A Note On Domination and Minus Domination Numbers in Cubic Graphs

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set if each vertex of $V - S$ has at least one neighbor in S. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A minus dominating function on G is a function $f: V \to \{-1, 0, 1\}$ such that $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ for each $v \in V$, where $N[v]$ is the closed neighborhood of v. The minus domination number of G is $\gamma^-(G) = min\{\sum_{v \in V} f(v) \mid f$ is a minus dominating function on $G\}$. It was incorrectly shown in [X. Yang, Q. Hou, X. Huang, H. Xuan, The difference between the domination number and minus domination number of a cubic graph, Applied Mathematics Letters 16(2003) 1089-1093] that there is an infinite family of cubic graphs in which the difference $\gamma - \gamma^-$ can be made arbitrary large. This note corrects the mistakes in the proof and poses a new problem on the upper bound for $\gamma - \gamma^-$ in cubic graphs.

Key words: Domination number, Minus domination number, Cubic graphs

1. Introduction

All the graphs considered in this paper are finite simple graphs without loops. Let $G = (V(G), E(G))$ be a graph. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $d(v) = |N(v)|$. The *minimum degree* of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. If each vertex of $V(G) - S$ has at least one neighbor in S, then we call S a *dominating set*. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A *minus dominating function* on G is a function $f: V(G) \to \{-1, 0, 1\}$ such that $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ for each $v \in V$. The *minus domination number* of G is $\gamma^-(G) = min\{\sum_{v \in V(G)} f(v) \mid f$ is a minus dominating function on $G\}$. The concept of minus domination was introduced by Dunbar et al. in [1]. A star of order n is denoted by S_n. Let $K_4(1)$ be the graph obtained from a complete graph K_4 on four vertices by subdividing one edge once. The *head* of the graph $K_4(1)$ is the only vertex of degree 2.

It is easy to see that $\gamma^-(G) \leq \gamma(G)$. Hedetniemi (see [2]) once asked the following question: Does there exist a cubic graph G for which $\gamma^-(G) < \gamma(G)$? In [2], Henning et al. answered the question in the affirmative by constructing a graph of order 52 with $\gamma^- = 14$ and $\gamma = 15$. However, it is not known that whether the difference $\gamma - \gamma^-$ can be made arbitrary large for cubic graphs.

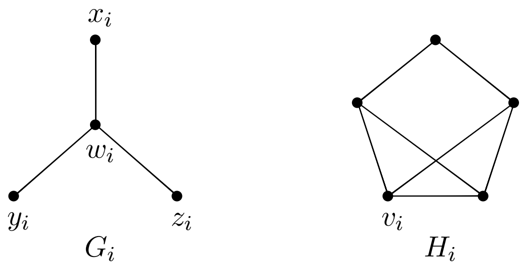

Figure 1.

Let $G_1, G_2, \ldots, G_k$ be k copies of S_4. Mark the vertices of each G_i $(1 \leq i \leq k)$ as shown in Figure 1. Let T be the tree obtained from $G_1, G_2, \ldots, G_k$ by identifying y_i and x_{i+1} for $1 \leq i \leq k-1$. It is easy to see that T contains k vertices of degree three, $k-1$ vertices of degree two and $k+2$ vertices of degree one. In [3], Yang et al. constructed an infinite family of cubic graphs $G(k)$ as follows: For a vertex v of T, if $d_T(v) = 1$, then take two copies of $K_4(1)$ and connect v to the heads of them, and if $d_T(v) = 2$, then take one copy of $K_4(1)$ and connect v to the head of it. Obviously, $G(k)$ contains $3k+3$ copies of $K_4(1)$, say $H_1, H_2, \ldots, H_{3k+3}$. Let v_i be a vertex of H_i as shown in Figure 1 and set $V_0 = \{v_i \mid 1 \leq i \leq 3k+3\}$. Furthermore, Yang et al. defined a "minus domination function" g on $V(G(k))$ as follows:

$$g(v) = \begin{cases} 1, & if\ v \in V_0\ or\ v \in V(T)\ and\ d_T(v) \neq 3, \\ -1, & if\ v \in V(T)\ and\ d_T(v) = 3, \\ 0, & otherwise. \end{cases}$$

They proved that $\gamma^-(G(k)) \leq \sum_{v \in V(G(k))} g(v) = 4k+4$ and $\gamma(G(k)) = 5k+4$, and then claimed that the difference $\gamma - \gamma^-$ can be made arbitrary large for cubic graphs. However, since $\sum_{u \in N[v]} g(u) = -1$ if $d_T(v) = 2$, g is not a minus domination function and hence the proof is incorrect. We do not know whether the graph $G(k)$ can show the result mentioned above. In this work we will show that the difference $\gamma - \gamma^-$ can be made arbitrary large by constructing a new infinite family of cubic graphs, and pose a new problem on the upper bound for $\gamma - \gamma^-$ in cubic graphs.

