support S_i. Comparing with the SDA, the effect of the SDA* is enhanced due to the replacement of the stochastic sampling process, but still is far from obtaining accurate values, especially for normal and Gaussian fuzzy numbers.

Example 3: A more complex function $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$ is employed in this example. Calculate the expected values of f_2 of triangular, normal, and Gaussian fuzzy numbers by the SDA, SDA*, and iSDA, respectively.

The simulation results of Example 3 are recorded in Table V, which share some similar conclusions with those of Example 2. From Tables IV and V, several remarks on the three algorithms used are outlined. First, the results of Examples 2 and 3 are similar with those derived from Example 1, as there still exist great differences in accuracy and time between the SDA and iSDA. Second, in terms of the parameter g, generally the performance of the SDA is barely acceptable for $g = 1$, but when g gets larger, the results become worse. Whereas the iSDA returns the simulation results of highest accuracy at $g = 6$, and the biggest error is 0.03%. Third, the SDA* reduces the error rate due to the incorporated uniform sampling process compared with the SDA. In summary, the results of three examples demonstrates that the iSDA works better regardless of different functions or kinds of fuzzy variables.

III. Special Numerical Integration Algorithm

With respect to the particular case of continuous and strictly monotone functions of regular fuzzy numbers (also called ordinary fuzzy variables in [14]), Li [14] proposed the *numerical integration algorithm* (NIA) to approximate the expected values by means of the concept of α-optimistic values. In this section, the NIA and its related principles and concepts are primarily recalled. Subsequently, after deriving the analytical expressions of α-optimistic values for regular fuzzy numbers, owing to the specific features of regular fuzzy numbers, we

TABLE IV
Comparative results among the SDA, SDA*, and iSDA for the case that $f_1 = x_1 + x_2 + \cdots + x_{10}$.

Algorithm (m/N)	Triangular	Normal			Gaussian		
SDA (3000/10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.7269	-3.5069	-4.2920	-7.5761	-4.3014	-5.6030	-9.2436
Error	0.59%	12.33%	7.30%	89.40%	7.54%	40.08%	131.09%
CPU Time (s)	0.527	0.476	0.479	0.498	0.468	0.498	0.495
SDA* (10000/10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4612	-3.8402	-3.6853	-3.7063	-3.8285	-3.7282	-3.8002
Error	0.10%	4.00%	7.87%	7.34%	4.29%	6.80%	5.00%
CPU Time (s)	1.777	1.188	1.413	1.816	1.411	1.583	1.807
iSDA (10000/none)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4999	-3.9983	-3.9995	-4.0000	-3.9985	-4.0000	-4.0000
Error	0.00%	0.04%	0.01%	0.00%	0.04%	0.00%	0.00%
CPU Time (s)	0.004	0.006	0.006	0.006	0.005	0.006	0.006

TABLE V
COMPARATIVE RESULTS AMONG THE SDA, SDA*, AND ISDA FOR THE CASE THAT $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$.

Algorithm (m/N)	Triangular	Normal			Gaussian		
SDA (3000/10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.4669	8.1734	12.9329	22.9128	8.0238	13.1042	23.4327
Error	6.67%	7.44%	46.47%	159.49%	2.93%	58.52%	183.47%
CPU Time (s)	0.783	0.576	0.565	0.608	0.534	0.591	0.600
SDA* (10000/10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2458	7.8663	8.7163	8.7785	7.9114	8.2307	8.2065
Error	0.13%	10.91%	1.29%	0.58%	4.29%	0.43%	0.72%
CPU Time (s)	1.753	1.565	1.688	1.844	1.682	1.713	1.810
iSDA (10000/none)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2500	7.8745	8.7462	8.8271	7.9176	8.2657	8.2652
Error	0.00%	10.82%	0.95%	0.03%	4.22%	0.01%	0.01%
CPU Time (s)	0.004	0.006	0.006	0.006	0.006	0.006	0.006

further propose the *special numerical integration algorithm* (NIA-S) to simplify the calculation procedure of the original NIA set forth by Li [14] (renamed as a general NIA, NIA-G for short, in this paper for being distinguishable).

A. General numerical integration algorithm

Before introducing the calculation procedure of the NIA-G proposed in [14], the relevant definitions and theorems are brought in.

Definition 2: (Liu [23]) The credibility distribution of a fuzzy variable ξ is defined as

$$\Phi(x) = \mathrm{Cr}\{\xi \le x\}, \quad \forall x \in \mathbb{R}. \tag{11}$$

Analogously, $\Psi(x) = \mathrm{Cr}\{\xi \ge x\}$ is denoted, and $\Psi + \Phi \equiv 1$ if ξ is a continuous fuzzy variable, which implies that

$$\Psi(x) = 1 - \Phi(x). \tag{12}$$

Definition 3: (Liu [23]) For any $\alpha \in (0, 1]$, the α-optimistic value of a fuzzy variable ξ is

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi \ge r\} \ge \alpha\}. \tag{13}$$

Theorem 2: (Li [14]) If ξ is a regular fuzzy number, for any $\alpha \in (0, 1]$, we have that

$$\xi_{\sup}(\alpha) = \Psi^{-1}(\alpha). \tag{14}$$

Assuming that the membership function, μ_ξ, of a regular fuzzy number ξ is known, Ψ can be deduced via μ_ξ as follows,

$$\Psi(x) = \begin{cases} \mu_\xi(x)/2, & \text{if } x \ge c \\ 1 - \mu_\xi(x)/2, & \text{if } x < c, \end{cases} \tag{15}$$

in which $\mu_\xi(c) = 1$.

According to the mathematical property of μ_ξ, we know that Ψ is continuous and strictly decreasing. Then in terms of Eqs. (14)-(15), Li [14] designed a bisection algorithm (see Algorithm 3 in Appendix D) to simulate $\xi_{\sup}(\alpha)$ for any given $\alpha \in (0, 1]$.

Now that the α-optimistic values are derived, and they can be further utilized to obtain mean values for continuous and strictly monotone functions of regular fuzzy numbers by the following theorem.

Theorem 3: (Li [14]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent regular fuzzy numbers. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0, 1]$, the expected value of $f(\boldsymbol{\xi}) = f(\xi_1, \xi_2, \cdots, \xi_n)$ is given by

$$E[f(\boldsymbol{\xi})] = \int_0^1 f\Big((\xi_1)_{\sup}(\alpha), \cdots, (\xi_h)_{\sup}(\alpha), (\xi_{h+1})_{\sup}(1-\alpha), \cdots, (\xi_n)_{\sup}(1-\alpha)\Big)\mathrm{d}\alpha. \tag{16}$$

According to Eq. (16), Li [14] designed an integration simulation algorithm NIA-G (see Algorithm 4 in Appendix D) to calculate $E[f(\boldsymbol{\xi})]$ by utilizing $\xi_{\sup}(\alpha)$ obtained from the bisection algorithm.

B. Special numerical integration algorithm

As a matter of fact, for the commonly used regular fuzzy numbers, we find that deriving the analytical expressions of their α-optimistic values is not difficult, and then the bisection procedure in the NIA-G could be replaced by the clear calculation formula of $\xi_{\sup}(\alpha)$. Based upon this concept, the NIA-S is thus put forward to improve NIA-G as follows.

As to a regular fuzzy number ξ, which is of LR-type with continuous and strictly decreasing shape functions L and R, in regard to Eqs. (38) and (15), we can get that

$$\Psi(x) = \begin{cases} \dfrac{1}{2}R\left(\dfrac{x-c}{\beta}\right), & \text{if } x \ge c \\ 1 - \dfrac{1}{2}L\left(\dfrac{c-x}{\gamma}\right), & \text{if } x < c. \end{cases} \tag{17}$$

Due to the strict monotonicity of L and R, their inverse functions exist and are denoted by L^{-1} and R^{-1}, respectively.