2. Our Examples

Let t be a positive integer, $n = 9t$, and $C_1 = u_1 u_2 \cdots u_n$ and $C_2 = v_1 v_2 \cdots v_n$ two cycles of length n. Define

- $A_i = \{a_{ij} \mid 1 \le j \le t\}$ for $i = 1, 2$,
- $B_i = \{u_l, v_l \mid l \equiv i \ (mod\ 3)\}$ for $0 \le i \le 2$,
- $X = \{x_i \mid 1 \le i \le n \ and \ i \equiv 0 \ (mod\ 3)\}$, and
- $Y = \{y_i \mid 1 \le i \le n \ and \ i \equiv 0 \ (mod\ 3)\}$.

Set $A = A_1 \cup A_2$ and $B = \cup_{i=0}^{2} B_i$. Let $H(n)$ be the graph with $V(H(n)) = A \cup B \cup X \cup Y$ and $E(H(n)) = E(C_1) \cup E(C_2) \cup E_1 \cup E_2 \cup E_3$, where

- $E_1 = \{u_i v_i \mid u_i, v_i \in B_1 \ and \ 1 \le i \le n\}$,
- $E_2 = \{u_i x_i, x_i y_i, y_i v_i \mid u_i, v_i \in B_0 \ and \ 1 \le i \le n\}$, and
- $E_3 = \{a_{1i} u_j, a_{2i} v_j \mid 1 \le i \le t,\ u_j, v_j \in B_2 \ and \ 9(i-1) + 2 \le j \le 9(i-1) + 8\}$.

The graph $H(18)$ is shown in Figure 2.

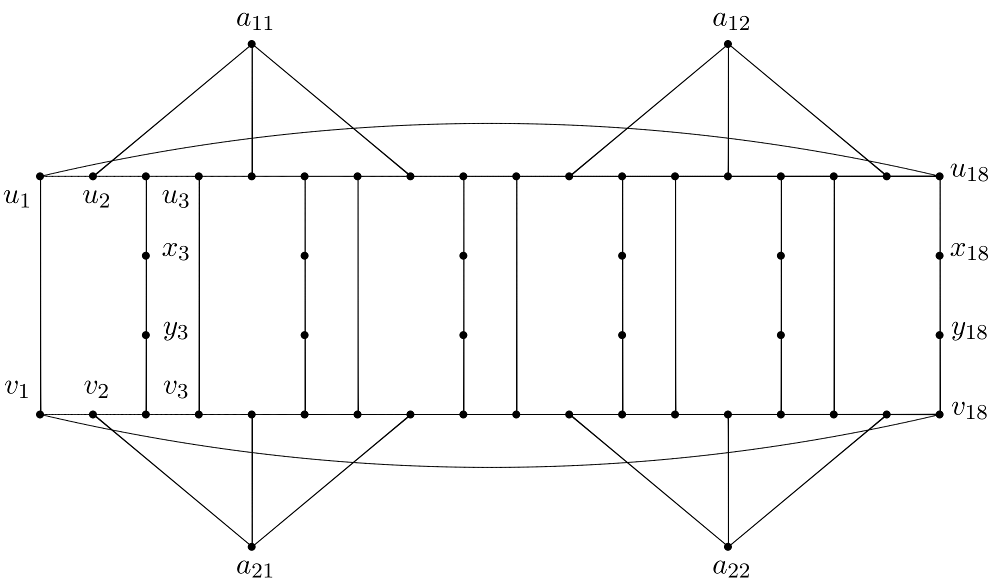

Figure 2.

Lemma 1. Let $H = H(n)[A \cup B]$ and S be a subset of $V(H)$. If S dominates $A \cup B - B_0$, then $|S| \geq 5n/9$.