Then the α-optimistic value of ξ, $\xi_{\sup}(\alpha)$, can be derived from Eqs. (14) and (17) as follows:

$$\xi_{\sup}(\alpha)=\Psi^{-1}(\alpha)=\begin{cases} c+\beta R^{-1}(2\alpha), & \text{if } 0<\alpha\leq 0.5 \\ c-\gamma L^{-1}(2-2\alpha), & \text{if } 0.5<\alpha\leq 1. \end{cases} \tag{18}$$

According to Eq. (18), the α-optimistic values of some commonly used regular fuzzy numbers enumerated in Examples 9-11 can be respectively obtained (details see Appendix C).

Consequently, based on Theorem 3 and Eq. (18) (e.g., Eqs. (41)-(43)), a special NIA is then set forth by using the analytical expressions of α-optimistic values of regular fuzzy numbers to substitute the bisection algorithm in NIA-G. The specific steps of NIA-S are described as follows.

Algorithm 5 (NIA-S)

Step 1. Initialize the number of integration points N. Let $E=0$ and $k=1$.

Step 2. Set $\alpha = k/N$. For each $1\leq i\leq n$, according to the calculation formula of α-optimistic values in Eq. (18), calculate

$$x_i=\begin{cases} (\xi_i)_{\sup}(\alpha), & \text{if } 1\leq i\leq h, \\ (\xi_i)_{\sup}(1-\alpha), & \text{if } h<i\leq n. \end{cases}$$

Step 3. Reset $E=E+f(x_1,x_2,\cdots,x_n)/N$ and $k=k+1$.

Step 4. If $k\leq N$, go to Step 2. Otherwise, return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.

As a general rule, the clear analytical expressions of the inverse functions of L and R are not difficult to obtain. Under this case, the NIA-S is more suitable to be chosen for the simulation of the expected value. But when it comes to a situation that the inverse functions are too complex to figure out, then the bisection algorithm is preferred to calculate the value of $\Psi^{-1}(\alpha)$ (see Algorithm 3) or the "polyfit" function of Matlab to generate approximate functions for $\Psi^{-1}(\alpha)$.

C. Comparative study with different functions of different fuzzy numbers

In this section, three numerical examples considering the expected values of continuous and strictly monotone functions of regular fuzzy numbers are conducted to compare the performance of the iSDA, NIA-G, and NIA-S based on the accuracy, stability, and operation speed measurements.

Example 4: According to the data and function given in Example 1, accomplish the expected value $E[\xi]$ of the fuzzy number $\xi=f_1(\eta_1,\eta_2,\cdots,\eta_{10})$ by means of the iSDA, NIA-G, and NIA-S, respectively, in which η_i, $i=1,2,\cdots,10$, are triangular fuzzy numbers.

The final simulation results of the iSDA, NIA-G, and NIA-S are obtained through altering the numbers of sample or integration points and reported in Table VI. Here the small enough number ϵ in the bisection part of the NIA-G is set to be 10^{-3} on account of the trade-off between accuracy and time. Meanwhile, the analytical expression of the α-optimistic value of a triangular fuzzy number in the NIA-S refers to Eq. (41).

From Table VI, we can see that along with the increasing number of integration points N, the accuracy degrees for the NIA-G and NIA-S are both greatly enhanced, and this point is not obviously reflected on the iSDA. On the whole, regardless of accuracy, stability, or operation speed, the performance of the iSDA in Example 4 is clearly superior among all the three algorithms compared.

Example 5: This example is designed to figure the expected value $E[\xi]$ of $\xi=f_1(\eta_1,\eta_2,\cdots,\eta_{10})$ out using the iSDA, NIA-G, and NIA-S, in which the function $f_1=x_1+x_2+\cdots+x_{10}$, and the fuzzy variables $\eta_i, i=1,2,\cdots,10$, included are respective triangular, normal, and Gaussian fuzzy numbers in Table I.

The simulation outcomes are summarized in Table VII, in which (m or N) after each algorithm represents the number of sample or integration points involved. It is observed that there is no g in the NIA-S since it utilizes the inverse functions directly other than the range of the support S_i. Not surprisingly, the iSDA with the setting $g=6$ still performs better either on the accuracy or with respect to the time.

Example 6: Other conditions stay unchanged, only substitute the function f_1 in Example 5 by $f_2=-(x_1\wedge x_2\wedge\cdots\wedge x_{10})$, and figure out the mean value $E[\xi]$ of $\xi=f_2(\eta_1,\eta_2,\cdots,\eta_{10})$ using the iSDA, NIA-G, and NIA-S.

The simulation results of Example 6 are enumerated in Table VIII for comparison purposes. Analogous to Section II-D, the computational time does not change so much for the iSDA, but the performance is quite well when $g=6$. The NIA-S is also effective except for its computational time which is several times longer than that of the iSDA. Certainly, the NIA-G is able to achieve a satisfactory result when $g=6$, nevertheless the time needed is hundreds times greater than the iSDA.

Notably, for the iSDA or NIA-G, when it comes to the function f_1, whatever the value g is, the simulation results are already good enough. However, as for the function f_2, only when $g=6$, a result of high accuracy can derive. The main cause of this difference may come from the features of these two functions, that is, f_1 focuses on the overall sum while f_2 aims at the minimum value only.

In summary, among the three algorithms, the iSDA, NIA-S, and NIA-G, everyone is much better than the SDA in accuracy, stability, or computational time, and their individual outputs are steady, unlike that of the SDA. Generally, the iSDA outperforms all the other algorithms in all aspects, i.e., it is highly efficient and time-saving. The NIA-S is slightly inferior to the iSDA from the aspect of time, but the good point is that its calculating procedure is not related to the range of the support (no altering of g). The main disadvantage of the NIA-G lies on the computational time, due to the reason that there exists a bisection circulation in its algorithm design.

IV. Extensions to Regular Fuzzy Intervals

It is clear that the regular fuzzy intervals are also of great importance no matter in theoretical developments like its variance research in [24] and the entropy calculation and simulation in [25], or in practical applications like the

TABLE VI
SIMULATION RESULTS FOR THE ISDA, NIA-G, AND NIA-S IN EXAMPLE 4

Number of Sample Points or Integration Points N	iSDA Simulation Value	iSDA CPU Time (s)	NIA-G Simulation Value	NIA-G CPU Time (s)	NIA-S Simulation Value	NIA-S CPU Time (s)
1000	38.4990	0.000	38.4870	0.010	38.4870	0.000
3000	38.4997	0.000	38.4957	0.046	38.4957	0.005
5000	38.4998	0.000	38.4974	0.070	38.4974	0.010
10000	38.4999	0.000	38.4987	0.140	38.4987	0.015
15000	38.4999	0.010	38.4991	0.202	38.4991	0.020
20000	38.5000	0.010	38.4994	0.265	38.4994	0.030
10^6	38.5000	0.330	38.5000	10.256	38.5000	0.883

TABLE VII
COMPARATIVE RESULTS AMONG THE ISDA, NIA-G, AND NIA-S FOR THE CASE THAT $f_1 = x_1 + x_2 + \cdots + x_{10}$.

	Triangular	Normal			Gaussian		
iSDA (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4999	-3.9983	-3.9995	-4.0000	-3.9985	-4.0000	-4.0000
Error	0.00%	0.04%	0.01%	0.00%	0.04%	0.00%	0.00%
CPU Time (s)	0.004	0.006	0.006	0.006	0.005	0.006	0.006
NIA-G (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4987	-3.9996	-3.9996	-3.9996	-3.9996	-3.9996	-3.9996
Error	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
CPU Time (s)	0.140	0.402	0.419	0.411	0.333	0.348	0.353
NIA-S (10000)							
Exact Value	38.5000		-4.0000			-4.0000	
Simulation Value	38.4987		-3.9996			-3.9996	
Error	0.00%		0.01%			0.01%	
CPU Time (s)	0.008		0.036			0.017	

portfolio optimization in [26]. One of the representative form of regular fuzzy interval is the commonly used trapezoidal fuzzy number. Scholars have continued interests in updating the fuzzy simulation of the expected value of functions that contain trapezoidal fuzzy numbers (from [6] to [7]). In this section, to calculate the expected value of a strictly monotone function f of regular fuzzy intervals $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$, the α-optimistic value $\widetilde{\xi}_{\sup}(\alpha)$ and α-pessimistic value $\widetilde{\xi}_{\inf}(\alpha)$ of regular fuzzy intervals are deduced. Then, Theorem 3 is further extended for the case of regular fuzzy intervals. On this basis, two extension algorithms called TiSDA and TNIA-S are proposed to simulate the expected value $E[f(\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)]$, respectively.

A. Regular fuzzy interval

The definition of regular fuzzy intervals based on LR fuzzy intervals (see Definitions 8 and 9 of Appendix B) is in accordance with that of regular fuzzy numbers, which is described as follows.

Definition 4: An LR fuzzy interval is said to be regular if the shape functions L and R are continuous and strictly decreasing functions on the open intervals $\{0 < L(x) < 1\}$ and $\{0 < R(x) < 1\}$, respectively.

In this paper, we consider the expected values of continuous and strictly monotone functions of regular fuzzy intervals. Before that, firstly, several properties of α-optimistic values of regular fuzzy intervals are elaborated as follows.

Theorem 4: Let $\widetilde{\xi}$ be a regular fuzzy interval. For any $\alpha \in (0, 1]$, we have that

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \alpha. \tag{19}$$

Proof: For any $\alpha \in (0, 1]$ and $\alpha \neq 0.5$, it follows from the continuity of the distribution function and the definition of optimistic value in Eq. (13) that

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha) - \frac{1}{n}\} \geq \alpha, \tag{20}$$

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha) + \frac{1}{n}\} \leq \alpha. \tag{21}$$

If $\alpha = 0.5$, we have that

$$\begin{aligned} &\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(0.5)\} \\ &= \frac{1}{2}\left(\mathrm{Pos}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(0.5)\} + 1 - \mathrm{Pos}\{\widetilde{\xi} < \widetilde{\xi}_{\sup}(0.5)\}\right) \\ &= \frac{1}{2}(0.5 + 1 - 0.5) = 0.5. \end{aligned} \tag{22}$$

Combining Eqs. (20) and (21) together with Eq. (22), the proof is complete. ■

TABLE VIII
COMPARATIVE RESULTS AMONG THE iSDA, NIA-G, AND NIA-S FOR THE CASE THAT $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$.

	Triangular	Normal			Gaussian		
iSDA (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2500	7.8745	8.7462	8.8271	7.9176	8.2657	8.2652
Error	0.00%	10.82%	0.95%	0.03%	4.22%	0.01%	0.01%
CPU Time (s)	0.004	0.006	0.006	0.006	0.006	0.006	0.006
NIA-G (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2497	7.8740	8.7455	8.8252	7.9170	8.2650	8.2650
Error	0.01%	10.83%	0.96%	0.05%	4.23%	0.02%	0.02%
CPU Time (s)	0.136	0.385	0.395	0.411	0.284	0.321	0.337
NIA-S (10000)							
Exact Value	3.2500		8.8300			8.2664	
Simulation Value	3.2499		8.8274			8.2650	
Error	0.00%		0.03%			0.02%	
CPU Time (s)	0.010		0.046			0.025	

Theorem 5: If $\widetilde{\xi}$ is a regular fuzzy interval, for any $\alpha \in (0,1]$, we obtain that

$$\widetilde{\xi}_{\sup}(\alpha) = \begin{cases} \Psi^{-1}(\alpha), & \text{if } \alpha \neq 0.5 \\ \overline{c}, & \text{if } \alpha = 0.5. \end{cases} \tag{23}$$

Proof: For any given $\alpha \in (0,1]$, denote $\Psi(x) = \mathrm{Cr}\{\widetilde{\xi} \geq x\} = \alpha$. Since $\Psi(x)$ is strictly decreasing in $\{x \leq \underline{c}\}$ and $\{x \geq \overline{c}\}$, for any $\alpha \in (0,1]$ and $\alpha \neq 0.5$, we have that $x = \Psi^{-1}(\alpha)$. Combining the above with Eq. (19) in Theorem 4, we get that $\widetilde{\xi}_{\sup}(\alpha) = \Psi^{-1}(\alpha)$. In addition, it follows immediately from the definition of $\widetilde{\xi}_{\sup}$ that $\widetilde{\xi}_{\sup}(\alpha) = \overline{c}$ when $\alpha = 0.5$. ■

According to Theorem 5, to obtain $\widetilde{\xi}_{\sup}(\alpha)$, we need to calculate $\Psi(x)$ of regular fuzzy interval $\widetilde{\xi}$ first. If the membership function $\mu_{\widetilde{\xi}}$ of a regular fuzzy interval $\widetilde{\xi}$ is attained, $\Psi(x)$ can be deduced via $\mu_{\widetilde{\xi}}$ as follows,

$$\Psi(x) = \begin{cases} 1 - \mu_{\widetilde{\xi}}(x)/2, & \text{if } x < \underline{c}, \\ \dfrac{1}{2}, & \text{if } \underline{c} \leq x \leq \overline{c} \\ \mu_{\widetilde{\xi}}(x)/2, & \text{if } x > \overline{c}. \end{cases} \tag{24}$$

Based on Eqs. (40) and (24), we have that

$$\Psi(x) = \begin{cases} 1 - \dfrac{1}{2} L\left(\dfrac{\underline{c} - x}{\gamma}\right), & \text{if } x < \underline{c} \\ \dfrac{1}{2}, & \text{if } \underline{c} < x \leq \overline{c} \\ \dfrac{1}{2} R\left(\dfrac{x - \overline{c}}{\beta}\right), & \text{if } x > \overline{c}. \end{cases} \tag{25}$$

Since the shape functions L and R are both continuous and strictly decreasing, the inverse functions L^{-1} and R^{-1} exist. Consequently, the analytical expression of $\widetilde{\xi}_{\sup}(\alpha)$ of a regular fuzzy interval in Eq. (23) is obtained as

$$\tilde{\xi}_{\sup}(\alpha) = \begin{cases} \beta R^{-1}(2\alpha) + \overline{c}, & \text{if } 0 < \alpha < 0.5 \\ \overline{c}, & \text{if } \alpha = 0.5 \\ \underline{c} - \gamma L^{-1}(2 - 2\alpha), & \text{if } 0.5 < \alpha \leq 1. \end{cases} \tag{26}$$

Further, for better understanding of the following theorems, the concept of α-pessimistic value is also introduced in this section.

Definition 5: (Liu [23]) For any $\alpha \in (0,1]$, the α-pessimistic value of a fuzzy variable ξ is

$$\xi_{\inf}(\alpha) = \inf\{r \mid \mathrm{Cr}\{\xi \leq r\} \geq \alpha\}. \tag{27}$$

Theorem 6: Let $\widetilde{\xi}$ be a regular fuzzy interval. For any $\alpha \in (0,1]$, we have that

$$\mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(\alpha)\} = \alpha. \tag{28}$$

Proof: Analogous to the proof of Theorem 4, we get that

$$\mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(\alpha) + \frac{1}{n}\} \geq \alpha, \tag{29}$$

$$\mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(\alpha) - \frac{1}{n}\} \leq \alpha, \tag{30}$$

and if $\alpha = 0.5$, it yields that

$$\begin{aligned} &\mathrm{Cr}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(0.5)\} \\ &= \frac{1}{2}\left(\mathrm{Pos}\{\widetilde{\xi} \leq \widetilde{\xi}_{\inf}(0.5)\} + 1 - \mathrm{Pos}\{\widetilde{\xi} > \widetilde{\xi}_{\inf}(0.5)\}\right) \\ &= \frac{1}{2}(0.5 + 1 - 0.5) = 0.5. \end{aligned} \tag{31}$$