Proof. Suppose S is a minimum subset of $V(H)$ that dominates $A \cup B - B_0$. Let $S \cap B_0 = S_1$ and $S - S_1 = S_2$. Choose S such that $|S_1|$ is as small as possible. Set $B_{0j} = \{u_i, v_i \mid u_i, v_i \in B_0 \text{ and } i \equiv 3(j-1) \ (mod\ 9)\}$, where $1 \leq j \leq 3$.

Claim 1. For any $u \in S_1$ and $v \in N[u]$, $N(v) \cap S_2 = \emptyset$.

Proof. Let $u \in S_1$ and $N(u) = \{u', u''\}$. Obviously, $u', u'' \notin B_0$. If $u' \in S_2$ or $N(u') \cap S_2 \neq \emptyset$, then $S \cup \{u''\} - \{u\}$ dominates $A \cup B - B_0$, which contradicts the choice of S. Thus we have $N(v) \cap S_2 = \emptyset$ for any $u \in S_1$ and $v \in N[u]$. ∎

Claim 2. $S \cap B_{02} = \emptyset$.

Proof. By symmetry, we need only to show that $u_3 \notin S$. If $u_3 \in S$, then by Claim 1 we have $u_2, u_4, u_5, a_{11} \notin S$. Since $N(u_5) = \{u_4, u_6, a_{11}\}$ and $N(a_{11}) = \{u_2, u_5, u_8\}$, in order to dominate u_5 and a_{11}, we have $u_6, u_8 \in S$, which contradicts Claim 1. ∎

Claim 3. $S \cap B_{03} = \emptyset$.

Proof. By symmetry, we need only to show that $u_6 \notin S$. If $u_6 \in S$, then by Claim 1 we have $u_4, u_5, u_7, u_8, v_7, a_{11} \notin S$. In order to dominate u_4, by Claim 2 we have $v_4 \in S$, which implies $v_6 \notin S$ by Claim 1. Since $N(v_7) = \{v_6, u_7, v_8\}$, we have $v_8 \in S$ in order to dominate v_7. In this case, $S \cup \{u_4, a_{21}, u_7\} - \{u_6, v_4, v_8\}$ dominates $A \cup B - B_0$, which contradicts the choice of S. ∎

Claim 4. $S \cap B_{01} = \emptyset$.

Proof. Let $U = \{a_{11}, a_{21}\} \cup \{u_i, v_i \mid i = 1, 2, 4, 5, 7, 8\}$. Similarly, we need only to show that $u_9 \notin S$. If $u_9 \in S$, then by Claim 1 we have $a_{11}, u_7, u_8 \notin S$. By Claim 2, we have $v_7 \in S$ in order to dominate u_7. By Claim 1, $v_9 \notin S$. If $v_8 \in S$ or $a_{21} \in S$, then $S \cup \{u_7\} - \{v_7\}$ dominates $A \cup B - B_0$, which contradicts Claim 1 and hence $a_{21}, v_8 \notin S$. Since $a_{11}, a_{21} \notin S$, by Claims 2 and 3, we have $|S \cap \{u_1, u_2\}| \geq 1$, $|S \cap \{v_1, v_2\}| \geq 1$ and $|S \cap \{u_4, u_5, v_4, v_5\}| \geq 2$ in order to dominate u_2, v_2, u_5, v_5. Thus we have $|S \cap U| \geq 5$. Obviously, $S' = (S - U) \cup \{a_{11}, a_{21}, u_1, u_4, u_7\}$ dominates $A \cup B - B_0$. Since $|S'| = |S|$ and $u_7, u_9 \in S'$, by Claim 1, this is a contradiction. ∎

By Claims 2-4, we have $S \subseteq A \cup B - B_0$. Let $P_3(6)$ be the graph obtained from three paths of order 6 by identifying their start vertices and end vertices, respectively. It is easy to see that $H[A \cup B - B_0]$ is the disjoint union of t copies of $P_3(6)$. Since $\gamma(P_3(6)) = 5$, we have $|S| \geq 5t = 5n/9$. ∎

We now begin to construct our examples $G(3, n)$. For each $v \in X \cup Y$, we let $H[v]$

be a graph that is isomorphic to $K_4(1)$ and $w(v)$ a given vertex of $H(v)$ that is not adjacent to the head of $H[v]$. Set $W = \{w(v) \mid v \in X \cup Y\}$. Let $G(3,n)$ be the graph obtained from $H(n)$ by connecting v to the head of $H(v)$ for each $v \in X \cup Y$. Set $F[v] = G(3,n)[V(H[v]) \cup \{v\}]$ and $F = \cup_{v \in X \cup Y} V(F[v])$.