Combining Eqs. (29) and (30) together with Eq. (31), the proof is complete. ■

Theorem 7: If $\widetilde{\xi}$ is a regular fuzzy interval, for any $\alpha \in (0,1]$, we have that

$$\widetilde{\xi}_{\inf}(\alpha) = \begin{cases} \Psi^{-1}(1-\alpha), & \text{if } \alpha \neq 0.5 \\ \underline{c}, & \text{if } \alpha = 0.5. \end{cases} \tag{32}$$

Proof: For any given $\alpha \in (0,1]$, denote $\Phi(x) = \mathrm{Cr}\{\widetilde{\xi} \leq x\} = \alpha$. Since $\Phi(x) = \mathrm{Cr}\{\xi \leq x\}$ is strictly increasing in $\{x \leq \underline{c}\}$ and $\{x \geq \overline{c}\}$, for $\alpha \in (0,1]$ and $\alpha \neq 0.5$, we have that $x = \Phi^{-1}(\alpha)$. Combining the above with Eq. (28) in Theorem 6, we get $\widetilde{\xi}_{\inf}(\alpha) = \Phi^{-1}(\alpha)$. In terms of Eq. (12), we get $\Phi(x) = 1 - \Psi(x) = \alpha$, which follows that $\widetilde{\xi}_{\inf}(\alpha) = \Psi^{-1}(1-\alpha)$. Besides, according to the definition of $\widetilde{\xi}_{\inf}$ that when $\alpha = 0.5$, we have $\widetilde{\xi}_{\inf}(\alpha) = \underline{c}$. ■

Analogously, the analytical expression of $\widetilde{\xi}_{\inf}(\alpha)$ in Eq. (32) is derived based on Eq. (26) as

$$\widetilde{\xi}_{\inf}(\alpha) = \begin{cases} \underline{c} - \gamma L^{-1}(2\alpha), & \text{if } 0 < \alpha < 0.5 \\ \underline{c}, & \text{if } \alpha = 0.5 \\ \beta R^{-1}(2-2\alpha) + \overline{c}, & \text{if } 0.5 < \alpha \leq 1. \end{cases} \tag{33}$$

On the basis of the above analytical analyses, some regular fuzzy intervals are illustrated in Examples 12-14, and their corresponding α-optimistic and α-pessimistic values are deduced respectively in light of Eqs. (26) and (33) (details see Appendix C).

Theorem 8: Let $\widetilde{\xi}$ be a regular fuzzy interval. Then

$$\widetilde{\xi}_{\inf}(\alpha) = \widetilde{\xi}_{\sup}(1-\alpha) \tag{34}$$

holds for $\alpha \in (0,1]$ except $\alpha = 0.5$. Especially, if $\widetilde{\xi}$ is a regular fuzzy number, Eq. (34) holds for $\alpha \in (0,1]$.

Proof: It follows immediately from Definitions 3 and 5, and Eqs. (23) and (32). The proof is complete. ■

Theorem 9: Assume that $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$ are independent regular fuzzy intervals. Denote $\widetilde{\boldsymbol{\xi}} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)$. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0,1]$, we have that

$$f(\widetilde{\boldsymbol{\xi}})_{\sup}(\alpha) = f\Big((\widetilde{\xi}_1)_{\sup}(\alpha), \cdots, (\widetilde{\xi}_h)_{\sup}(\alpha), (\widetilde{\xi}_{h+1})_{\inf}(\alpha), \cdots, (\widetilde{\xi}_n)_{\inf}(\alpha)\Big).$$

Proof: Without loss of generality, we will merely prove the case of $h = 1$ and $n = 2$. On the basis that $\widetilde{\xi}_1$ and $\widetilde{\xi}_2$ are independent regular fuzzy intervals, for any $\alpha \in (0,1]$, we have that

$$\begin{aligned} &\mathrm{Cr}\{f(\widetilde{\xi}_1, \widetilde{\xi}_2) \geq f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha))\} \\ &\geq \mathrm{Cr}\{\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha)\} \cap \{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha)\}\} \\ &= \mathrm{Cr}\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha)\} \wedge \mathrm{Cr}\{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha)\} \\ &= \alpha \wedge \alpha \\ &= \alpha. \end{aligned}$$

Then again, since the function f is continuous, for any $\epsilon > 0$, there exists a real number $\delta > 0$ such that if

$$|x_1 - (\widetilde{\xi}_1)_{\sup}(\alpha)| + |x_2 - (\widetilde{\xi}_2)_{\inf}(\alpha)| \leq \delta,$$

we have $|f(x_1, x_2) - f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha))| < \epsilon$. By taking advantage of the independence, we get

$$\begin{aligned} &\mathrm{Cr}\{f(\widetilde{\xi}_1, \widetilde{\xi}_2) \geq f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha)) + \epsilon\} \\ &\leq \mathrm{Cr}\{\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha) + \delta\} \cup \{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha) - \delta\}\} \\ &= \mathrm{Cr}\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha) + \delta\} \vee \mathrm{Cr}\{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha) - \delta\} \\ &< \alpha. \end{aligned}$$

Eventually, we obtain that

$$f(\widetilde{\boldsymbol{\xi}})_{\sup}(\alpha) = f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha)).$$

The proof is complete. ■

Theorem 10: Let $\widetilde{\xi}$ be a regular fuzzy interval. If its expected value exists, then

$$E[\widetilde{\xi}] = \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha. \tag{35}$$

Proof: Denote $\widetilde{\xi} = (\underline{c}, \overline{c}, \gamma, \beta)_{LR}$. Providing that $\underline{c} \geq 0$, it follows from the definition of the expected value operator in Eq. (1) and the credibility distribution in Eq. (11) that

$$\begin{aligned} E[\widetilde{\xi}] &= \int_0^{+\infty} \mathrm{Cr}\{\widetilde{\xi} \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathrm{Cr}\{\widetilde{\xi} \leq x\}\mathrm{d}x \\ &= \int_0^{\underline{c}} (1 - \Phi(x))\mathrm{d}x + \int_{\underline{c}}^{\overline{c}} (1 - \Phi(x))\mathrm{d}x \\ &\quad + \int_{\overline{c}}^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{\underline{c}} x\mathrm{d}\Phi(x) + \int_{\overline{c}}^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) \\ &= \int_{\Phi(0)}^{0.5} \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_{\alpha\downarrow 0.5}^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha \\ &\quad + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha. \end{aligned} \tag{36}$$

Combining with Theorem 8, then Eq. (36) can be further written as

$$\begin{aligned} E[\widetilde{\xi}] &= \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(1-\alpha)\mathrm{d}\alpha \\ &= -\int_1^0 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha. \end{aligned}$$

Similar proof procedure can be provided to derive Eq. (35) if $\underline{c} \leq 0$. The proof is complete. ■

Using the results presented in Theorems 9 and 10, the calculation formula of the expected values of strictly monotone functions of regular fuzzy intervals is provided in Theorem 11.

Theorem 11: Suppose that $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$ are independent regular fuzzy intervals. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0,1]$, the expected value of $f(\widetilde{\boldsymbol{\xi}}) = f(\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)$ is

$$E[f(\widetilde{\boldsymbol{\xi}})] = \int_0^1 f\Big((\widetilde{\xi}_1)_{\sup}(\alpha), \cdots, (\widetilde{\xi}_h)_{\sup}(\alpha), (\widetilde{\xi}_{h+1})_{\inf}(\alpha), \cdots, (\widetilde{\xi}_n)_{\inf}(\alpha)\Big)\mathrm{d}\alpha.$$

Proof: The proof derives directly from Theorems 9 and 10, thus it is omitted. ■

Notably, on the basis of Theorem 8, Theorem 11 will be directly transformed into Theorem 3 as the version for obtaining expected values of functions of a series of independent regular fuzzy numbers.

As illustrated in this paper, regular fuzzy numbers can be observed as a special case of regular fuzzy intervals. This means all the definitions and theorems raised in this section for regular fuzzy intervals also hold for regular fuzzy numbers, which is consistent with the results presented in Li [14].

B. Simulation algorithms

For the purpose of carrying out the expected values of strictly monotone functions of regular fuzzy intervals, we extend the iSDA and NIA-S from regular fuzzy numbers to their relevant interval versions, called TiSDA and TNIA-S.

First, the basic concept of the TiSDA resembles that of the iSDA except that we do not consider the interval range where the membership degree equals to 1 in regular fuzzy intervals. Analogously, discretize a continuous regular fuzzy interval according to the extended version of Theorem 1 in [16] at first place. Without loss of generality, as to a regular fuzzy interval $\widetilde{\xi}_i$, its closure of the support is denoted by $S_i = [a_i, b_i]$. Since ξ_i is a regular fuzzy interval, there exists an interval $[\underline{c}_i, \overline{c}_i] \in S_i$ such that their membership degrees correspond to 1 and $a_i < \underline{c}_i < \overline{c}_i < b_i$. Then, equally divide the left part of $\underline{c}_i$ and the right part of $\overline{c}_i$ in S_i (i.e., $[a_i, \underline{c}_i]$ and $[\overline{c}_i, b_i]$) into k pieces, respectively. Setting the jth point of the left part as x_{ij}^L and the $(k-j)$th point of the right part as x_{ij}^R for $i = 1, 2, \cdots, n$, we get

$$\begin{aligned} x_{ij}^L &= a_i + (\underline{c}_i - a_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \\ x_{ij}^R &= b_i - (b_i - \overline{c}_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1. \end{aligned} \tag{37}$$

The forms of $\boldsymbol{X}_j^L$ and $\boldsymbol{X}_j^R$ of regular fuzzy intervals are identical to those in Eq. (5), in which x_{ij}^L and x_{ij}^R refer to Eq. (37). Additionally, $\underline{\boldsymbol{c}} = (\underline{c}_1, \cdots, \underline{c}_h, \overline{c}_{h+1}, \cdots, \overline{c}_n)$ and $\overline{\boldsymbol{c}} = (\overline{c}_1, \cdots, \overline{c}_h, \underline{c}_{h+1}, \cdots, \underline{c}_n)$ are included in the discretization procedure. Similarly to Eq. (6), the discrete fuzzy interval $f^*(\widetilde{\boldsymbol{\xi}})$ is defined, where $f(\boldsymbol{X}_j^L)$ and $f(\boldsymbol{X}_j^R)$ are with membership degrees $\mu_1(x_{1j}^L)$ and $\mu_1(x_{1j}^R)$ for $j = 0, 1, \cdots, k-1$, respectively and $f(\underline{\boldsymbol{c}})$ and $f(\overline{\boldsymbol{c}})$ are with the membership degree 1.

Next, figure out the mean value for $f^*(\widetilde{\boldsymbol{\xi}})$. We only need to replace $w_k f(\boldsymbol{c})$ in Eq. (7) by $w_{k_1} f(\underline{\boldsymbol{c}}) + w_{k_2} f(\overline{\boldsymbol{c}})$, where

$$w_{k_1} = \frac{1}{2}\big(1 - \mu_1(x_{1,k-1}^L)\big), \quad w_{k_2} = \frac{1}{2}\big(1 - \mu_1(x_{1,k-1}^R)\big).$$

The calculation of other w_j for $j = 1, 2, \cdots, m$ are based on Eq. (9).

As a result, through substituting Steps 2 and 5 of the iSDA in Algorithm 2 with the above discretization procedure of regular fuzzy intervals and replacing "Reset $E = E + w_k f(\boldsymbol{c})$" in Step 8 by "Reset $E = E + (w_{k_1} f(\underline{\boldsymbol{c}}) + w_{k_2} f(\overline{\boldsymbol{c}}))$", we constitute a new simulation algorithm TiSDA for regular fuzzy intervals. It is noted that there are some differences between the iSDA and TiSDA. The peak value $\boldsymbol{c}$ in the iSDA is extended to $\underline{\boldsymbol{c}}$ and $\overline{\boldsymbol{c}}$ in TiSDA. Meanwhile, the number of discrete points in the iSDA is $2k+1$, while that of the TiSDA is $2k+2$.

Further, on the basis of Theorem 11 and the analytical expressions of $\widetilde{\xi}_{\sup}(\alpha)$ in Eq. (23) and $\widetilde{\xi}_{\inf}(\alpha)$ in Eq. (32), the TNIA-S is proposed to approximate mean values for strictly monotone functions of regular fuzzy intervals, which shares similar concept with the NIA-S. Likewise, when the inverse functions for L and R are not easy to derive in some situations, we can complete them with the aid of the "polyfit" function in Matlab or take advantage of the bisection algorithm (see Algorithm 3) to obtain $\Psi^{-1}(\alpha)$ directly.

C. Comparative study among the SDA, TiSDA, and TNIA-S

Two numerical examples regarding the widely used trapezoidal fuzzy number and other two regular fuzzy intervals are implemented in this section to indicate the efficiencies of the TiSDA and TNIA-S. Since the SDA is suitable for simulating mean values of general functions containing all kinds of fuzzy numbers, here the simulation results of the SDA are also taken into account for the purpose of comparison.

Example 7: Assume that $\widetilde{\eta}_i, i = 1, 2, \cdots, 10$, are independent trapezoidal fuzzy numbers summarized in Table IX involved in two continuous and strictly monotone functions $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$, and $f_3 = x_1 + \cdots + x_5 - x_6 - \cdots - x_{10}$. We need to accomplish the expected value $E[\widetilde{\xi}]$ of the fuzzy number $\widetilde{\xi}_j = f_j(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_{10}), j = 2, 3$.

TABLE IX
DIFFERENT KINDS OF REGULAR FUZZY INTERVALS UTILIZED IN EXAMPLES

Index	Trapezoidal fuzzy number	Two regular fuzzy intervals	
$\widetilde{\eta}_1$	$\mathcal{A}(2,3,5,8)$	$\mathcal{B}(2,3,5,8)$	$\mathcal{C}(2,3,5,8)$
$\widetilde{\eta}_2$	$\mathcal{A}(4,6,7,9)$	$\mathcal{B}(4,6,7,9)$	$\mathcal{C}(4,6,7,9)$
$\widetilde{\eta}_3$	$\mathcal{A}(5,6,7,8)$	$\mathcal{B}(5,6,7,8)$	$\mathcal{C}(5,6,7,8)$
$\widetilde{\eta}_4$	$\mathcal{A}(2,4,5,6)$	$\mathcal{B}(2,4,5,6)$	$\mathcal{C}(2,4,5,6)$
$\widetilde{\eta}_5$	$\mathcal{A}(3,5,6,9)$	$\mathcal{B}(3,5,6,9)$	$\mathcal{C}(3,5,6,9)$
$\widetilde{\eta}_6$	$\mathcal{A}(6,7,9,10)$	$\mathcal{B}(6,7,9,10)$	$\mathcal{C}(6,7,9,10)$
$\widetilde{\eta}_7$	$\mathcal{A}(-5,-3,-2,-1)$		
$\widetilde{\eta}_8$	$\mathcal{A}(2,6,8,9)$		
$\widetilde{\eta}_9$	$\mathcal{A}(0,1,2,4)$		
$\widetilde{\eta}_{10}$	$\mathcal{A}(-1,0,2,5)$		

Initially, we obtain the exact value of $E[\widetilde{\xi}_3]$ on the basis of the linearity of the expected value operator E, that is,

$$E[\widetilde{\xi}_3] = E[\widetilde{\eta}_1] + \cdots + E[\widetilde{\eta}_5] - E[\widetilde{\eta}_6] - \cdots - E[\widetilde{\eta}_{10}] = 12.75.$$

While the exact values of $E[\widetilde{\xi}_2]$ is a little more challenging to derive, which are calculated with Matlab and recorded in Table X.