Lemma 2. $\gamma(G(3,n)) = 17n/9$.

Proof. Let S be a minimum dominating set of $G(3,n)$ and $S \cap F = S_1$. It is easy to see that $|S \cap F[v]| \geq 2$ for each $v \in X \cup Y$, and hence $|S_1| \geq 4n/3$. Since S_1 cannot dominate any vertex of $A \cup B - B_0$, in order to dominate $A \cup B - B_0$, we have $|S \cap (A \cup B)| \geq 5n/9$ by Lemma 1. Thus we have $\gamma(G(3,n)) \geq 4n/3 + 5n/9 = 17n/9$. On the other hand, since $S' = A \cup X \cup Y \cup W \cup \{u_i \mid u_i \in B_1 \textit{ and } 1 \leq i \leq n\}$ is a dominating set of $G(3,n)$ and $|S'| = 17n/9$, we have $\gamma(G(3,n)) \leq 17n/9$, and hence $\gamma(G(3,n)) = 17n/9$. ∎

The following lemma was established independently by Dunbar et al. in [4] and Zelinka in [5].

Lemma 3. Let G be a cubic graph of order n. Then $\gamma^-(G) \geq n/4$.

By the definitions of minus domination function and minus domination number, it is easy to show that the equality in Lemma 3 holds if and only if $n \equiv 0 \pmod 4$, and there is a minus domination function f on G such that $\sum_{u \in N[v]} f(u) = 1$ for each $v \in V(G)$.

Lemma 4. $\gamma^-(G(3,n)) = 14n/9$.

Proof. Let f be a function on $V(G(3,n))$ defined as follows:

$$f(v) = \begin{cases} 1, & \textit{if } v \in A \cup B_1 \cup X \cup Y \cup W, \\ -1, & \textit{if } v \in B_0, \\ 0, & \textit{otherwise}. \end{cases}$$

It is easy to check that $\sum_{u \in N[v]} f(u) = 1$ for each $v \in V(G(3,n))$, and hence f is a minus domination function of $G(3,n)$. Since $\sum_{v \in V(G(3,n))} f(v) = 4n/3 + 2n/9 = 14n/9$, we have $\gamma^-(G(3,n)) \leq 14n/9$. On the other hand, noting that $G(3,n)$ is a graph of order $56n/9$, we have $\gamma^-(G(3,n)) \geq 14n/9$ by Lemma 3, and hence $\gamma^-(G(3,n)) = 14n/9$. ∎

Remark. From the proof of Lemma 4, we see that the lower bound of γ^- in Lemma 3 is the best possible.

Theorem 1. For any positive k, there is a cubic graph G such that $\gamma(G) - \gamma^-(G) \geq k$.

Proof. Take $G = G(3,n)$. By Lemmas 2 and 4, we have $\gamma(G) - \gamma^-(G) = n/3$. Since $n/3 \to \infty$ as $n \to \infty$, we see that the conclusion holds. ∎

3. Problem

Let G be a graph of order n. It is well known that $\gamma(G) \le n/2$. Reed [6, 7] proved that $\gamma(G) \le 3n/8$ if $\delta(G) \ge 3$, and conjectured $\gamma(G) \le \lceil n/3 \rceil$ if G is cubic. For the difference $\gamma(G) - \gamma^-(G)$, it was shown in [1] that $\gamma(G) - \gamma^-(G) \le (n-4)/5$ if G is a tree and the upper bound is sharp. If G is cubic, then by Lemma 3 and Reed's result, we have $\gamma(G) - \gamma^-(G) \le n/8$. Furthermore, if Reed's conjecture is true, then $\gamma(G) - \gamma^-(G) \le n/12$. Our problem is the following.

Problem 1. For a cubic graph G of order n, what is the best possible upper bound for $\gamma(G) - \gamma^-(G)$?

The graph $G(3, n)$ shows that the upper bound of $\gamma(G) - \gamma^-(G)$ is at least $3n/56$.

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