The approximation results of the SDA, TiSDA, and TNIA-S are also listed in Table X, in which (m/N) indicates the numbers of sample or integration points involved in the experiment. Similarly as before, the outputs of the SDA are unsteady, and thus the average values of ten times outputs are taken in the table, while the simulation results of the TiSDA and TNIA-S are identical every time. As to the two types of functions in Example 7, it is explicit that the TiSDA and TNIA-S are reliable and stable no matter the accuracy or the operation speed. In contrast, the largest error degree of the SDA is 3.45% when it comes to the function f_2 and the time consumed is hundreds of times larger than the

TABLE X
COMPARATIVE RESULTS AMONG THE SDA, TISDA, AND TNIA-S FOR THE CASE OF f_2 AND f_3.

Algorithm	f_2 $-(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$	f_3 $x_1 + x_2 + \cdots - x_{10}$
SDA (3000/10000)		
Exact Value	2.7500	12.7500
Simulation Value	2.8448	12.6033
Error	3.45%	1.15%
CPU Time (s)	0.551	0.585
TiSDA (10000/none)		
Exact Value	2.7500	12.7500
Simulation Value	2.7500	12.7500
Error	0.00%	0.00%
CPU Time (s)	0.001	0.000
TNIA-S (none/10000)		
Exact Value	2.7500	12.7500
Simulation Value	2.7499	12.7477
Error	0.00%	0.02%
CPU Time (s)	0.007	0.008

other two algorithms. Although both the TiSDA and TNIA-S return satisfactory simulation results at the end, the overall performance of the TiSDA is still better than the TNIA-S. It is expected that the accuracy of the TNIA-S will be further enhanced as the number of integration points N rises, but the time needed will grow as well which surely decreases its competitiveness. Overall, the TiSDA outperforms the SDA in every aspects, and it is able to compute a precise enough value in a relatively short time period.

Example 8: Three types of regular fuzzy intervals of Examples 12-14 are listed in Table IX, which are respectively incorporated in a continuous and strictly increasing function $f_4 = \sqrt{x_1^2 + x_2^2 + \cdots + x_6^2}, x_i \geq 0, i = 1, 2, \cdots, 6$, and another continuous and strictly monotone function $f_5 = x_1 x_2 x_3/(x_4 x_5 x_6)$. Calculate the corresponding expected value of $E[\widetilde{\xi}]$ of the fuzzy number $\widetilde{\xi} = f_4(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_6)$ or $\widetilde{\xi} = f_5(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_6)$ for three types of regular fuzzy intervals.

Six kinds of outputs of three regular fuzzy intervals under two continuous and strictly monotone functions are clearly illustrated in Table XI. Firstly, as to different functions, the SDA returns better computations in f_4 than f_5 for the former two regular fuzzy intervals. Combining this with the two functions f_2 and f_3 in Example 7, we know that the SDA is not reliable when encountering different functions. In contrast, the TiSDA and TNIA-S are more dependable, flexible, and adaptable to changeable functions, and can both return perfect simulation results. Secondly, as to different regular fuzzy intervals, it is explicit that the error degree in the SDA becomes larger as the form of the membership function gets complicated, especially in $\mathcal{C}(a, b, c, d)$. This situation also happens in the TiSDA and TNIA-S. However, their simulation outcomes are still quite satisfactory. The performances of the TiSDA and TNIA-S in this example are comparable as well whereas the TiSDA is more time-saving, which are consistent with previous analyses in numerical examples.

V. CONCLUSION

The regular fuzzy numbers which include triangular, normal and Gaussian fuzzy numbers, and the regular fuzzy intervals which contain trapezoidal fuzzy numbers are appeared in many real-world applications. In the corresponding literature, there exist two mainstream fuzzy simulation algorithms in approximating mean values for fuzzy numbers. The first one, namely SDA, was proposed by Liu and Liu [4], and it follows the concept that stochastically discretize continuous fuzzy numbers. The SDA is capable of simulating the expected value for general functions containing different fuzzy numbers. The second algorithm, namely NIA-G, was formulated by Li [14], and it is based on the integration simulation and the bisection procedure.

In this paper, we put forward two novel simulation techniques of calculating expected values for strictly monotone functions of regular fuzzy numbers. First, the iSDA was proposed to revise the stochastic discretization procedure and the calculation formula of the expected value of the SDA. These two parts were substituted by a novel uniform sampling process that we initiated and another calculation formula applying to discrete fuzzy numbers, respectively. Second, the NIA-S took advantage of the analytical expressions of α-optimistic values of regular fuzzy numbers directly in its algorithm design to replace the bisection procedure in NIA-G. From the results obtained for the regular fuzzy numbers, we observed that although the iSDA and NIA-S are based on distinct simulation concepts, they both performed better in accuracy, stability, and computational time when they compared with SDA. In addition, as to regular fuzzy intervals, the iSDA and NIA-S were extended to the TiSDA and TNIA-S according to a series of regular fuzzy interval related theorems, respectively. These novel theorems are not only the foundation of this paper, but also will serve future research. The simulation results demonstrated that either the TiSDA or TNIA-S outperforms the SDA. Besides, it is noted that the continuous and strictly monotone functions f in the examples of this paper are simple. For every f, whether it is challenging to write the expression of f, we are able to conduct the expected value simulation using the proposed novel techniques.

Although significant progress has been conducted on improving the existing simulation algorithms of the expected values for fuzzy numbers and intervals, future research is also required. First, since the fuzzy numbers incorporated in our examples are of the same type, like triangular, normal, or Gaussian fuzzy numbers, it is worth exploring that the simulation results of the expected value of different type of fuzzy numbers. Second, the inclusive functions are restricted to strictly monotone functions. However, not all the applications in practice satisfy this condition, some of them may follow monotone or other types of functions. Finally, in this paper, the novel simulation algorithms are not only designed for expected value models, but it should be also extended to cover other fuzzy simulations approaches, such as calculating the credibility of fuzzy events or critical values.

TABLE XI
COMPARATIVE RESULTS AMONG THE SDA, TISDA, AND TNIA-S FOR THE CASE OF f_4 AND f_5 OF THREE REGULAR FUZZY INTERVALS.

Algorithm	Trapezoidal fuzzy number		$\mathcal{B}(a,b,c,d)$		$\mathcal{C}(a,b,c,d)$	
SDA (3000/10000)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.9497	1.8682	15.2901	2.0975	14.9121	2.3484
Error	0.36%	97.40%	0.20%	118.58%	5.39%	134.51%
CPU Time (s)	0.176	0.137	0.233	0.162	0.228	0.159
TiSDA (10000/none)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.8956	0.9464	15.2597	0.9596	15.7566	1.0013
Error	0.00%	0.00%	0.00%	0.00%	0.03%	0.01%
CPU Time (s)	0.001	0.000	0.002	0.001	0.002	0.001
TNIA-S (none/10000)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.8956	0.9465	15.2593	0.9597	15.7594	1.0015
Error	0.00%	0.01%	0.00%	0.01%	0.02%	0.01%
CPU Time (s)	0.004	0.002	0.007	0.002	0.009	0.003

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SUPPLEMENTARY MATERIAL

APPENDIX A
ACRONYMS UTILIZED IN THIS PAPER

TABLE XII
THE ACRONYMS OF ALGORITHMS AND FUZZY VARIABLES

Acronyms for algorithms
SDS: stochastic discretization simulation (Liu and Iwamura (1998) [18])
SDA: stochastic discretization algorithm (Liu and Liu (2002) [4])
HIA: hybrid intelligent algorithm (Liu (2002) [17])
UDA: uniform discretization algorithm (Liu (2006) [13])
NIA-G: general numerical integration algorithm (Li 2015 [14])
iSDA: improved stochastic discretization algorithm
SDA*: an intermediate algorithm between SDA and iSDA
NIA-S: special numerical integration algorithm
TiSDA: extended version of iSDA for regular fuzzy intervals
TNIA-S: extended version of NIA-S for regular fuzzy intervals
Acronyms for fuzzy variables
$\mathcal{T}(a,b,c)$: triangular fuzzy number
$\mathcal{N}(c,\sigma)$: normal fuzzy number
$\mathcal{G}(c,b)$: Gaussian fuzzy number
$\mathcal{A}(a,b,c,d)$: trapezoidal fuzzy number
$\mathcal{B}(a,b,c,d), \mathcal{C}(a,b,c,d)$: two specified regular fuzzy intervals

APPENDIX B
SOME PRELIMINARIES

The credibility measure was proposed based on the possibility measure and the necessity measure, which are introduced as follows.

Suppose that ξ is a fuzzy variable, μ is the membership function of ξ, and r is a real number. Then the fuzzy event $\{\xi \le r\}$ has the following possibility [27] and necessity [28],

$$\mathrm{Pos}\{\xi \le r\} = \sup_{x \le r} \mu(x),$$
$$\mathrm{Nec}\{\xi \le r\} = 1 - \mathrm{Pos}\{\xi > r\} = 1 - \sup_{x > r} \mu(x).$$

To overcome the absence of self-duality in the possibility or necessity measure, the credibility measure was further proposed by Liu and Liu in [4] as follows,

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}\big(\mathrm{Pos}\{\xi \le r\} + \mathrm{Nec}\{\xi \le r\}\big).$$

The definition of LR fuzzy numbers is due to Dubois and Prade [1], and it derives as follows.

Definition 6: (Dubois and Prade [29]) A shape function L (or R) is a decreasing function from $\mathbb{R}^+ \to [0,1]$ such that

(1) $L(0) = 1$;
(2) $L(x) < 1,\ \forall x > 0$;
(3) $L(x) > 0,\ \forall x < 1$;
(4) $L(1) = 0$ [*or* $L(x) > 0,\ \forall x$ *and* $L(+\infty) = 0$].

Definition 7: (Dubois and Prade [1]) A fuzzy number ξ is of LR-type if there exist shape functions L (for left) and R (for right), and scalers $\gamma > 0$, $\beta > 0$ with membership function

$$\mu_\xi(x) = \begin{cases} L\left(\dfrac{c-x}{\gamma}\right), & \text{if } x \le c \\ R\left(\dfrac{x-c}{\beta}\right), & \text{if } x > c, \end{cases} \tag{38}$$

where the real number c is called the mean value or peak of ξ, and γ and β are called the left and right spreads, respectively. Symbolically, ξ is denoted by $(c,\gamma,\beta)_{LR}$.

A generalized definition for the fuzzy intervals together with the LR fuzzy intervals were proposed by Dubois and Prade [30], which are reviewed as follows.

Definition 8: (Dubois and Prade [30]) A fuzzy interval $\widetilde{\xi}$ is a quantity with a quasi-concave membership function μ, i.e., a convex fuzzy subset of the real line $\mathbb{R}$ such that

$$\mu(z) \ge \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in \mathbb{R}, z \in [x, y]. \tag{39}$$

Definition 9: (Dubois and Prade [30]) A fuzzy interval $\widetilde{\xi}$ is of LR-type if there exist shape functions L (for left), R (for right) and four parameters $(\underline{c}, \overline{c}) \in \mathbb{R}^2 \bigcup \{-\infty, +\infty\}, \gamma > 0, \beta > 0$ with membership function

$$\mu_{\widetilde{\xi}}(x) = \begin{cases} L\left(\dfrac{\underline{c}-x}{\gamma}\right), & \text{if } x \le \underline{c} \\ 1, & \text{if } \underline{c} < x \le \overline{c} \\ R\left(\dfrac{x-\overline{c}}{\beta}\right), & \text{if } x > \overline{c}, \end{cases} \tag{40}$$

and the fuzzy interval is represented as $\widetilde{\xi} = (\underline{c}, \overline{c}, \gamma, \beta)_{LR}$.

It is obtained that when $\overline{c} = \underline{c}$, an LR fuzzy interval is turned to be an LR fuzzy number, which means an LR fuzzy number can be regarded to be a degradation form to an LR fuzzy interval.

APPENDIX C
SOME COMMONLY USED REGULAR FUZZY NUMBERS AND INTERVALS WITH THEIR α-OPTIMISTIC AND α-PESSIMISTIC VALUES

Three commonly used regular fuzzy numbers are utilized in the numerical examples of this paper, including the triangular, normal, and Gaussian fuzzy numbers.

Example 9: When the shape functions L and R are written by the following form,

$$L(x) = R(x) = \max\{0, 1-x\},$$

the corresponding LR fuzzy number is a triangular fuzzy number, whose membership function is determined by the triplet (a, c, b) with $a < c < b$ as

$$\mu_{\mathcal{T}}(x) = \begin{cases} \dfrac{x-a}{c-a}, & \text{if } a \le x \le c \\ \dfrac{x-b}{c-b}, & \text{if } c < x \le b \\ 0, & \text{otherwise,} \end{cases}$$

and we can also denote $\xi = (c, c-a, b-c)_{LR}$ or $\xi \sim \mathcal{T}(a,c,b)$. Based on Eq. (17), its α-optimistic value is obtained as

$$\xi_{\mathcal{T}\sup}(\alpha) = \begin{cases} 2\alpha c + (1-2\alpha)b, & \text{if } 0 < \alpha \le 0.5 \\ (2\alpha-1)a + (2-2\alpha)c, & \text{if } 0.5 < \alpha \le 1. \end{cases} \tag{41}$$

Example 10: When the shape functions L and R are written by the following form,

$$L(x) = R(x) = 2\left(1+\exp(\pi x/\sqrt{6})\right)^{-1},$$

the corresponding LR fuzzy number is a normal fuzzy number, whose membership function is known as

$$\mu_{\mathcal{N}}(x) = 2\left(1+\exp\left(\pi|x-c|/\sqrt{6}\sigma\right)\right)^{-1}, \quad x\in\mathbb{R}, \sigma>0,$$

which can also be denoted by $\xi=(c,\sigma,\sigma)_{LR}$ or $\xi\sim\mathcal{N}(c,\sigma)$. Based on Eq. (17), its α-optimistic value is obtained as

$$\xi_{\mathcal{N}\sup}(\alpha) = c+(\ln(1-\alpha)-\ln\alpha)\sqrt{6}\sigma/\pi, \quad \alpha\in(0,1). \tag{42}$$

Example 11: When the shape functions L and R are written by the following form,

$$L(x) = R(x) = e^{-x^2},$$

the LR fuzzy number is a Gaussian fuzzy number, whose membership function is expressed as

$$\mu_{\mathcal{G}}(x) = e^{-(\frac{x-c}{b})^2}, \quad x\in\mathbb{R}, b>0,$$

and can also be represented by $\xi=(c,b,b)_{LR}$ or $\xi\sim\mathcal{G}(c,b)$. Based on Eq. (17), its α-optimistic value is obtained as

$$\xi_{\mathcal{G}\sup}(\alpha) = \begin{cases} c+b\sqrt{-\ln(2\alpha)}, & \text{if } 0<\alpha\le 0.5\\ c-b\sqrt{-\ln(2-2\alpha)}, & \text{if } 0.5<\alpha<1. \end{cases} \tag{43}$$

The following are three regular fuzzy intervals adopted for numerical examples in this paper.

Example 12: When the shape functions L and R are

$$L(x) = R(x) = \max\{0, 1-x\},$$

the corresponding LR fuzzy interval $\widetilde{\xi}$ is a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number $\widetilde{\xi}$ with $a<b<c<d$ is

$$\mu_{\mathcal{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a\le x<b\\ 1, & \text{if } b\le x\le c\\ \dfrac{d-x}{d-c}, & \text{if } c<x\le d\\ 0, & \text{otherwise,} \end{cases} \tag{44}$$

which is denoted by $\widetilde{\xi}\sim\mathcal{A}(a,b,c,d)$, and is illustrated in Fig. 2.

Further, in light of Eqs. (26) and (33), the α-optimistic and α-pessimistic values of a trapezoidal fuzzy number $\widetilde{\xi}\sim\mathcal{A}(a,b,c,d)$ are derived as follows,

$$\widetilde{\xi}_{\mathcal{A}\sup}(\alpha) = \begin{cases} d-2(d-c)\alpha, & \text{if } 0<\alpha\le 0.5\\ 2b-a-2(b-a)\alpha, & \text{if } 0.5<\alpha\le 1 \end{cases} \tag{45}$$

$$\widetilde{\xi}_{\mathcal{A}\inf}(\alpha) = \begin{cases} a+2(b-a)\alpha, & \text{if } 0<\alpha\le 0.5\\ 2c-d+2(d-c)\alpha, & \text{if } 0.5<\alpha\le 1, \end{cases} \tag{46}$$

which are depicted in Figs. 3 and 4, respectively.

Example 13: When the shape functions L and R are

$$L(x) = \max\{0,1-x\}, R(x) = \max\{0,1-x^2\},$$

a new LR fuzzy interval $\widetilde{\xi}$ is established, whose membership function with $a<b<c<d$ is

$$\mu_{\mathcal{B}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a\le x<b\\ 1, & \text{if } b\le x\le c\\ \dfrac{(x+d-2c)(d-x)}{(d-c)^2}, & \text{if } c<x\le d\\ 0, & \text{otherwise,} \end{cases} \tag{47}$$

which is denoted by $\widetilde{\xi}\sim\mathcal{B}(a,b,c,d)$. Similarly, its α-optimistic and α-pessimistic values are respectively derived as

$$\widetilde{\xi}_{\mathcal{B}\sup}(\alpha) = \begin{cases} c+(d-c)\sqrt{1-2\alpha}, & \text{if } 0<\alpha\le 0.5\\ 2b-a-2(b-a)\alpha, & \text{if } 0.5<\alpha\le 1 \end{cases} \tag{48}$$

$$\widetilde{\xi}_{\mathcal{B}\inf}(\alpha) = \begin{cases} a+2(b-a)\alpha, & \text{if } 0<\alpha\le 0.5\\ c+(d-c)\sqrt{2\alpha-1}, & \text{if } 0.5<\alpha\le 1. \end{cases} \tag{49}$$

Example 14: When the shape functions L and R are

$$L(x) = \max\{0,1-x^2\}, R(x) = e^{-x},$$

another new LR fuzzy interval $\widetilde{\xi}$ is built, whose membership function with $a<b<c<d$ is

$$\mu_{\mathcal{C}}(x) = \begin{cases} \dfrac{(2b-a-x)(x-a)}{(b-a)^2}, & \text{if } a\le x<b\\ 1, & \text{if } b\le x\le c\\ e^{\frac{c-x}{d-c}}, & \text{if } c<x\le d\\ 0, & \text{otherwise,} \end{cases} \tag{50}$$

which is written as $\widetilde{\xi}\sim\mathcal{C}(a,b,c,d)$. Correspondingly, we get

$$\widetilde{\xi}_{\mathcal{C}\sup}(\alpha) = \begin{cases} c-(d-c)\ln(2\alpha), & \text{if } 0<\alpha\le 0.5\\ b-(b-a)\sqrt{2\alpha-1}, & \text{if } 0.5<\alpha\le 1 \end{cases} \tag{51}$$

$$\widetilde{\xi}_{\mathcal{C}\inf}(\alpha) = \begin{cases} b-(b-a)\sqrt{1-2\alpha}, & \text{if } 0\le\alpha\le 0.5\\ c-(d-c)\ln(2-2\alpha), & \text{if } 0.5<\alpha<1. \end{cases} \tag{52}$$

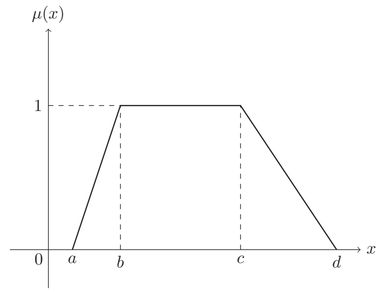

Fig. 2. The membership function of $\mathcal{A}(a,b,c,d)$ in Eq. (44).

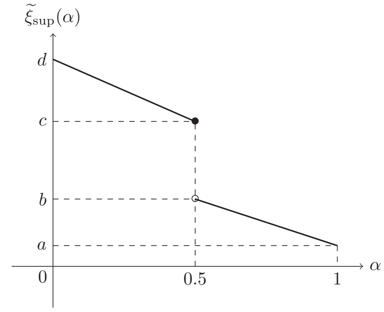

Fig. 3. The $\widetilde{\xi}_{\sup}(\alpha)$ value of $\mathcal{A}(a,b,c,d)$ in Eq. (45).

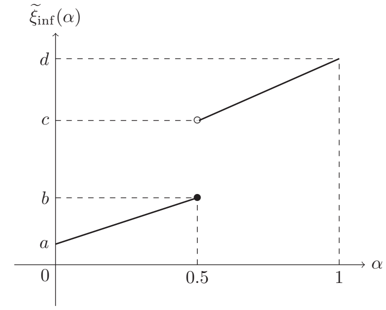

Fig. 4. The $\widetilde{\xi}_{\inf}(\alpha)$ value of $\mathcal{A}(a,b,c,d)$ in Eq. (46).

Appendix D
Algorithms: SDA and NIA-G

The detailed steps of *stochastic discretization algorithm* (SDA) are illustrated as follows.

Algorithm 1 (SDA of Liu and Liu [4])

Step 1. Initialize the numbers of sample points m and integration points N, and a sufficient small number ϵ. Set $E = 0$.
Step 2. Randomly generate $u_1^j, u_2^j, \cdots, u_n^j$ from the ϵ-level sets of $\xi_1, \xi_2, \cdots, \xi_n$, respectively, and denote $\boldsymbol{u}_j = (u_1^j, u_2^j, \cdots, u_n^j)$ for $j = 1, 2, \cdots, m$.
Step 3. Identify the minimal and maximal values $p = f(\boldsymbol{u}_1) \wedge f(\boldsymbol{u}_2) \wedge \cdots \wedge f(\boldsymbol{u}_m)$ and $q = f(\boldsymbol{u}_1) \vee f(\boldsymbol{u}_2) \vee \cdots \vee f(\boldsymbol{u}_m)$, respectively.
Step 4. Randomly generate a real number r from $[p, q]$.
Step 5. If $r \geq 0$, reset $E = E + E^R(r)$.
Step 6. If $r < 0$, reset $E = E - E^L(r)$.
Step 7. Repeat the 4th to 6th steps for N times.
Step 8. Return $E[f(\boldsymbol{\xi})] = p \vee 0 + q \wedge 0 + E \cdot (q - p)/N$.

The detailed steps of *general numerical integration algorithm* (NIA-G) incorporating a bisection algorithm are presented as follows.

Algorithm 3 (Bisection Algorithm of Li [14])

Step 1. Initialize a small enough number $\epsilon > 0$, and $[a, b]$ such that $\Psi(a) > \alpha > \Psi(b)$.
Step 2. Denote $d = (a + b)/2$.
Step 3. If $\Psi(d) > \alpha$, reset $a = d$. If $\Psi(d) < \alpha$, reset $b = d$. Otherwise, stop and return d.
Step 4. If $|\Psi(b) - \Psi(a)| \leq \epsilon$, return $(a + b)/2$. Otherwise, go to Step 2.

Algorithm 4 (NIA-G of Li [14])

Step 1. Initialize the number of integration points N. Set $E = 0$ and $k = 1$.
Step 2. Let $\alpha = k/N$. Using Algorithm 3, for each $1 \leq i \leq n$, calculate

$$x_i = \begin{cases} (\xi_i)_{\sup}(\alpha), & \text{if } 1 \leq i \leq h \\ (\xi_i)_{\sup}(1 - \alpha), & \text{if } h < i \leq n. \end{cases}$$

Step 3. Reset $E = E + f(x_1, x_2, \cdots, x_n)/N$ and $k = k + 1$.
Step 4. If $k \leq N$, go to Step 2. Otherwise return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.